Advanced Studies in Contemporary Mathematics
28 (2018), No. 1, pp. 13 - 25

www.jangjeon.or.kr
http://dx.doi.org/10.17777/ascm2018.28.1.13

A New Combined Homotopy-Laplace Decomposition Method for Solving DDEs of Order $(1, 2)$

Ananth Kumar S. R.[1] and R. Rangarajan [2]
[1] Department of Mathematics,
Govt. First Grade College, Channagiri,
Davanagere-577 213, INDIA

[2] Department of Studies in Mathematics,
University of Mysore,
Manasagangotri,
Mysore - 570 006, INDIA
e-mail: ananthkumar.s.r@gmail.com, rajra63@gmail.com

Abstract

In the recent literature, nonlinear problems are solved by two powerful decomposition methods, namely, Laplace decomposition method and Homotopy analysis methods. In the present paper a new method is proposed motivated by the above two methods to solve both nonlinear differential-difference equations and integro-differential-difference equations of order $(1, 2)$.

Keywords: Differential-difference equation, Integro-differential-difference equation, Laplace transform, Adomian polynomials, Laplace decomposition method and Homotopy Analysis method.

AMS classification : 34K05, 3KK30, 44A10, 49M27, 65H20.

1 Introduction

The Laplace decomposition method is indeed a fine blend of both Laplace transform method and Adomian decomposition method [1, 2, 4]. Laplace decomposition method shows flexibility as well as provides convenience in the computation of analytical solutions for both linear and nonlinear problems [3, 5]. Homotopy analysis method describes a whole class of topologically deformed solutions including both approximate and exact solutions of a given nonlinear problem [8, 9]. This method is motivated by both perturbation methods and Adomian decomposition method which has also flexibility to handle both linear and nonlinear problems effectively. In the present paper, we workout a new combined Homotopy-Laplace Decomposition Method for solving Differential-Difference Equations (DDEs) with differential order one and difference of order two [6, 7, 10]. The motivation for this new strategy is to compute not just one approximate or exact solution but to compute a class of topologically deformed solutions including both approximate and exact solutions of a given nonlinear problem. The present computational result will extend the results already published in [3]. The test problems are also selected from the same reference. In this way the present paper enhances our previous work.

2 Description of the Homotopy-Laplace decomposition method

Let us consider the following differential-difference equation with differential order one and difference of order two:

$$\begin{aligned} u'(t) = f(t) \quad &+ \quad F_1(u(t-\omega), u'(t-\omega)) + F_2(u(t-2\omega), u'(t-2\omega)) \\ &+ \quad \int_0^t G(u(t_1 - 2\omega))dt_1 \qquad t > 2\omega, \end{aligned} \tag{2.1}$$

and the following initial constant interval condition:

$$u(t) \quad = \quad u_0, \qquad\qquad 0 \le t \le 2\omega. \tag{2.2}$$

In the above equations (2.1) and (2.2), $\omega > 0$ and u_0 are known constants. The known functions $f(t)$, $F_1(x, y)$, $F_2(x, y)$ and $G(x, y)$ are selected as either linear or nonlinear functions depending upon the particular problem discussed. Further they can be selected in such a way that, they can be approximated by Adomian polynomials suitable for iterative computation of Laplace transform as well as inverse Laplace transform [2, 3, 8].

Let us workout a standard homotopy analysis method for finding approximate solution for (2.1) and (2.2). Following Liao [9], let us systematically set the following ingredients.

1. The linear operator: $\mathcal{L}[\phi(t;q)] = \dfrac{\partial \phi(t;q)}{\partial t}$.
2. The nonlinear operator:

$$\begin{aligned} \mathcal{N}[\phi(t;q)] \quad = \quad & \frac{\partial \phi(t;q)}{\partial t} - f(t) - F_1(\phi(t-\omega;q), \phi'(t-\omega;q)) \\ & - F_2(\phi(t-2\omega;q), \phi'(t-2\omega;q)) - \int_0^t G(\phi(t_1 - 2\omega;q))dt_1. \end{aligned}$$

3. The embedding parameter: $q \in [0, 1]$ varies continuously from 0 to 1.

4. The auxiliary parameter: $h \neq 0$ is chosen to be equal to 1.
5. The auxiliary function: $H(x) \neq 0$ is chosen to be equal to -1.
6. The topologically deformed solution:

