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FOURIER SERIES OF r-DERANGEMENT AND HIGHER-ORDER DERANGEMENT FUNCTIONS

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Abstract. A derangement is a permutation that has no fixed points and the derangement number d_m is the number of fixed point-free permutations on an m element set. In this paper, we consider two Appell polynomials, called r-derangement and higher-order derangement polynomials which are natural companions for the two generalizations of the derangement numbers, namely r-derangement and higher-order derangement numbers. Then we extend the restrictions of those polynomials on the unit interval by periodicity of the period 1 to the whole real line which are called the r-derangement and higher-order derangement functions. The aim of this paper is to derive Fourier series expansions of those functions and express them in term of Bernoulli functions. As immediate corollaries, we will be able to express the two Appell polynomials as linear combinations of Bernoulli polynomials.

1. Introduction and preliminaries

A derangement is a permutation that has no fixed points. The derangement number d_m (also denoted by $!m$ and called m subfactorial) is the number of fixed point-free permutations on an m element set (see [1,2]).

The problem of counting derangements was initiated by Pierre Rémond de Montmort in 1708 (see [5]). The first few of the derangement numbers $\{d_m\}_{m=0}^{\infty}$ are

$$d_0 = 1, d_1 = 0, d_2 = 1, d_3 = 2, d_4 = 9, d_5 = 44, d_6 = 265, d_7 = 1854, d_8 = 14833,$$
$$d_9 = 133496, d_{10} = 1334961, \cdots .$$

In fact, d_m is given by the closed form formula

$$d_m = m! \sum_{k=0}^{m} \frac{(-1)^k}{k!}. \tag{1.1}$$

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Also, there are recursive formulas given by

$$d_m = (m-1)!(d_{m-1} + d_{m-2}),\ (m \geq 2), \tag{1.2}$$

with $d_0 = 1$, $d_1 = 0$, and

$$d_m = md_{m-1} + (-1)^m,\ (m \geq 1), \tag{1.3}$$

with $d_0 = 1$.

In addition, the exponential generating function for the derangement numbers d_m is given by

$$\frac{1}{1-t}e^{-t} = \sum_{m=0}^{\infty} d_m \frac{t^m}{m!}. \tag{1.4}$$

Here we introduce the derangement polynomial $d_m(x)$ in terms of generating function by

$$\frac{1}{1-t}e^{(x-1)t} = \sum_{m=0}^{\infty} d_m(x) \frac{t^m}{m!}. \tag{1.5}$$

Note here that $d_m(x+1)$, viewed as a function of the real variable x, is the same as the function $D_m(x) = m! \sum_{i=0}^{m} \frac{x^i}{i!}$, defined in [3, p.6]. Clearly, $d_m(0) = d_m$.

Also, for any positive integer r, we will define the derangement number $d_n^{(r)}$ of order r and the derangement polynomials $d_n^{(r)}(x)$ of order r respectively by

$$\left(\frac{1}{1-t}\right)^r e^{-t} = \sum_{m=0}^{\infty} d_m^{(r)} \frac{t^m}{m!}, \tag{1.6}$$

$$\left(\frac{1}{1-t}\right)^r e^{(x-1)t} = \sum_{m=0}^{\infty} d_m^{(r)}(x) \frac{t^m}{m!}. \tag{1.7}$$

It is obvious that $d_m^{(r)}(0) = d_m^{(r)}$, $d_m^{(1)} = d_m$, and $d_m^{(1)}(x) = d_m(x)$.

From (1.7), we see that

$$\frac{d}{dx} d_m^{(r)}(x) = md_{m-1}^{(r)}(x),\ (m \geq 1), \tag{1.8}$$

$$d_m^{(r)}(x+1) - d_m^{(r)}(x) = \sum_{k=0}^{m-1} \binom{m}{k} d_k^{(r)}(x),\ (m \geq 1). \tag{1.9}$$

In turn, these imply that

$$d_m^{(r)}(1) - d_m^{(r)} = \sum_{k=0}^{m-1} \binom{m}{k} d_k^{(r)} \quad (m \geq 1), \tag{1.10}$$

$$\begin{aligned} \int_0^1 d_m^{(r)}(x)dx &= \frac{1}{m+1}\left(d_{m+1}^{(r)}(1) - d_{m+1}^{(r)}\right) \\ &= \frac{1}{m+1}\sum_{k=0}^{m}\binom{m+1}{k} d_k^{(r)}. \end{aligned} \tag{1.11}$$

