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INDEX MATRICES AND OLAP-CUBE PART 2: AN PRESENTATION OF THE OLAP-ANALYSIS BY INDEX MATRICES

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Abstract. In the current paper an interpretation of the OLAP cube using the apparatus of index matrices is presented. The main properties of OLAP cubes and the operations, providing access to data are discussed. The OLAP operations "slice", "dice" and "pivot" are defined by index matrices.

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1. Introduction

The architecture of On-line Analytical Processing (OLAP) is introduced by Codd in [10]. Its aim is to extract knowledge from data warehouses and to provide users with the ability to perform dynamic data analytics. The OLAP databases are designed primarily for performing exploratory queries rather than updates. They are focused on the problem of data analysis. They give rapid answers to complex queries involving large amounts of data and perform some of the summary calculations before the user requests them. An OLAP environment can pre-calculate totals and averages that enable the system to respond quickly to the user. OLAP databases are normally stored in a multidimensional data structure. The data are arranged into multiple dimensions. Each dimension have multiple levels of abstraction defined by concept hierarchies. Navigating through these hierarchies, different types of OLAP operations can be executed. The basic unit of the OLAP analysis is the cube. A cube is a storage unit that combines a number of dimensions and their measures [1, 11, 14, 15, 16, 17, 21].

The Index Matrices (IM) were introduced in 1984 in [2]. The apparatus of 3D-Extended Index Matrices (3D-EIMs) was defined in a series of papers [3, 22, 24].

For the needs of the present research we will repeat the definition of the 3D-index matrix and some operations over them in section 2. In section 3 we will recall the definitions of operations over OLAP-cubes and their properties will be discussed. In this section 3 we will discuss applications of the apparatus of the index matrices for presentation of basic operations in OLAP-cube.

2. Short remarks on 3D-Extended index matrix

Let us start with a definition of a 3D-extended index matrix from [4, 8], which was extended in [22].

2.1. **Definition of 3D-extended index matrix and some operations over them.**

2.1.1. *Definition of 3D-extended index matrix.* The Intuitionistic Fuzzy Pair (IFP) [7] is an object with the form $\langle a, b \rangle$, where $a, b \in [0, 1]$ and $a+b \leq 1$, which is used to evaluate objects or processes. Its components (a and b) are interpreted as degrees of membership and non-membership, or degrees of validity and non-validity, or degrees of correctness and non-correctness, etc. Let I be a fixed set of indices,

$$I^n = \{\langle i_1, i_2, ..., i_n \rangle | (\forall j : 1 \leq j \leq n)(i_j \in \ I)\}$$

and

$$I^* = \bigcup_{1 \leq n \leq \infty} I^n.$$

Let X be a fixed set of objects. In particular, they can be either real numbers, or only the numbers 0 or 1, or logical variables, propositions or predicates, IFPs, functions etc.

A "3D-extended Index Matrix" (3D-EIM) with index sets K, L and H ($K, L, H \subset I^*$) and elements from set X is called the object:

$$[K, L, H, \{a_{k_i,l_j,h_g}\}] = \left\{ \begin{array}{c|ccccc} h_g & l_1 & \ldots & l_j & \ldots & l_n \\ \hline k_1 & a_{k_1,l_1,h_g} & \vdots & a_{k_1,l_j,h_g} & \ldots & a_{k_1,l_n,h_g} \\ \vdots & \vdots & \ldots & \vdots & \ldots & \vdots \\ k_i & a_{k_i,l_1,h_g} & \ldots & a_{k_i,l_j,h_g} & \ldots & a_{k_i,l_n,h_g} \\ \vdots & \vdots & \ldots & \vdots & \ldots & \vdots \\ k_m & a_{k_m,l_1,h_g} & \ldots & a_{k_m,l_j,h_g} & \ldots & a_{k_m,l_n,h_g} \end{array} \quad | \ h_g \in H \right\},$$

where $K = \{k_1, k_2, \ldots, k_m\}$, $L = \{l_1, l_2, \ldots, l_n\}$, $H = \{h_1, h_2, \ldots, h_f\}$, and for $1 \leq i \leq m,\ 1 \leq j \leq n,\ 1 \leq g \leq f : a_{k_i,l_j,h_g} \in X$.

Following [4, 22], let 3D-EIM$_R$ be the set of all 3D-EIMs with elements being real numbers; 3D-EIM$_{\{0,1\}}$ be the set of all $(0,1)$-3D-EIMs with elements being 0 or 1; 3D-EIM$_P$ be the set of all 3D-EIMs with elements – predicates; 3D-EIM$_{IFP}$ be the set of all 3D-EIMs with elements – IFPs and 3D-EIM$_{FE}$ – the set of all 3D-EIMs with elements – one-argument functions $\in F^{\ 1}$.

2.1.2. *Projection.* Let us have 3D-EIM $A = [K, L, H, \{a_{k_i,l_j,h_g}\}]$ and let $M \subseteq K$, $N \subseteq L$ and $U \subseteq H$. Then,

$$pr_{M,N,U} A = [M, N, U, \{b_{k_i,l_j,h_g}\}],$$

where for each $k_i \in M$, $l_j \in N$ and $h_g \in U$ $b_{k_i,l_j,h_g} = a_{k_i,l_j,h_g}$.

2.1.3. *Transposition.* As we saw in [4, 22], there are 5 (= 3!-1) EIMs, related to this operation: five transposed EIM. The analytical forms of the separate transposed 3D-EIMs are the following.