$\phi(x,q)$ is such that $\phi(t,0) = u_0(t) = k$, $\phi(t,1) = u(t)$

$$(1-q)\mathcal{L}[\phi(t;q) - u_0(t)] = h\ q\ H(t)\ \mathcal{N}[\phi(t;q)]. \tag{2.3}$$

Now (2.3) becomes,

$$\begin{aligned}(1-q)\left[\frac{\partial\phi(t;q)}{\partial t} - 0\right] &= -q\left[\frac{\partial\phi(t;q)}{\partial t} - f(t) - F_1(\phi(t-\omega;q),\phi'(t-\omega;q))\right.\\ &- F_2(\phi(t-2\omega;q),\phi'(t-2\omega;q))\\ &- \left.\int_0^t G(\phi(t_1-2\omega;q))dt_1\right].\end{aligned} \tag{2.4}$$

subjected to the initial interval condition

$$\phi(t,q) = k, \qquad 0 \le t \le 2\omega. \tag{2.5}$$

First we note that,

$\displaystyle\int_0^{2\omega} \frac{\partial\phi(t;q)}{\partial t}\ e^{-st}\ dt = 0$ as a result we have $\displaystyle\int_{2\omega}^{\infty} \frac{\partial\phi(t;q)}{\partial t}\ e^{-st}\ dt = L\left\{\frac{\partial\phi(t;q)}{\partial t}\right\}.$

Hence multiply both sides of (2.4) by e^{-st} and integrate between 2ω and ∞, we obtain

$$\begin{aligned}\int_{2\omega}^{\infty} \frac{\partial\phi(t;q)}{\partial t}\ e^{-st}\ dt &= q\int_{2\omega}^{\infty} f(t)\ e^{-st}\ dt\\ &+ q\int_{2\omega}^{\infty} F_1(\phi(t-\omega;q),\phi'(t-\omega;q))\ e^{-st}\ dt\\ &+ q\int_{2\omega}^{\infty} F_2(\phi(t-2\omega;q),\phi'(t-2\omega;q))\ e^{-st}\ dt\\ &+ q\int_{2\omega}^{\infty}\int_0^t G(\phi(t_1-2\omega;q))dt_1\ e^{-st}\ dt.\end{aligned}$$

Let us apply suitable shifting of variables,

$$\begin{aligned}L\{\phi(t;q)\} &= \frac{k}{s} - \frac{q\lambda\ e^{-\omega s}}{s^2} + \frac{q\lambda\ e^{-2\omega s}}{s^2} + \frac{qe^{-2\omega s}}{s}L\{f(t+2\omega)\}\\ &+ \frac{qe^{-\omega s}}{s}L\{F_1(\phi(t;q),\phi'(t;q))\}\\ &+ \frac{qe^{-2\omega s}}{s}L\{F_2(\phi(t;q),\phi'(t;q))\}\\ &+ \frac{qe^{-2\omega s}}{s^2}L\{G(\phi(t;q))\},\end{aligned} \tag{2.6}$$

where $\lambda = F_1(k,0)$ and note that, $\int_{2\omega}^{t+2\omega} G(\phi(t_1-2\omega;q))dt_1 = \int_0^t G(\phi(t_1;q))dt_1$.

Now consider the n^{th} order derivative with respect to the embedding parameter q :

$$i.e. \quad u_0^{[n]}(t) = \left.\frac{\partial^n \phi(t;q)}{\partial q^n}\right|_{q=0} \tag{2.7}$$

exists, where $n = 1,2,3,\ldots,$ $u_0^{[n]}(t)$ is called the n^{th} order deformation derivative

$$u_n(t) = \frac{u_0^{[n]}(t)}{n!} = \left.\frac{1}{n!}\frac{\partial^n \phi(t;q)}{\partial q^n}\right|_{q=0}. \tag{2.8}$$