In [4], the notion of derangement numbers was generalized to r-derangement numbers by imposing further restrictions. The r-derangement number, denoted by $d_r(m)$ $(0 \leq r \leq m)$, is the number of derangements on an $m+r$ element set such that the first r elements appear in distinct cycles in its cycle decomposition. We observe here that $d_0(m) = d_m$, and $d_r(m) = 0$, if $r > m$.

A closed form formula for $d_r(m)$ is given by

$$d_r(m) = \sum_{j=r}^{m} \binom{j}{r} \frac{m!}{(m-j)!}(-1)^{m-j}, \ (m \geq r \geq 0). \tag{1.12}$$

The r-derangement numbers $d_r(m)$ can also be recursively determined by

$$\begin{aligned} d_r(m) &= r d_{r-1}(m-1) + (m-1)d_r(m-2) \\ &\quad + (m+r-1)d_r(m-1), \ (m \geq 3, r \geq 1), \end{aligned} \tag{1.13}$$

with $d_1(m) = d_{m+1}$, $d_r(r) = r!$ $(r \geq 1)$, $d_r(r+1) = r(r+1)!$ $(r \geq 2)$, and

$$d_{r+1}(m) = \frac{m-r}{r+1} d_r(m) + \frac{m}{r+1} d_r(m-1), \ (m \geq 1, r \geq 0). \tag{1.14}$$

Moreover, the exponential generating function for the r-derangement numbers $d_r(m)$ is given by

$$\frac{t^r}{(1-t)^{r+1}} e^{-t} = \sum_{m=0}^{\infty} d_r(m) \frac{t^m}{m!}. \tag{1.15}$$

For all of the above results on r-derangement numbers $d_r(m)$, we let the reader refer to [4,6].

Here we define the r-derangement polynomials $d_r(m,x)$ by

$$\frac{t^r}{(1-t)^{r+1}} e^{(x-1)t} = \sum_{n=0}^{\infty} d_r(m,x) \frac{t^m}{m!.} \tag{1.16}$$

Clearly, $d_r(m,0) = d_r(m)$, and $d_0(m,x) = d_m(x)$.
From (1.16), we observe that

$$\frac{d}{dx}d_r(m,x) = md_r(m-1,x), \ (m \geq 1), \tag{1.17}$$

$$d_r(m,x+1) - d_r(m,x) = \sum_{k=0}^{m-1}\binom{m}{k}d_r(k,x), \ (m \geq 1). \tag{1.18}$$

These in turn imply that

$$d_r(m,1) - d_r(m) = \sum_{k=0}^{m-1}\binom{m}{k}d_r(k), \ (m \geq 1), \tag{1.19}$$

$$\begin{aligned}\int_0^1 d_r(m,x)dx &= \frac{1}{m+1}\big(d_r(m+1,1) - d_r(m+1)\big) \\ &= \frac{1}{m+1}\sum_{k=1}^{m}\binom{m+1}{k}d_r(k).\end{aligned} \tag{1.20}$$

For any real number x, the fractional part of x is denoted by

$$< x >= x - \lfloor x \rfloor \in [0,1). \tag{1.21}$$

As is well known, the Bernoulli polynomials $B_m(x)$ are given by

$$\frac{t}{e^t - 1}e^{xt} = \sum_{m=0}^{\infty} B_m(x)\frac{t^m}{m!}. \tag{1.22}$$

We also recall the following facts about Bernoulli functions $B_m(< x >)$:

(a) for $m \geq 2$,

$$B_m(< x >) = -m! \sum_{\substack{n=-\infty \\ n\neq 0}}^{\infty} \frac{e^{2\pi inx}}{(2\pi in)^m}, \tag{1.23}$$

(b) for $m = 1$,

$$-\sum_{\substack{n=-\infty \\ n\neq 0}}^{\infty} \frac{e^{2\pi inx}}{2\pi in} = \begin{cases} B_1(< x >), & \text{for } x \notin \mathbb{Z}, \\ 0, & \text{for } x \in \mathbb{Z}. \end{cases} \tag{1.24}$$

In this paper, we will consider the functions $d_r(m, < x >)$ and $d_m^{(r)}(< x >)$ defined on the real line, derive their Fourier series expansions, and express them in terms of Bernoulli functions. As immediate corollaries, we will be able to express $d_m(r,x)$ and $d_m^{(r)}(x)$ as linear combinations of Bernoulli polynomials.