$[1,3,2]$**-transposition**

$$[K, L, H, \{a_{k_i,l_j,h_g}\}]^{[1,3,2]} = [K, H, L, \{a_{k_i,h_g,l_j}\}];$$

$[2,1,3]$**-transposition**

$$[K, L, H, \{a_{k_i,l_j,h_g}\}]^{[2,1,3]} = [L, K, H, \{a_{l_j,k_i,h_g}\}];$$

$[2,3,1]$**-transposition**

$$[K, L, H, \{a_{k_i,l_j,h_g}\}]^{[2,3,1]} = [L, H, K, \{a_{l_j,h_g,k_i}\}];$$

$[3,1,2]$**-transposition**

$$[K, L, H, \{a_{k_i,l_j,h_g}\}]^{[3,1,2]} = [H, K, L, \{a_{h_g,k_i,l_j}\}];$$

$[3,2,1]$**-transposition**

$$[K, L, H, \{a_{k_i,l_j,h_g}\}]^{[3,2,1]} = [H, L, K, \{a_{h_g,l_j,k_i}\}].$$

2.1.4. *Aggregation operations.* Let the 3D-EIM A $= [K, L, H, \{a_{k_i,l_j,h_g}\}]$ $(K, L, H \subset I^*)$ be given and let for $1 \le i \le m, 1 \le j \le n, 1 \le g \le f$, $\{k_{i,0}, k_0\} \notin K$, $\{l_{j,0}, l_0\} \notin L$, $\{h_{g,0}, h_0\} \notin H$ and $a_{k_i,l_j,h_g} \in X$. Let $\circ : X \times X \longrightarrow X$ and $* : X \times X \longrightarrow X$.

Let

$$\circ \in \begin{cases} \{\text{“}+\text{”}, \text{“}\times\text{”}, \text{“}average\text{”}, \text{“}max\text{”}, \text{“}min\text{”}\}, & \text{if } A \in 3D-EIM_R \\ & \text{or } A \in 3D-EIM_{FE}; \\ \{\text{“}max\text{”}, \text{“}min\text{”}\}, & \text{if } A \in 3D-EIM_{\{0,1\}} \\ \{\text{“}\wedge\text{”}, \text{“}\vee\text{”}\}, & \text{if } A \in 3D-EIM_P \\ & \text{or } A \in 3D-EIM_{IFP} \end{cases}$$

In the case of 3D-EIM$_{IFP}$, in aggregation operations can participate aggregating pair operations $(\circ, *)$ whose elements are applied respectively to the first and second elements of IFP, where

$$(\circ, *) \in \{(min, max), (min, average), (min, min), (average, average),$$

$$(average, min), (max, min)\}.$$

The aggregation operations have the forms following [23]:

$(\circ) - \alpha_K$**-aggregation**

$$\alpha_{(K,\circ)}(A, k_0)$$

$$= \left\{ \begin{array}{c|cccc} h_g & l_1 & l_2 & \dots & l_n \\ \hline k_0 & \underset{1\le i\le m}{\circ} a_{k_i,l_1,h_g} & \underset{1\le i\le m}{\circ} a_{k_i,l_2,h_g} & \dots & \underset{1\le i\le m}{\circ} a_{k_i,l_n,h_g} \end{array} | h_g \in H \right\};$$

$(\circ)-\alpha_L$**-aggregation**

$$\alpha_{(L,\circ)}(A,l_0)=\left\{\begin{array}{c|c} h_g & l_0 \\ \hline k_1 & \underset{1\le j\le n}{\circ} a_{k_1,l_j,h_g} \\ k_2 & \underset{1\le j\le n}{\circ} a_{k_2,l_j,h_g} \\ \vdots & \vdots \\ k_m & \underset{1\le j\le n}{\circ} a_{k_m,l_j,h_g} \end{array} \;\middle|\; h_g\in H\right\};$$

$(\circ)-\alpha_H$**-aggregation**

$$\alpha_{(H,\circ)}(A,h_0)=\left\{\begin{array}{c|c} l_j & h_0 \\ \hline k_1 & \underset{1\le g\le f}{\circ} a_{k_1,l_1,h_g} \\ k_2 & \underset{1\le g\le f}{\circ} a_{k_2,l_2,h_g} \\ \vdots & \vdots \\ k_m & \underset{1\le g\le f}{\circ} a_{k_m,l_n,h_g} \end{array} \;\middle|\; l_j\in L\right\};$$

$(\circ)-\alpha_{(K,L)}$**-aggregation**

$$\alpha_{(K,L,\circ)}(A,\langle k_0,l_0\rangle)$$

$$=\begin{array}{c|cccc} & h_1 & h_2 & \dots & h_f \\ \hline \langle k_0,l_0\rangle & \underset{\substack{1\le i\le m\\1\le j\le n}}{\circ} a_{k_i,l_j,h_1} & \underset{\substack{1\le i\le m\\1\le j\le n}}{\circ} a_{k_i,l_j,h_2} & \dots & \underset{\substack{1\le i\le m\\1\le j\le n}}{\circ} a_{k_i,l_j,h_f} \end{array};$$

$(\circ)-\alpha_{(K,H)}$**-aggregation**

$$\alpha_{(K,H,\circ)}(A,\langle k_0,h_0\rangle)$$

$$=\begin{array}{c|cccc} & l_1 & l_2 & \dots & l_n \\ \hline \langle k_0,h_0\rangle & \underset{\substack{1\le i\le m\\1\le g\le f}}{\circ} a_{k_i,l_1,h_g} & \underset{\substack{1\le i\le m\\1\le g\le f}}{\circ} a_{k_i,l_2,h_g} & \dots & \underset{\substack{1\le i\le m\\1\le g\le f}}{\circ} a_{k_i,l_n,h_g} \end{array};$$