By using Taylor's theorem, we can expand $\phi(t;q)$ in a power series of the embedding parameter q, given by

$$\phi(t;q) = \phi(t;0) + \sum_{n=1}^{\infty} \frac{1}{n!}\left.\frac{\partial^n \phi(t-n\omega;q)}{\partial q^n}\right|_{q=0} q^n \tag{2.9}$$

$$\phi(t;q) = u_0(t) + \sum_{n=1}^{\infty} u_n(t-n\omega)e(t-n\omega)q^n, \tag{2.10}$$

where $e(t)$ is a unit step function, given by

$$\begin{aligned} e(t-c) &= 0, \qquad t<c,\\ e(t-c) &= 1, \qquad t>c. \end{aligned}$$

By using (2.10), $\phi(t;q)$ takes the following form in each of the following intervals:

$$\phi(t;q) = \sum_{n=0}^{N} u_n(t-n\omega)q^n, \qquad N\omega \le t \le (N+1)\omega \tag{2.11}$$
$$N = 0,1,2,\ldots\ .$$

On applying Laplace transformation, we obtain the following Laplace decompositions:

$$L\{\phi(t;q)\} = \sum_{n=0}^{\infty} e^{-n\omega s} L\{u_n(t)\} q^n, \tag{2.12}$$

$$L\{F_1(\phi(t;q),\phi'(t;q))\} = \sum_{n=0}^{\infty} e^{-n\omega s} L\{A_n(t)\} q^n, \tag{2.13}$$

$$L\{F_2(\phi(t;q),\phi'(t;q))\} = \sum_{n=0}^{\infty} e^{-n\omega s} L\{B_n(t)\} q^n \tag{2.14}$$

$$\text{and} \quad L\{G(\phi(t;q))\} = \sum_{n=0}^{\infty} e^{-n\omega s} L\{C_n(t)\} q^n. \tag{2.15}$$

In (2.13), A_n $'s$ are the n^{th} Adomian Polynomials [5] of $F_1(x(t), y(t))$ as given below:

$$
\begin{aligned}
A_0(t) &= F_1(x,y)\,|_{(u_0(t),u_0'(t))},\\
A_1(t) &= \left.\frac{\partial F_1}{\partial x}\right|_{(u_0(t),u_0'(t))} u_1(t) \;+\; \left.\frac{\partial F_1}{\partial y}\right|_{(u_0(t),u_0'(t))} u_1'(t),\\
A_2(t) &= \left.\frac{\partial F_1}{\partial x}\right|_{(u_0(t),u_0'(t))} u_2(t) \;+\; \left.\frac{\partial F_1}{\partial y}\right|_{(u_0(t),u_0'(t))} u_2'(t)\\
&+ \frac{1}{2!}\left[\left.\frac{\partial^2 F_1}{\partial x^2}\right|_{(u_0(t),u_0'(t))} u_1^2(t) \;+\; 2\left.\frac{\partial^2 F_1}{\partial x \partial y}\right|_{(u_0(t),u_0'(t))} u_1(t)\; u_1'(t)\right.\\
&+ \left.\left.\frac{\partial^2 F_1}{\partial y^2}\right|_{(u_0(t),u_0'(t))} (u_1'(t))^2\right]
\end{aligned}
$$

and so on. In (2.8), B_n $'s$ are the n^{th} Adomian Polynomials [5] of $F_2(x(t), y(t))$. Let us note that $B_0, B_1, B_2, \ldots$ are same as $A_0, A_1, A_2, \ldots$ except for the fact that F_1 should be replaced by F_2 throughout. In (2.15), C_n $'s$ are the n^{th} Adomian Polynomials [5] of $G(\phi(t;q))$ as given below:

$$
\begin{aligned}
C_0(t) &= G(u_0(t)),\\
C_1(t) &= G'((u_0(t)))u_1(t),\\
C_2(t) &= G'((u_0(t)))u_2(t) + \frac{1}{2!}G''((u_0(t)))u_1^2(t)
\end{aligned}
$$

and so on.