2. Fourier series of r-derangement functions

Here we will investigate the r-derangement functions $d_r(m, < x >)$, $(r \geq 0)$, derive their Fourier series expansions and express them in terms of Bernoulli functions.

From (1.11), we note that

$$d_r(m,x) = 0, \text{ for } 0 \leq m < r, \ d_r(r,x) = r!. \tag{2.1}$$

So, throughout this section, we will assume that $m > r$.

For convenience, we let

$$\begin{aligned}\Delta_m^{(r)} = d_r(m,1) - d_r(m) &= m!\binom{m}{r} - d_r(m)\\ &= \sum_{k=0}^{m-1}\binom{m}{k} d_r(k)\\ &= \sum_{k=r}^{m-1}\binom{m}{k} d_r(k), \ (m > r).\end{aligned} \tag{2.2}$$

We note here that

$$\begin{aligned}d_r(m,1) = d_r(m) \Leftrightarrow \Delta_m^{(r)} &= 0\\ \Leftrightarrow \sum_{k=r}^{m-1}\binom{m}{k} d_r(k) &= 0,\end{aligned} \tag{2.3}$$

and

$$\int_0^1 d_m^{(r)}(x)dx = \frac{1}{m+1}\Delta_{m+1}^{(r)}. \tag{2.4}$$

It is important to observe that $\Delta_m^{(r)} > 0$ (and hence $\Delta_m^{(r)} \neq 0$), for $m > r$. To see this, we recall the following identity from [2, (8)]:

$$d_r(m) = \sum_{j=r}^{m}\binom{j-1}{r-1}\frac{m!}{(m-j)!}d_{m-j}, \ (m \geq r). \tag{2.5}$$

The equation (2.5) shows that $d_r(m) > 0$, for $m \geq r$, which implies, in view of (2.2), that $\Delta_m^{(r)} > 0$, for $m > r$.

The Fourier series of $d_r(m,<x>)$ is

$$\sum_{n=-\infty}^{\infty} A_n^{(m,r)} e^{2\pi i n x}, \tag{2.6}$$

where

$$\begin{aligned} A_n^{(m)} = A_n^{(m,r)} &= \int_0^1 d_r(m,<x>)e^{-2\pi i n x}dx \\ &= \int_0^1 d_r(m,x)e^{-2\pi i n x}dx. \end{aligned} \tag{2.7}$$

Case 1: $n \neq 0$

$$\begin{aligned} A_n^{(m)} &= \int_0^1 d_r(m,x)e^{-2\pi i n x}dx \\ &= -\frac{1}{2\pi i n}[d_r(m,x)e^{-2\pi i n x}]_0^1 + \frac{m}{2\pi i n}\int_0^1 d_r(m-1,x)e^{-2\pi i n x}dx \\ &= \frac{m}{2\pi i n}A_n^{(m-1)} - \frac{1}{2\pi i n}\Delta_m^{(r)}. \end{aligned}$$

So we have shown that

$$A_n^{(m)} = \frac{m}{2\pi i n}A_n^{(m-1)} - \frac{1}{2\pi i n}\Delta_m^{(r)}, \tag{2.8}$$

from which by induction on m we can show that

$$A_n^{(m)} = -\frac{1}{m+1}\sum_{j=1}^{m-r}\frac{(m+1)_j}{(2\pi i n)^j}\Delta_{m-j+1}^{(r)}, \tag{2.9}$$

where $(x)_n = x(x-1)\cdots(x-n+1)$, for $n \geq 1$, and $(x)_0 = 1$.