$(\circ)-\alpha_{(L,H)}$**-aggregation**

$$\alpha_{(L,H,\circ)}(A,\langle l_0,h_0\rangle)$$

$$=\begin{array}{c|cccc} & k_1 & k_2 & \dots & k_m \\ \hline \langle l_0,h_0\rangle & \underset{\substack{1\le j\le n\\1\le g\le f}}{\circ} a_{k_1,l_j,h_g} & \underset{\substack{1\le j\le n\\1\le g\le f}}{\circ} a_{k_2,l_j,h_g} & \dots & \underset{\substack{1\le j\le n\\1\le g\le f}}{\circ} a_{k_m,l_j,h_g} \end{array}.$$

2.1.5. *Generalized aggregation operations over 3D-EIMs.* Let us recall the generalized aggregation operations from [24] as follows:
Let 3D-EIM A be given, that

$$[K, L, H, \{a_{k_{i,d},l_{j,b},h_{g,c}}\}]$$

$$= \left\{ \begin{array}{c|ccccccc} H_g & L_1 & \ldots & l_{j_1} & \ldots & l_{j_J} & \ldots & L_n \\ \hline K_1 & a_{K_1,L_1,H_g} & \vdots & a_{K_1,l_{j_1},H_g} & \ldots & a_{K_1,l_{j,J},H_g} & \ldots & a_{K_1,L_n,H_g} \\ \vdots & \vdots & \ldots & \vdots & \ldots & \vdots & \ldots & \\ k_{i,1} & a_{k_{i,1},L_1,H_g} & \ldots & a_{k_{i,1},l_{j,1},H_g} & \ldots & a_{k_{i,1},l_{j,J},H_g} & \ldots & a_{k_{i,1},L_n,H_g} \\ \vdots & \vdots & \ldots & \vdots & \ldots & \vdots & \ldots & \vdots \\ k_{i,I} & a_{k_{i,I},L_1,H_g} & \ldots & a_{k_{i,I},l_{j,1},H_g} & \ldots & a_{k_{i,I},l_{j,J},H_g} & \ldots & a_{k_{i_I},L_n,H_g} \\ \vdots & \vdots & \ldots & \vdots & \ldots & \vdots & \ldots & \vdots \\ K_m & a_{K_m,L_1,H_g} & \ldots & a_{K_m,l_{j,1},H_g} & \ldots & a_{K_m,l_{j,J},H_g} & \ldots & a_{K_m,L_n,H_g} \end{array} | H_g \in H \right\},$$

where

$$K = \{K_1, K_2, \ldots, K_i, \ldots, K_m\}, K_i = \{k_{i,1}, k_{i,2}, \ldots, k_{i,I}\} \text{ for } 1 \le i \le m,$$
$$L = \{L_1, L_2, \ldots, L_j, \ldots, L_n\}, L_j = \{l_{j,1}, l_{j,2}, \ldots, l_{j,J}\} \text{ for } 1 \le j \le n,$$
$$H = \{H_1, H_2, \ldots, H_g, \ldots, H_f\}, H_g = \{h_{g,1}, h_{g,2}, \ldots, h_{g,G}\} \text{ for } 1 \le g \le f$$

and $(K, L, H \subset I^*)$, and for $1 \le i \le m,\ 1 \le j \le n,\ 1 \le g \le f,\ 1 \le d \le I,\ 1 \le b \le J,\ 1 \le c \le G : a_{k_{i,d},l_{j,b},h_{g,c}} \in X$.

The generalized aggregation operations over the given matrix A have the forms:
$(\circ) - \alpha_{(K,K_i)}$**-aggregation** – it is aggregation of index K_i of dimension K

$$\alpha_{(K,K_i,\circ)}(A, K_{i,0})$$

$$= \left\{ \begin{array}{c|ccccc} H_g & L_1 & \ldots & L_j & \ldots & L_n \\ \hline K_1 & a_{K_1,L_1,H_g} & \ldots & a_{K_1,L_j,H_g} & \ldots & a_{K_1,L_n,H_g} \\ \vdots & \vdots & \ldots & \vdots & \ldots & \vdots \\ K_{i,0} & \underset{i,1 \le i \le i,I}{\circ} a_{k_i,L_1,H_g} & \ldots & \underset{i,1 \le i \le i,I}{\circ} a_{k_i,L_j,H_g} & \ldots & \underset{i,1 \le i \le i,I}{\circ} a_{k_i,L_n,H_g} \\ \vdots & \vdots & \ldots & \vdots & \ldots & \vdots \\ K_m & a_{K_m,L_1,H_g} & \ldots & a_{K_m,L_j,H_g} & \ldots & a_{K_m,L_n,H_g} \end{array} | H_g \in H \right\},$$

where $K_i \subset K$ and $1 \le i \le m$

Let index set $K_* \subseteq K$ be given and $K_* = \{K_{v_1}, \ldots, K_{v_x}, \ldots, K_{v_t}\}, 1 \le v_x \le m$ for $1 \le x \le t$; $V_* = \{K_{v_1,0}, \ldots, K_{v_x,0}, \ldots, K_{v_t,0}\}, K_{v_x,0} \notin K$ for $1 \le x \le t$.