Applying the Laplace decompositions (2.12) – (2.15) in (2.6), we obtain

$$
\begin{aligned}
\sum_{n=0}^{\infty} e^{-n\omega s} L\{u_n(t)\}\, q^n &= \frac{k}{s} - \frac{q\lambda\, e^{-\omega s}}{s^2} + \frac{q\lambda\, e^{-2\omega s}}{s^2} + \frac{q e^{-2\omega s}}{s} L\{f(t+2\omega)\}\\
&+ \frac{q e^{-\omega s}}{s} \sum_{n=0}^{\infty} e^{-n\omega s} L\{A_n(t)\}\, q^n\\
&+ \frac{q e^{-2\omega s}}{s} \sum_{n=0}^{\infty} e^{-n\omega s} L\{B_n(t)\}\, q^n\\
&+ \frac{q e^{-2\omega s}}{s^2} \sum_{n=0}^{\infty} e^{-n\omega s} L\{C_n(t)\}\, q^n. \qquad (2.16)
\end{aligned}
$$

One may compute $L\{u_n(t)\}$ iteratively as follows :

$$
\begin{aligned}
L\{u_0(t)\} &= \frac{k}{s} \quad ; \quad L\{u_1(t)\} = -\frac{\lambda}{s^2} + \frac{1}{s} L\{A_0(t)\}\\
L\{u_2(t)\} &= \frac{\lambda}{qs^2} + \frac{1}{qs} L\{f(t+2\omega)\} + \frac{1}{s} L\{A_1(t)\}\\
&+ \frac{1}{qs} L\{B_0(t)\} + \frac{1}{qs^2} L\{C_0(t)\}
\end{aligned}
$$

$$L\{u_{n+1}(t)\} = \frac{1}{s}L\{A_n(t)\} + \frac{1}{qs}L\{B_{n-1}(t)\} + \frac{1}{qs^2}L\{C_{n-1}(t)\}, \quad n = 2,3,4,\ldots .$$

The exact or approximate solution is obtained by using inverse Laplace transform by giving different values for $q \in [0,1]$. When $q = 1$ we get intervalwise exact solution.

3 Test Problems

In this Section, three test problems [3] are worked out to illustrate the applicability of the method.

Test Problem 1. Consider the following linear differential-difference equation with differential order one and difference of order two:

$$2u'(t) - u(t-\omega) = u(t-2\omega), \qquad t > 2\omega, \tag{3.1}$$

with the initial interval condition

$$u(t) = 1, \qquad 0 \le t \le 2\omega. \tag{3.2}$$

Let $q \in [0,1]$ denote the embedding parameter. The homotopy analysis method is based on a kind of continuous mapping $u(t) \to \phi(t;q)$ such that, as the embedding parameter q increases from 0 to 1 , $\phi(t;q)$ varies from the initial condition $u_0(t)$ to the exact solution $u(t)$.
Now choose an auxiliary linear operator as

$$\mathcal{L}[\phi(t;q)] = 2\frac{\partial \phi(t;q)}{\partial t} \tag{3.3}$$

and we define nonlinear operator

$$\mathcal{N}[\phi(t;q)] = 2\frac{\partial \phi(t;q)}{\partial t} - \phi(t-\omega;q) - \phi(t-2\omega;q). \tag{3.4}$$

Let $h = 1$ and $H(t) = -1$ denote the auxiliary parameter and auxiliary function respectively. Using the embedding parameter $q \in [0,1]$, we construct an equation

$$(1-q)\mathcal{L}[\phi(t;q) - u_0(t)] = h\ q\ H(t)\ \mathcal{N}[\phi(t;q)]. \tag{3.5}$$

Now by using (3.3) and (3.4) in (3.5) we have,

$$(1-q)\left[2\frac{\partial \phi(t;q)}{\partial t} - 0\right] = -q\left[2\frac{\partial \phi(t;q)}{\partial t} - \phi(t-\omega;q) - \phi(t-2\omega;q)\right]$$

subjected to the initial interval condition

$$\phi(t,q)=1, \qquad 0\le t\le 2\omega.$$

Following the initial steps of the method, we arrive at

$$\begin{aligned} L\{\phi(t,q)\} &= \frac{1}{s}-\frac{q}{2s^2}e^{-\omega s}+\frac{q}{2s^2}e^{-2\omega s}+\frac{q}{2s}e^{-\omega s}L\{\phi(t,q)\} \\ &\quad +\frac{q}{2s}e^{-2\omega s}L\{\phi(t,q)\}. \end{aligned} \tag{3.6}$$

The next step is to apply the following Laplace decomposition series for $L\{\phi(t,q)\}$:

$$L\{\phi(t,q)\} = L\{u_0(t)\}+\sum_{n=1}^{\infty}e^{-n\omega s}L\{u_n(t)\}q^n. \tag{3.7}$$