Case 2: $n = 0$

$$A_0^{(m)} = \int_0^1 d_r(m,x)dx = \frac{1}{m+1}\Delta_{m+1}^{(r)}. \tag{2.10}$$

As we noted earlier, $\Delta_m^{(r)} \neq 0$, for $m > r$, and hence $d_r(m,1) \neq d_r(m)$. Then $d_r(m,<x>)$ is piecewise C^∞, and discontinuous with jump discontinuities at integers. The Fourier series of $d_r(m,<x>)$ converges pointwise to $d_r(m,<x>)$, for $x \notin \mathbb{Z}$, and converges to

$$\tfrac{1}{2}\big(d_r(m,1) + d_r(m)\big) = d_r(m) + \tfrac{1}{2}\Delta_m^{(r)}, \tag{2.11}$$

for $x \in \mathbb{Z}$.

Now, the Fourier series of $d_r(m,<x>)$ is

$$\begin{aligned}
&\frac{1}{m+1}\Delta_{m+1}^{(r)}+\sum_{\substack{n=-\infty\\ n\neq 0}}^{\infty}\left(-\frac{1}{m+1}\sum_{j=1}^{m-r}\frac{(m+1)_j}{(2\pi i n)^j}\Delta_{m-j+1}^{(r)}\right)e^{2\pi i n x}\\
&=\frac{1}{m+1}\Delta_{m+1}^{(r)}+\frac{1}{m+1}\sum_{j=1}^{m-r}\binom{m+1}{j}\Delta_{m-j+1}^{(r)}\left(-j!\sum_{\substack{n=-\infty\\ n\neq 0}}^{\infty}\frac{e^{2\pi i n x}}{(2\pi i n)^j}\right)\\
&=\frac{1}{m+1}\Delta_{m+1}^{(r)}+\frac{1}{m+1}\sum_{j=2}^{m-r}\binom{m+1}{j}\Delta_{m-j+1}^{(r)}B_j(<x>)\\
&\quad+\Delta_m^{(r)}\times\begin{cases}B_1(<x>), & \text{for } x\notin\mathbb{Z},\\ 0, & \text{for } x\in\mathbb{Z}\end{cases}\\
&=\frac{1}{m+1}\sum_{\substack{j=0\\ j\neq 1}}^{m-r}\binom{m+1}{j}\Delta_{m-j+1}^{(r)}B_j(<x>)\\
&\quad+\Delta_m^{(r)}\times\begin{cases}B_1(<x>), & \text{for } x\notin\mathbb{Z},\\ 0, & \text{for } x\in\mathbb{Z}.\end{cases}
\end{aligned}\tag{2.12}$$

We are now ready to state our result.

Theorem 2.1. *Assume that $m>r$, and let*

$$\Delta_m^{(r)}=d_r(m,1)-d_r(m)=m!\binom{m}{r}-d_r(m)=\sum_{k=r}^{m-1}\binom{m}{k}d_r(k).$$

Then we have the following.,

(a) $$\frac{1}{m+1}\Delta_{m+1}^{(r)}+\sum_{\substack{n=-\infty\\ n\neq 0}}^{\infty}\left(-\frac{1}{m+1}\sum_{j=1}^{m-r}\frac{(m+1)_j}{(2\pi i n)^j}\Delta_{m-j+1}^{(r)}\right)e^{2\pi i n x}$$

$$=\begin{cases}d_r(m,<x>), & \textit{for } x\notin\mathbb{Z},\\ d_r(m)+\frac{1}{2}\Delta_m^{(r)}, & \textit{for } x\in\mathbb{Z}.\end{cases}$$

(b) $$\frac{1}{m+1}\sum_{j=0}^{m-r}\binom{m+1}{j}\Delta_{m-j+1}^{(r)}B_j(<x>)=d_r(m,<x>),\ \textit{for } x\notin\mathbb{Z};$$

$$\frac{1}{m+1}\sum_{\substack{j=0\\ j\neq 1}}^{m-r}\binom{m+1}{j}\Delta_{m-j+1}^{(r)}B_j(<x>)=d_r(m)+\frac{1}{2}\Delta_m^{(r)},\ \textit{for } x\in\mathbb{Z}.$$

Corollary 2.2. *Assume that $m > r$, and let $\Delta_m^{(r)}$ be as in Theorem 2.1. Then we have the following polynomial identity*

$$d_r(m,x) = \frac{1}{m+1}\sum_{j=1}^{m-r}\binom{m+1}{j}\Delta_{m-j+1}^{(r)}B_j(x).$$

3. Fourier series of higher-order derangement functions

In this section, we will consider the higher-order derangement functions $d_m^{(r)}(x)$, derive their Fourier series expansions and express them in terms of Bernoulli functions. Here r is a fixed positive integer.