Let us recall the following definitions: $(\circ) - \alpha_{(K,K_*)}$**-aggregation**

$$\alpha_{(K,K_*,\circ)}(A, V_*) = \alpha_{(K,K_*,\circ)}(A, \langle K_{v_{1,0}}, \ldots, K_{v_{x,0}}, \ldots, K_{v_{p,0}} \rangle)$$
$$= \alpha_{(K,K_{V_t},\circ)}((\ldots \alpha_{(K,K_{v_1},\circ)}(A, K_{v_{1,0}}) \ldots), K_{v_{t,0}}).$$

Analogously are constructed the definitions of the operations:

$\{(\circ) - \alpha_{(L,L_j)}\}, \{(\circ) - \alpha_{(H,H_g)}\}$-aggregation and their summaries.

Let us perform the definitions for other aggregating operations from [24]:

$(\circ) - \alpha_{(\langle K,K_i\rangle,\langle L,L_j\rangle)}$**-aggregation**

$$\alpha_{(\langle K,K_i\rangle,\langle L,L_j\rangle,\circ)}(A,\langle K_{i,0},L_{j,0}\rangle)$$

$$= \left\{ \begin{array}{c|ccccc} H_g & L_1 & \dots & L_{j,0} & \dots & L_n \\ \hline K_1 & a_{K_1,L_1,H_g} & \dots & \underset{j_1\le j\le j_J}{\circ} a_{K_1,l_j,H_g} & \dots & a_{K_1,L_n,H_g} \\ \vdots & \vdots & \dots & \vdots & \dots & \vdots \\ K_{i,0} & \underset{i_1\le i\le i_I}{\circ} a_{k_i,L_1,H_g} & \dots & \underset{\substack{i_1\le i\le i_I \\ j_1\le j\le j_J}}{\circ} a_{k_i,l_j,H_g} & \dots & \underset{i_1\le i\le i_I}{\circ} a_{k_i,L_n,H_g} \\ \vdots & \vdots & \dots & \vdots & \dots & \vdots \\ K_m & a_{K_m,L_1,H_g} & \dots & \underset{j_1\le j\le j_J}{\circ} a_{K_m,L_j,H_g} & \dots & a_{K_m,L_n,H_g} \end{array} \middle| H_g \in H \right\},$$

where $K_i \subset K$ and $L_j \subset L$.

Let index sets $K_* \subseteq K$ and $L_* \subseteq L$ be given, and $K_* = \{K_{v_1},\dots,K_{v_x},\dots,K_{v_t}\}$, $1 \le v_x \le m$ for $1 \le x \le t$ and $L_* \subseteq L$ and $L_* = \{L_{u_1},\dots,L_{u_y},\dots,L_{u_s}\}$, $1 \le u_y \le n$ for $1 \le y \le s$. Let be given $V_* = \{K_{v_1,0},\dots,K_{v_x,0},\dots,K_{v_t,0}\}$, $K_{v_x,0} \notin K$ for $1 \le x \le t$; $W_* = \{L_{u_1,0},\dots,L_{u_y,0},\dots,L_{u_s,0}\}$, $L_{u_y,0} \notin L$ for $1 \le y \le s$.

Let us recall the definition of:

$(\circ) - \alpha_{(\langle K,K_*\rangle,\langle L,L_*\rangle)}$**-aggregation**

$$\alpha_{(\langle K,K_*\rangle,\langle L,L_*\rangle,\circ)}(A,V_*,W_*)$$
$$= \alpha_{(\langle K,K_*\rangle,\langle L,L_*\rangle,\circ)}(A,\langle K_{v_{1,0}},\dots,K_{v_{x,0}},\dots,K_{v_{t,0}}\rangle,\langle L_{u_{1,0}},\dots,L_{u_{y,0}},\dots,L_{u_{s,0}}\rangle)$$
$$= \alpha_{(\langle K,K_{v_p}\rangle,\langle L,L_{u_s}\rangle,\circ)}((\dots\alpha_{(\langle K,K_{v_1}\rangle,\langle L,L_{u_1}\rangle,\circ)}(A,\langle K_{v_{1,0}},L_{u_{1,0}}\rangle)\dots),\langle K_{v_{t,0}},L_{u_{s,0}}\rangle).$$

Analogously are constructed the definitions of following aggregation operations: $\{(\circ) - \alpha_{(\langle K,K_i\rangle,\langle H,H_g\rangle)}\}$, $\{(\circ) - \alpha_{(\langle L,L_j\rangle,\langle H,H_g\rangle)}\}$-aggregation and their summaries.

2.2. Definition of 3D-Multilayer extended index matrix and some operations over them.

2.2.1. *Definition of 3D-multilayer extended index matrix {3D-MLEIM}.* Let us begin the section with a definition of 3D-multilayer extended index matrix A (3D-MLEIM, see [24]) with P-levels (layers) of use of a dimension K, Q-levels (layers) of use of a dimension L and R-levels (layers) of use of a dimension H as follows:

$$A = [K, L, H, \{a_{K^{(p)}_{i,d}, L^{(q)}_{j,b}, H^{(r)}_{g,c}}\}]$$

$$= \left\{ \begin{array}{c|ccccc} H_g^{(R)} & L_1^{(Q)} & \dots & L_j^{(Q)} & \dots & L_n^{(Q)} \\ \hline K_1^{(P)} & a_{K_1^{(P)},L_1^{(Q)},H_g^{(R)}} & \dots & a_{K_1^{(P)},L_j^{(Q)},H_g^{(R)}} & \dots & a_{K_1^{(P)},L_n^{(Q)},H_g^{(R)}} \\ \vdots & \vdots & \dots & \vdots & \dots & \vdots \\ K_i^{(P)} & a_{K_i^{(P)},L_1^{(Q)},H_g^{(R)}} & \dots & a_{K_i^{(P)},L_j^{(Q)},H_g^{(R)}} & \dots & a_{K_i^{(P)},L_n^{(Q)},H_g^{(R)}} \\ \vdots & \vdots & \dots & \vdots & \dots & \vdots \\ K_m^{(P)} & a_{K_m^{(P)},L_1^{(Q)},H_g^{(R)}} & \dots & a_{K_m^{(P)},L_j^{(Q)},H_g^{(R)}} & \dots & a_{K_m^{(P)},L_n^{(Q)},H_g^{(R)}} \end{array} |H_g^{(R)} \in H \right\},$$