Now by using (3.7) in (3.6), we get,

$$\begin{aligned} \sum_{n=0}^{\infty}e^{-n\omega s}L\{u_n(t)\}q^n &= \frac{1}{s}-\frac{q}{2s^2}e^{-\omega s}+\frac{q}{2s}e^{-\omega s}\sum_{n=0}^{\infty}e^{-n\omega s}L\{u_n(t)\}q^n \\ &\quad +\frac{q}{2s^2}e^{-2\omega s}+\frac{q}{2s}e^{-2\omega s}\sum_{n=0}^{\infty}e^{-n\omega s}L\{u_n(t)\}q^n. \end{aligned} \tag{3.8}$$

Equating the terms with co-efficient of $e^{-n\omega s}$ on both sides of (3.8) we get $L\{u_n(t)\}$. An application of inverse Laplace transform will yield $u_n(t)$:

$$u_0(t)=1 \quad ; \quad u_1(t) = 0 \text{ and for } n\ge 2 \text{ we have,}$$

$$u_n(t) = \sum_{r=1}^{\lfloor \frac{n}{2}\rfloor}\binom{n-r-1}{r-1}\frac{t^{n-r}}{2^{n-r-1}\cdot(n-r)!q^r}, \qquad n\ge 2. \tag{3.9}$$

As the next step of the method, using (3.9) for $t>0$, we have

$$\phi(t,q) = 1+\sum_{n=2}^{\infty}\sum_{r=1}^{\lfloor \frac{n}{2}\rfloor}\binom{n-r-1}{r-1}\frac{(t-n\omega)^{n-r}e(t-n\omega)}{2^{n-r-1}\cdot(n-r)!}q^{n-r}. \tag{3.10}$$

Further, using (2.5) we can find the exact solution of (3.1) in the interval wise:

$$\begin{aligned} u(t) &= 1+\sum_{n=2}^{N}\sum_{r=1}^{\lfloor \frac{n}{2}\rfloor}\binom{n-r-1}{r-1}\frac{(t-n\omega)^{n-r}}{2^{n-r-1}\cdot(n-r)!}q^{n-r}, \; N\omega\le t\le (N+1)\omega \\ &\qquad N=2,3,4,\dots\;. \end{aligned} \tag{3.11}$$

If $q \to 1$ then $\phi(t,q) \to u(t)$ which is the same exact solution given in [3]. Further the equation (3.1) becomes first order differential equation and (3.10) yields

$$\begin{aligned} u(t) &= 1+\sum_{n=1}^{\infty}\left[\binom{n-1}{0}+\binom{n-1}{0}+\cdots+\binom{n-1}{n-1}\right]\frac{t^n}{2^{n-1}\cdot n!} \\ &= 1+\sum_{n=1}^{\infty}\frac{t^n}{n!} = e^t. \end{aligned}$$

Test Problem 2. Consider the following nonlinear differential-difference equation with differential order one and difference of order two.

$$u'(t) = 2-u(t-\omega)+u^3(t-2\omega), \qquad t>2\omega, \tag{3.12}$$

with the initial interval condition:

$$u(t) = 1, \qquad 0\le t\le 2\omega. \tag{3.13}$$

Let $q \in [0,1]$ denote the embedding parameter. The homotopy analysis method is based on a kind of continuous mapping $u(t) \to \phi(t;q)$ such that, as the embedding parameter q increases from 0 to 1 , $\phi(t;q)$ varies from the initial condition $u_0(t)$ to the exact solution $u(t)$.
Now choose an auxiliary linear operator as

$$\mathcal{L}[\phi(t;q)] = \frac{\partial\phi(t;q)}{\partial t} \tag{3.14}$$

and we define nonlinear operator

$$\mathcal{N}[\phi(t;q)] = \frac{\partial\phi(t;q)}{\partial t} - 2 + \phi(t-\omega;q) - \phi^3(t-2\omega;q). \tag{3.15}$$

Let $h=1$ and $H(t)=-1$ denote the auxiliary parameter and auxiliary function respectively. Using the embedding parameter $q \in [0,1]$, we construct an equation

$$(1-q)\mathcal{L}[\phi(t;q)-u_0(t)] = h\ q\ H(t)\ \mathcal{N}[\phi(t;q)] \tag{3.16}$$