We first let

$$\Omega_m^{(r)} = d_m^{(r)}(1) - d_m^{(r)} = (r+m-1)_m - d_m(r) = \sum_{k=0}^{m-1}\binom{m}{k}d_k^{(r)},\ (m \geq 1). \quad (3.1)$$

From (1.6), we immediately see that

$$d_0^{(r)} = 1,\ d_1^{(r)} = r-1,\ d_m^{(r)} > 0,\ \text{for } m \geq 2. \quad (3.2)$$

Indeed, from (1.6), we have

$$\begin{aligned}\sum_{m=0}^{\infty} d_m^{(r)}\frac{t^m}{m!} &= \left(\frac{1}{1-t}\right)^{r-1}\left(\frac{1}{1-t}e^{-t}\right)\\ &= \left(\sum_{l=0}^{\infty}(r+l-2)_l\frac{t^l}{l!}\right)\left(\sum_{k=0}^{\infty}d_k\frac{t^k}{k!}\right)\\ &= \sum_{m=0}^{\infty}\left(\sum_{l=0}^{m}\binom{m}{l}(r+l-2)_l d_{m-l}\right)\frac{t^m}{m!},\end{aligned}$$

and hence, for $m \geq 2$

$$d_m^{(r)} = \sum_{l=0}^{m}\binom{m}{l}(r+l-2)_l d_{m-l} = d_m + \cdots > 0.$$

Now, using (2.12) we easily see from (2.11) that $\Omega_m^{(r)} > 0$, for all $m \geq 1$.

Throughout this section, we will assume that $m \geq 1$. We note that

$$\int_0^1 d_m^{(r)}(x)dx = \frac{1}{m+1}\Omega_{m+1}^{(r)}. \quad (3.3)$$

The Fourier series of $d_m^{(r)}(< x >)$ is

$$\sum_{n=-\infty}^{\infty} C_n^{(m,r)} e^{2\pi i n x},$$

where

$$c_n^{(m)} = c_n^{(m,r)} = \int_0^1 d_m^{(r)}(< x >)e^{-2\pi inx}dx$$
$$= \int_0^1 d_m^{(r)}(x)e^{-2\pi inx}dx.$$

Case 1: $n \neq 0$

$$C_n^{(m)} = \int_0^1 d_m^{(r)}(x)e^{-2\pi inx}dx$$
$$= -\frac{1}{2\pi in}[d_m^{(r)}(x)e^{-2\pi inx}]_0^1 + \frac{m}{2\pi in}\int_0^1 d_{m-1}^{(r)}(x)e^{-2\pi inx}dx$$
$$= \frac{m}{2\pi in}C_n^{(m-1)} - \frac{1}{2\pi in}\Omega_m^{(r)}.$$

Then we have shown that

$$C_n^{(m)} = \frac{m}{2\pi in}C_n^{(m-1)} - \frac{1}{2\pi in}\Omega_m^{(r)}, \tag{3.4}$$

from which by induction on m we can show

$$C_n^{(m)} = -\frac{1}{m+1}\sum_{j=1}^{m}\frac{(m+1)_j}{(2\pi in)^j}\Omega_{m-j+1}^{(r)}. \tag{3.5}$$

Case 2: $n = 0$

$$C_0^{(m)} = \int_0^1 d_m^{(r)}(x)dx = \frac{1}{m+1}\Omega_{m+1}^{(r)}. \tag{3.6}$$

We noted in the above that $\Omega_m^{(r)} \neq 0$, for $m \geq 1$, and therefore $d_m^{(r)}(1) \neq d_m^{(r)}(0)$. Then $d_m^{(r)}(< x >)$ is piecewise C^∞, and discontinuous with jump discontinuities at integers. The Fourier series of $d_m^{(r)}(< x >)$ converges pointwise to $d_m^{(r)}(< x >)$, for $x \notin \mathbb{Z}$, and converges to

$$\tfrac{1}{2}\left(d_m^{(r)}(1) + d_m^{(r)}\right) = d_m^{(r)} + \tfrac{1}{2}\Omega_m^{(r)}, \tag{3.7}$$

for $x \in \mathbb{Z}$.