where

$$K = \{K_1^{(P)}, K_2^{(P)}, \dots, K_i^{(P)}, \dots, K_m^{(P)}\},$$

$$K_i^{(P)} = \left\{K_{i,1}^{(P-1)}, K_{i,2}^{(P-1)}, \dots, K_{i,x}^{(P-1)}, \dots, K_{i,I}^{(P-1)}\right\} \text{ for } 1 \le i \le m$$

$$\dots$$

$$K_u^{(1)} = \left\{K_{u,1}^{(0)}, K_{u,2}^{(0)}, \dots, K_{u,U}^{(0)}\right\}$$

i.e. the $p-$th layer of dimension K of the multilayer matrix, where $(1 \le p \le P)$, is performed by

$$K_{u_*}^{(p)} = \left\{K_{u_*,1}^{(p-1)}, K_{u_*,2}^{(p-1)}, \dots, K_{u_*,U_*}^{(p-1)}\right\} \text{ for } 1 \le p \le P$$

$$L = \{L_1^{(Q)}, L_2^{(Q)}, \dots, L_j^{(Q)}, \dots, L_n^{(Q)}\},$$

$$L_j^{(Q)} = \{L_{j,1}^{(Q-1)}, L_{j,2}^{(Q-1)}, \dots, L_{j,y}^{(Q-1)}, \dots, L_{j,J}^{(Q-1)}\} \text{ for } 1 \le j \le n$$

$$\dots$$

$$L_v^{(1)} = \left\{L_{v,1}^{(0)}, L_{v,2}^{(0)}, \dots, l_{v,V}^{(0)}\right\}$$

i.e. the $q-$th layer of dimension Q of the multilayer matrix is performed by

$$L_{v,*}^{(q)} = \left\{L_{v_*,1}^{(q-1)}, L_{v_*,2}^{(q-1)}, \dots, L_{v_*,V_*}^{(q-1)}\right\} \text{ for } 1 \le q \le Q$$

$$H = \{H_1^{(R)}, H_2^{(R)}, \dots, H_g^{(R)}, \dots, H_f^{(R)}\},$$

$$H_g^{(R)} = \{H_{g,1}^{(R-1)}, H_{g,2}^{(R-1)}, \dots, H_{g,z}^{(R-1)}, \dots, H_{g,G}^{(R-1)}\} \text{ for } 1 \le g \le f$$

$$\dots$$

$$H_w^{(1)} = \left\{H_{w,1}^{(0)}, H_{w,2}^{(0)}, \dots, H_{w,W}^{(0)}\right\}$$

i.e. the $r-$th layer of dimension H of the multilayer matrix is performed by

$$H_{w_*}^{(r)} = \left\{H_{w_*,1}^{(r-1)}, H_{w_*,2}^{(r-1)}, \dots, H_{w_*,W_*}^{(r-1)}\right\} \text{ for } 1 \le r \le R$$

and $(K, L, H \subset I^*)$, and for $1 \le i \le I,\ 1 \le j \le J,\ 1 \le g \le G,\ 1 \le p \le P,\ 1 \le q \le Q,\ 1 \le r \le R,\ 1 \le d \le I,\ 1 \le b \le J,\ 1 \le c \le G : a_{K_{i,d}^{(p)},L_{j,b}^{(q)},H_{g,c}^{(r)}} \in X,\ K_{i,0}^{(p)} \notin K$, $L_{j,0}^{(q)} \notin L$ and $H_{g,0}^{(r)} \notin H$.

2.2.2. *Generalized projection operations over 3D-multilayer extended index matrix {3D-MLEIM}.* Let us have 3D-MLEIM $A = [K, L, H, \{a_{K_{i,d}^{(p)}, L_{j,b}^{(q)}, H_{g,c}^{(r)}}\}]$. We will extend the definitions of the operation "Projection" over matrix A.

$$pr_{(K_i^{(P)}, p\text{-layer}), (L_j^{(Q)}, q\text{-layer}), (H_g^{(R)}, r\text{-layer})} A$$

$$= [(K_i^{(P)}, p\text{-layer}), (L_j^{(Q)}, q\text{-layer}), (H_g^{(R)}, r\text{-layer}), \{b_{K_{i,d}^{(p)}, L_{j,b}^{(q)}, H_{g,c}^{(r)}}\}],$$

where for each $K_{i,d}^{(p)} \in \{K_i^{(P)}, p\text{-layer}\}$, $L_{j,b}^{(q)} \in \{L_{j^{(Q)}}, q\text{-layer}\}$ and $H_{g,c}^{(r)} \in \{H_{g^{(R)}}, r\text{-layer}\}$

$$b_{K_{i,d}^{(p)}, L_{j,b}^{(q)}, H_{g,c}^{(r)}} = a_{K_{i,d}^{(p)}, L_{j,b}^{(q)}, H_{g,c}^{(r)}},$$

where $K_i^{(P)} \subset K, 1 \leq p \leq P, L_j^{(Q)} \subset L, 1 \leq q \leq Q$ and $H_g^{(R)} \subset H, 1 \leq r \leq R$.