Now using (3.14) and (3.15) in (3.16) we obtain,

$$(1-q)\left[\frac{\partial\phi(t;q)}{\partial t}-0\right] = -q\left[\frac{\partial\phi(t;q)}{\partial t}-2+\phi(t-\omega;q)-\phi^3(t-2\omega;q)\right] \tag{3.17}$$

subjected to the initial interval condition

$$\phi(t,q)=1, \qquad 0\le t\le 2\omega.$$

Following the initial steps of the method, we arrive at

$$L\{\phi(t,q)\} = \frac{1}{s} + \frac{q\ e^{-\omega s}}{s^2} + \frac{q\ e^{-2\omega s}}{s^2} - \frac{q\ e^{-\omega s}}{s} L\{\phi(t;q)\} + \frac{q\ e^{-2\omega s}}{s} L\{\phi^3(t;q)\}. \tag{3.18}$$

The next step is to apply the following Laplace decomposition series for $L\{\phi(t,q)\}$ and $L\{\phi^3(t;q)\}$:

$$L\{\phi(t,q)\} = L\{u_0(t)\} + \sum_{n=1}^{\infty} e^{-n\omega s} L\{u_n(t)\} q^n. \tag{3.19}$$

Now let us expanding $L\{\phi^3(t,q)\}$, by using Laplace decomposition as follows

$$L\{\phi^3(t,q)\} = \sum_{n=0}^{\infty} e^{-n\omega s} L\{A_n(t)\} q^n, \tag{3.20}$$

where $A_i\ 's$ are Adomian Polynomials,

$$\begin{aligned} A_0(t) &= u_0^3(t), \\ A_1(t) &= 3u_0^2(t)u_1(t), \\ A_2(t) &= 3u_0^2(t)u_2(t) + 3u_0(t)u_1^2(t), \\ A_3(t) &= 3u_0^2(t)u_3(t) + 3u_0(t)u_1(t)u_2(t) + u_1^3(t) \quad \text{and so on.} \end{aligned}$$

Now by using (3.19) and (3.20) in (3.18), we get,

$$\begin{aligned} L\{u_0(t)\} + \sum_{n=1}^{\infty} e^{-n\omega s} L\{u_n(t)\} q^n &= \frac{1}{s} + \frac{q\ e^{-\omega s}}{s^2} + \frac{q\ e^{-2\omega s}}{s^2} \\ &\quad - \frac{q\ e^{-\omega s}}{s} \sum_{n=0}^{\infty} e^{-n\omega s} L\{u_n(t)\} q^n \\ &\quad + \frac{q\ e^{-2\omega s}}{s} \sum_{n=0}^{\infty} e^{-n\omega s} L\{A_n(t)\} q^n. \end{aligned} \tag{3.21}$$

Equating the terms with co-efficient of $e^{-n\omega s}$ on both sides of (3.21) we get $L\{u_n(t)\}$.

$$\begin{aligned} L\{u_0(t)\} &= \frac{1}{s}, \qquad L\{u_1(t)\} = \frac{1}{s^2} - \frac{1}{s} L\{u_0(t)\} \\ L\{u_2(t)\} &= \frac{1}{qs^2} - \frac{1}{s} L\{u_1(t)\} + \frac{1}{qs} L\{A_0(t)\} \\ L\{u_n(t)\} &= -\frac{1}{s} L\{u_{n-1}(t)\} + \frac{1}{qs} L\{A_{n-2}(t)\} \quad \text{for } n \geq 3. \end{aligned}$$

An application of inverse Laplace transform to (3.19) will yield $\phi(t,q)$, over any general interval.
In particular, for $5\omega \le t \le 6\omega$, the approximate solution is

$$\begin{aligned}\phi(t,q) &= \sum_{n=0}^{5} u_n(t-n\omega)q^n \\ &= 1+2(t-2\omega)q-2\frac{(t-3\omega)^2}{2!}q^2+\ 2\frac{(t-4\omega)^3}{3!}q^3 \\ &\quad +6\frac{(t-4\omega)^2}{2!}q^2-2\frac{(t-5\omega)^4}{4!}q^4-12\frac{(t-5\omega)^3}{3!}q^3. \quad (3.22)\end{aligned}$$

If $q \to 1$ then $\phi(t,q) \to u(t)$ which is the same approximate solution given in [3].