Now, the Fouries series of $d_m^{(r)}(< x >)$ is

$$
\begin{aligned}
&\frac{1}{m+1}\Omega_{m+1}^{(r)}+\sum_{\substack{n=-\infty\\ n\neq 0}}^{\infty}\left(-\frac{1}{m+1}\sum_{j=1}^{m}\frac{(m+1)_j}{(2\pi in)^j}\Omega_{m-j+1}^{(r)}\right)e^{2\pi inx}\\
&=\frac{1}{m+1}\Omega_{m+1}^{(r)}+\frac{1}{m+1}\sum_{j=1}^{m}\binom{m+1}{j}\Omega_{m-j+1}^{(r)}\left(-j!\sum_{\substack{n=-\infty\\ n\neq 0}}^{\infty}\frac{e^{2\pi inx}}{(2\pi in)^j}\right)\\
&=\frac{1}{m+1}\Omega_{m+1}^{(r)}+\frac{1}{m+1}\sum_{j=2}^{m}\binom{m+1}{j}\Omega_{m-j+1}^{(r}B_j(<x>)\\
&\quad+\Omega_m^{(r)}\times\begin{cases}B_1(<x>), & \text{for } x\notin\mathbb{Z},\\ 0, & \text{for } x\in\mathbb{Z}\end{cases}\\
&=\frac{1}{m+1}\sum_{\substack{j=0\\ j\neq 1}}^{m}\binom{m+1}{j}\Omega_{m-j+1}^{(r)}B_j(<x>)+\Omega_m^{(r)}\times\begin{cases}B_1(<x>), & \text{for } x\notin\mathbb{Z},\\ 0, & \text{for } x\in\mathbb{Z}.\end{cases}
\end{aligned}
\tag{3.8}
$$

Now, we are ready to state our result.

Theorem 3.1. *Assume that $m \geq 1$, and put*

$$
\Omega_m^{(r)}=d_m^{(r)}(1)-d_m^{(r)}=(r+m-1)_m-d_m(r)=\sum_{k=0}^{m-1}\binom{m}{k}d_k^{(r)}.
$$

Then we have the following.

$$
\begin{aligned}
(a)\quad &\frac{1}{m+1}\Omega_{m+1}^{(r)}+\sum_{\substack{n=-\infty\\ n\neq 0}}^{\infty}\left(-\frac{1}{m+1}\sum_{j=1}^{m}\frac{(m+1)_j}{(2\pi in)^j}\Omega_{m-j+1}^{(r)}\right)e^{2\pi inx}\\
&=\begin{cases}d_m^{(r)}(<x>), & \textit{for } x\notin\mathbb{Z},\\ d_m^{(r)}+\frac{1}{2}\Omega_m^{(r)}, & \textit{for } x\in\mathbb{Z}.\end{cases}\\
(b)\quad &\frac{1}{m+1}\sum_{j=0}^{m}\binom{m+1}{j}\Omega_{m-j+1}^{(r)}B_j(<x>)=d_m^{(r)}(<x>),\ \textit{for } x\notin\mathbb{Z};\\
&\frac{1}{m+1}\sum_{\substack{j=0\\ j\neq 1}}^{m}\binom{m+1}{j}\Omega_{m-j+1}^{(r)}B_j(<x>)=d_m^{(r)}+\tfrac{1}{2}\Omega_m^{(r)},\ \textit{for } x\in\mathbb{Z}.
\end{aligned}
$$

Corollary 3.2. *Assume that $m \geq 1$, and let $\Omega_m^{(r)}$ be as in Theorem 3.1. Then we have the following polynomial identity.*

$$d_m^{(r)}(x) = \frac{1}{m+1}\sum_{j=0}^{m}\binom{m+1}{j}\Omega_{m-j+1}^{(r)}B_j(x).$$

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