Let, there be given index sets, whose members are also index sets:

$$K_* \subseteq K \text{ and } K_* = \{K_{v_1}^{(P)}, \ldots, K_{v_x}^{(P)}, \ldots, K_{v_t}^{(P)}\},$$

$$P_* = \{p_1, \ldots, p_x, \ldots, p_t\}, \text{ where } 1 \leq p_x \leq P \text{ for } 1 \leq x \leq t,$$

$$L_* \subseteq L \text{ and } L_* = \{L_{u_1}^{(Q)}, \ldots, L_{u_y}^{(Q)}, \ldots, L_{u_s}^{(Q)}\},$$

$$Q_* = \{q_1, \ldots, q_y, \ldots, q_s\}, \text{ where } 1 \leq q_y \leq Q \text{ for } 1 \leq y \leq s,$$

$$H_* \subseteq H \text{ and } H_* = \{H_{w_1}^{(R)}, \ldots, H_{w_z}^{(R)}, \ldots, H_{w_e}^{(R)}\},$$

$$R_* = \{r_1, \ldots, r_z, \ldots, r_e\}, \text{ where } 1 \leq r_z \leq R \text{ for } 1 \leq z \leq e.$$

We denote the dimension of some index set G by $dim(G) = u$. Let $dim(K_*) = dim(P_*) = t, dim(L_*) = dim(Q_*) = s, dim(H_*) = dim(R_*) = e$.

Then

$$pr_{(K_*, P_*), (L_*, Q_*), (H_*, R_*)} A = [(K_*, P_*), (L_*, Q_*), (H_*, R_*), \{b_{K_{i,d}^{(p)}, L_{j,b}^{(q)}, H_{g,c}^{(r)}}\}],$$

where for each $K_{i,d}^{(p)} \in \{K_{v_x}^{(P)}, p_x - \text{ layer }\}$ for $1 \leq x \leq t$, $L_{j,b}^{(q)} \in \{L_{u_y}^{(Q)}, q_y - \text{ layer }\}$ for $1 \leq y \leq s$ and $H_{g,c}^{(r)} \in \{H_{w_z}^{(R)}, r_z - \text{ layer }\}$ for $1 \leq z \leq e$, $b_{K_{i,d}^{(p)}, L_{j,b}^{(q)}, H_{g,c}^{(r)}} = a_{K_{i,d}^{(p)}, L_{j,b}^{(q)}, H_{g,c}^{(r)}}$.

2.2.3. *Generalized aggregation operations over 3D-multilayer extended index matrix {3D-MLEIM} (*[24]*).* The definition of the generalized aggregation operation, which performs aggregation on the p-th layer of dimension K of the matrix A, which is 3D-MLEIM, is:

$(\circ) - \alpha_{(K, K_i^{(P)}, \text{ p-layer})}$**-aggregation**

$$\alpha_{(K, K_i^{(P)}, \text{ p-layer }, \circ)}(A, K_{i,0}^{(p)})$$

$$
= \left\{ \begin{array}{c|cccccc}
H_g^{(R)} \in H & L_1^{(Q)} & \dots & \dots & L_j^{(Q)} & \dots & L_n^{(Q)} \\ \hline
K_1^{(P)} & a_{K_1^{(P)},L_1^{(Q)},H_g^{(R)}} & \dots & \dots & a_{K_1^{(P)},L_j^{(Q)},H_g^{(R)}} & \dots & a_{K_1^{(P)},L_n^{(Q)},H_g^{(R)}} \\
\vdots & \vdots & \ddots & \ddots & \vdots & \ddots & \vdots \\
K_i^P & \dots & \ddots & & & & \\
\vdots & \vdots & \ddots & & & & \\
\{K_i^P,\ \text{p-layer}\}K_{i,0}^{(p)} & \mathop{\circ}\limits_{\substack{1 \le \rho \le p-1 \\ K_u^{(\rho)} \in K_{i*}^{(p)}}} a_{K_u^{(\rho)},L_1^{(Q)},H_g^{(R)}} & \dots & \dots & \mathop{\circ}\limits_{\substack{1 \le \rho \le p-1 \\ K_u^{(\rho)} \in K_{i*}^{(p)}}} a_{K_u^{(\rho)},L_j^{(Q)},H_g^{(R)}} & \dots & \mathop{\circ}\limits_{\substack{1 \le \rho \le p-1 \\ K_u^{(\rho)} \in K_{i*}^{(p)}}} a_{K_u^{(\rho)},L_n^{(Q)},H_g^{(R)}} \\
\vdots & \vdots & \ddots & \ddots & \vdots & \ddots & \vdots \\
K_m^{(P)} & a_{K_m^{(P)},L_1^{(Q)},H_g^{(R)}} & \dots & \dots & a_{K_m{}^{(P)},L_j^{(Q)},H_g^{(R)}} & \dots & a_{K_m^{(P)},L_n^{(Q)},H_g^{(R)}}
\end{array} \right\},
$$

where $K_i^{(P)} \subset K, 1 \le p \le P$

Let index set $K_* \subseteq K$ be given and $K_* = \{K_{v_1}^{(P)}, \dots, K_{v_x}^{(P)}, \dots, K_{v_t}^{(P)}\}$, $V_* = \{K_{v_1,0}^{(p)}, \dots, K_{v_x,0}^{(p)}, \dots, K_{v_t,0}^{(p)}\} \notin K$ for $1 \le p \le P$.