Test Problem 3. Consider the following integro-differential-difference equation with differential order one and difference of order two.

$$u'(t) \quad = \quad u(t-\omega)u'(t-\omega)\ +\ \int_0^t \sin(u(t_1-2\omega))\ dt_1, \qquad t>2\omega, \tag{3.23}$$

with the initial interval condition:

$$u(t) \quad = \quad 1, \qquad 0 \le t \le 2\omega. \tag{3.24}$$

Let $q \in [0,1]$ denote the embedding parameter. The homotopy analysis method is based on a kind of continuous mapping $u(t) \to \phi(t;q)$ such that, as the embedding parameter q increases from 0 to 1 , $\phi(t;q)$ varies from the initial condition $u_0(t)$ to the exact solution $u(t)$.
Now choose an auxiliary linear operator as

$$\mathcal{L}[\phi(t;q)] = \frac{\partial\phi(t;q)}{\partial t} \tag{3.25}$$

and we define nonlinear operator

$$\mathcal{N}[\phi(t;q)] = \frac{\partial\phi(t;q)}{\partial t} - \phi(t-\omega;q)\phi'(t-\omega;q)\ -\ \int_0^t \sin(\phi(t_1-2\omega;q))\ dt_1. \tag{3.26}$$

Let $h=1$ and $H(t)=-1$ denote the auxiliary parameter and auxiliary function respectively. Using the embedding parameter $q \in [0,1]$, we construct an equation

$$(1-q)\mathcal{L}[\phi(t;q)-u_0(t)] = h\ q\ H(t)\ \mathcal{N}[\phi(t;q)] \tag{3.27}$$

Now using (3.25) and (3.26) in (3.27) we obtain,

$$(1-q)\left[\frac{\partial \phi(t;q)}{\partial t} - 0\right] = -q\left[\frac{\partial \phi(t;q)}{\partial t} - \phi(t-\omega;q)\phi'(t-\omega;q) - \int_0^t \sin(\phi(t_1-2\omega;q))\,dt_1\right] \quad (3.28)$$

subjected to the initial interval condition

$$\phi(t,q) = 1, \qquad 0 \le t \le 2\omega.$$

Following the initial steps of the method, we arrive at

$$L\{\phi(t;q)\} = \frac{1}{s} + \frac{q\,e^{-\omega s}}{s} L\{\phi(t;q)\phi'(t;q)\} + \frac{q\,e^{-2\omega s}}{s^2} L\{\sin(\phi(t;q))\}. \quad (3.29)$$

The next step is to compute the following Laplace decomposition series for $L\{\phi(t;q)\phi'(t;q)\}$ and $L\{\sin(\phi(t;q))\}$:

$$L\{\phi(t;q)\phi'(t;q)\} = \sum_{n=0}^{\infty} e^{-n\omega s} L\{B_n(t)\}\,q^n, \quad (3.30)$$

where $B_i\ 's$ are Adomian Polynomials,

$$B_n(t) = u_0(t)u_n'(t) + u_1(t)u_{n-1}'(t) + \cdots + u_n(t)u_0'(t), \qquad \text{for } n \ge 0$$

$$\text{and} \qquad L\{\sin(\phi(t;q))\} = \sum_{n=0}^{\infty} e^{-n\omega s} L\{C_n(t)\}\ q^n, \quad (3.31)$$

where $C_i\ 's$ are Adomian Polynomials given below,

$$\begin{aligned}
C_0(t) &= \sin(u_0(t)).\\
C_1(t) &= u_1(t)\cos(u_0(t)).\\
C_2(t) &= u_2(t)\cos(u_0(t)) - \frac{1}{2}u_1^2(t)\sin(u_0(t)).\\
C_3(t) &= u_3(t)\cos(u_0(t)) - u_1(t)u_2(t)\sin(u_0(t)) - \frac{1}{6}u_1^3(t)\cos(u_0(t))
\end{aligned}$$

and so on.