In this case let us recall the definition of the aggregation operation:

$(\circ) - \alpha_{(K,K_*,p\textbf{-layer})}$**-aggregation**

$$
\alpha_{(K,K_*,p\text{-layer},\circ)}(A, V_*) = \alpha_{(K,K_*,p\text{-layer},\circ)}(A, \langle K_{v_1,0}^{(p)}, \dots, K_{v_x,0}^{(p)}, \dots, K_{v_t,0}^{(p)} \rangle)
$$

$$
= \alpha_{(K,K_{v_t}^{(P)},\circ)}((\dots \alpha_{(K,K_{v_1}^{(P)},\circ)}(A, K_{v_1,0}^{(p)}) \dots), K_{v_t,0}^{(p)}).
$$

Let there be given index set $K_* \subseteq K$ and $K_* = \{K_{v_1}^{(P)}, \dots, K_{v_x}^{(P)}, \dots, K_{v_t}^{(P)}\}$, $V_* = \{K_{v_1,0}^{(p_1)}, \dots, K_{v_x,0}^{(p_x)}, \dots, K_{v_t,0}^{(p_t)}\} \notin K$ and $P_* = \{p_1, \dots, p_x, \dots, p_t\}$, where $1 \le p_x \le P$ for $1 \le x \le t$. Let $dim(K_*) = dim(P_*) = dim(V_*) = t$.

In this case we will recall the definition of aggregation operation as follows:

$(\circ) - \alpha_{(K,K_*,P_*)}$**-aggregation**

$\alpha_{(K,K_*,P_*,\circ)}(A, V_*) = \alpha_{(K,K_*,P_*,\circ)}(A, \langle K_{v_1,0}^{(p_1)}, \dots, K_{v_x,0}^{(p_x)}, \dots, K_{v_t,0}^{(p_t)} \rangle)$
$= \alpha_{(K,K_{v_t}^{(P)},\circ)}((\dots \alpha_{(K,K_{v_1}^{(P)},\circ)}(A, K_{v_1,0}^{(p_1)}) \dots), K_{v_t,0}^{(p_t)})$.

Similar are the definition of following operations: $(\circ) - \alpha_{(L,L_j^{(Q)},q\text{-layer})}$, $(\circ) - \alpha_{(H,H_g^{(R)},r\text{-layer})}$-aggregation and their generalizations.

Let us recall the definitions for other aggregating operations from [24]: $\alpha_{(\langle K,K_i^{(P)},p\text{-layer}\rangle,\langle L,L_j^{(Q)},q\text{-layer}\rangle,\circ)}$, $\alpha_{(\langle K,K_i^{(P)},p\text{-layer}\rangle,\langle H,H_g^{(R)},r\text{-layer}\rangle,\circ)}$ and $\alpha_{(\langle L,L_j^{(Q)},q\text{-layer}\rangle,\langle H,H_g^{(R)},r\text{-layer}\rangle,\circ)}$-aggregation.

The definition of $\{(\circ)-\alpha_{(\langle K,K_i^{(P)},\boldsymbol{p}\textbf{-layer}\rangle,\langle L,L_j^{(Q)},\boldsymbol{q}\textbf{-layer}\rangle,\circ)}$**-aggregation**$\}$ is:

$$\alpha_{(\langle K,K_i^{(P)},p\text{-layer}\rangle,\langle L,L_j^{(Q)},q\text{-layer}\rangle,\circ)}(A,\langle K_{i,0}^{(p)},L_{j,0}^{(q)}\rangle)$$

$$= \left\{ \begin{array}{c|cccccc} & L_1^{(Q)} & \dots & L_{j_0}^{(q)} & \dots \\ \hline K_1^{(P)} & a_{K_1^{(P)},L_1^{(Q)},H_g^{(R)}} & \dots & \underset{\substack{1\le\sigma\le q-1\\ L_v^{\sigma}\in L_{j*}^{(q)}}}{\circ} a_{K_1^{(P)},L_j^{(\sigma)},H_g^{(R)}} & \dots \\ \vdots & \vdots & \ddots & \vdots & \ddots \\ K_{i,0}^{(p)} & \underset{\substack{1\le\rho\le p-1\\ K_u^{\rho}\in K_{i*}^{(p)}}}{\circ} a_{K_u^{(\rho)},L_1^{(Q)},H_g^{(R)}} & \dots & \underset{\substack{1\le\rho\le P-1\\ K_u^{\rho}\in K_i^{(p)}\\ 1\le\sigma\le Q-1\\ L_v^{\sigma}\in L_j^{(q)}}}{\circ} a_{K_u^{(\rho)},L_j^{(\sigma)},H_g^{(R)}} & \dots \\ \vdots & \vdots & \ddots & \vdots & \ddots \\ K_m^{(P)} & a_{K_m^{(P)},L_1^{(Q)},H_g^{(R)}} & \dots & \underset{\substack{1\le\sigma\le Q-1\\ L_v^{\sigma}\in L_j^{(q)}}}{\circ} a_{K_m^{(P)},L_j^{(\sigma)},H_g^{(R)}} & \dots \end{array} \right.$$

$$\left. \begin{array}{cc} \dots & L_n^{(Q)} \\ \hline \dots & a_{K_1^{(P)},L_n^{(Q)},H_g^{(R)}} \\ \ddots & \vdots \\ \dots & \underset{\substack{1\le\rho\le P-1\\ K_u^{\rho}\in K_{i*}^{(p)}}}{\circ} a_{K_u^{(\rho)},L_n^{(Q)},H_g^{(R)}} \\ \ddots & \vdots \\ \dots & a_{K_m^{(P)},L_n^{(Q)},H_g^{(R)}} \end{array} | H_g^{(R)} \in H \right\},$$

where $K_i^{(P)} \subset K$ for $1 \le p \le P$, $L_j^{(Q)} \subset L$ for $1 \le q \le Q$.
Similar are the definitions: $(\circ)-\alpha_{(\langle K,K_i^{(P)},p\text{-layer}\rangle,\langle H,H_g^{(R)},r\text{-layer}\rangle)}$,

$(\circ)-\alpha_{(\langle H,H_g^{(R)},r\text{-layer}\rangle,\langle L,L_j^{(Q)},q\text{-layer}\rangle)}$ and their generalizations.