Now by using (3.19), (3.30) and(3.31) in (3.29), we get,

$$\sum_{n=0}^{\infty} e^{-n\omega s} L\{u_n(t)\} = \frac{1}{s} + \frac{e^{-\omega s}}{s}\sum_{n=0}^{\infty} e^{-n\omega s} L\{B_n(t)\} + \frac{e^{-2\omega s}}{s^2}\sum_{n=0}^{\infty} e^{-n\omega s} L\{C_n(t)\}. \quad (3.32)$$

Equating the terms with co-efficient of $e^{-n\omega s}$ on both sides of (3.32) we get $L\{u_n(t)\}$.

$$\begin{aligned} L\{u_0(t)\} &= \frac{1}{s}, \qquad L\{u_1(t)\} = \frac{1}{s}L\{B_0(t)\} \\ L\{u_n(t)\} &= \frac{1}{s}L\{B_{n-1}(t)\} + \frac{1}{q\ s^2}L\{C_{n-2}(t)\} \qquad \text{for } n \geq 2. \end{aligned}$$

An application of inverse Laplace transform will yield $u_n(t)$.
For $4\omega \leq t \leq 5\omega$, the approximate solution is

$$\begin{aligned} \phi(t;q) &\approx \sum_{n=0}^{4} u_n(t-n\omega)q^n \\ &= 1 + \sin(1)\frac{(t-2\omega)^2}{2!}\ q + \sin(1)\frac{(t-3\omega)^2}{2!}\ q^2 \\ &\quad + \sin(1)\frac{(t-4\omega)^2}{2!}\ q^3 + \sin(1)\cos(1)\frac{(t-4\omega)^4}{4!}\ q^2. \quad (3.33) \end{aligned}$$

If $q \to 1$ then $\phi(t,q) \to u(t)$ which is the same approximate solution given in [3].

In the above computational work, an innovative strategy, namely, combining Laplace decomposition method and Liao's Homotopy analysis method is implemented on the above three test problems. The described numerical analytic solutions exhibit all qualitative nature of exact solution whenever the parameters q and ω are varied. when $q = 1$ they yield exact solutions. When $\omega = 0$ and $q = 1$ they yield exact or approximate solution for the corresponding differential equation (means equation without difference argument). So the Homotopy-Laplace decomposition method is really promoting with nice applicability to variety of nonlinear problems. The method is powerful so that it can be extended to solve many types of differential-difference equations with higher difference orders.

References

[1] G. Adomian, *Solving Frontier Problems of Physics: The Decomposition Method*, Kluwer Academic Publishers, Boston and London, 1994.

[2] G. Adomian, A review of the decomposition method and some recent results for nonlinear equation, *Math.Comput.Modelling*, **13(7)**, (1992), 17-43.

[3] S. R. Ananth Kumar and R. Rangarajan, Applicability of Laplace decomposition method for solving certain differential-difference equation of order (1,2), *Bulletin of the International Mathematical Virtual Institute*, **3** (2013), 103-111.

[4] E. Babolian, A. R. Vahidi and A. Shoja, An efficient method for nonlinear fractional differential equations: Combination of the Adomian decomposition method and spectral method, *Indian J. Pure Appl. Math.*, (2014), 1017-1028.

[5] D. Bahuguna, A. Ujlayan and D. N. Pandey, A comparative study of numerical methods for solving an integro-differential equation, *Computers and Mathematics with Applications*, **57** (2009), 1485-1493.

[6] R. Bellman and K. L. Cooke, *Differential-Difference Equations*, Academic press, New York, 1963.

[7] B. Deka and R. C. Deka, Finite element method for a class of parabolic integro-differential equations with interfaces, *Indian J. Pure Appl. Math.*, **(44)** (2013), 823-847.

[8] A. Ghorbani and J. S. Nadjafi, He's Homotopy perturbation method for calculating Adomian polynomials, *International Journal of Nonlinear Sciences and Numerical Simulation*, **8(2)** (2007), 229-232.

[9] S. Liao, Beyond Perturbation. *Introduction to Homotopy Analysis Method*, Chapman and Hall/CRC, New York 2004.

[10] A. M. Wazwaz, Analytical Approximation and Pade Approximation for Volterra's population model, *Appl. Math. and Comput.*, **100** (1999), 13-25.