3. An implementation of the OLAP operations by index matrices

In the current section we present an implementation of the OLAP operations by index matrices. Particular attention is paid to the operations “slice”, “dice” and "pivot". Operation "slice" selects one dimension of the cube and reduces its dimensionality. The result is a sub-cube containing a subset of dimensions for selected values. Operation "dice" selects condition on one dimension or more than one dimension. In this way the operation "dice" reduces the number of member values of one or more dimensions. The result is a sub-cube. The operation "pivot" rotates the dimensions of the OLAP cube. The operations "slice" and "dice" project some data of the dimensions. When the dimension has more than one value, it is necessary to have used in advance the operations "roll-up" or "dice" to obtain a cube with only one value in the selected dimension [9, 12, 13, 18, 19, 20, 22].

3.1. **Operation "slice".**

3.1.1. *Definition.* The "slice" operation performs selection of one dimension on an OLAP cube using a criterion. The result is a sub-cube. The dimensions K and H are selected from the cube presented on the Fig.1

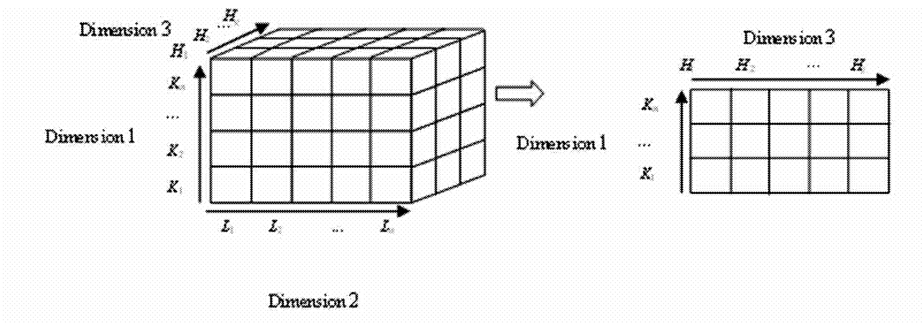

FIG. 1. Operation "slice"

3.1.2. *Presentation of the operation "slice" by the index matrices.* Definition 1: In the case of 3D-EIM A: Let there be given a 3D-EIM $A = [K, L, H, \{a_{k_i,l_j,h_g}\}]$. Let K_*, L_* and H_* are index sets so that:

$$K_* \subseteq K \text{ and } K_* = \{K_{v_1}, \ldots, K_{v_x}, \ldots, K_{v_t}\},$$
$$L_* \subseteq L \text{ and } L_* = \{L_{u_1}, \ldots, L_{u_y}, \ldots, L_{u_s}\},$$
$$H_* \subseteq H \text{ and } H_* = \{H_{w_1}, \ldots, H_{w_z}, \ldots, H_{w_d}\},$$

Then operation "slice" is expressed by:

$$pr_{K_*,L,H}A \oplus_{\vee} pr_{K,L_*,H}A \oplus_{\vee} pr_{K,L,H_*}A.$$

Definition 2: In the case of 3D-MLEIM $A = [K, L, H, \{a_{K_{i,d}^{(p)},L_{j,b}^{(q)},H_{g,c}^{(r)}}\}]$:
Operation "slice" is expressed by:
$pr_{(K_i^{(P)},p\text{-layer}),L,H}A \oplus_{\vee} pr_{K,(L_j^{(Q)},q\text{-layer}),H}A$
$\oplus_{\vee} pr_{K,L,(H_g^{(R)},r\text{-layer})}A,$
where $K_i^{(P)} \subset K, 1 \le p \le P, L_j^{(Q)} \subset L, 1 \le q \le Q$ and $H_g^{(R)} \subset H$,
$1 \le r \le R$.

Let, there be given index sets of index sets

$$K_* \subseteq K \text{ and } K_* = \{K_{v_1}^{(P)}, \ldots, K_{v_x}^{(P)}, \ldots, K_{v_t}^{(P)}\},$$

$$P_* = \{p_1, \dots, p_x, \dots, p_t\}, \text{ where } 1 \le p_x \le P \text{ for } 1 \le x \le t,$$
$$L_* \subseteq L \text{ and } L_* = \{L_{u_1}^{(Q)}, \dots, L_{u_y}^{(Q)}, \dots, L_{u_s}^{(P)}\},$$
$$Q_* = \{q_1, \dots, q_y, \dots, q_s\}, \text{ where } 1 \le q_y \le Q \text{ for } 1 \le y \le s,$$
$$H_* \subseteq H \text{ and } H_* = \{H_1^{(R)}, \dots, H_{w_z}^{(R)}, \dots, H_{w_e}^{(R)}\},$$
$$R_* = \{r_1, \dots, r_z, \dots, r_e\}, \text{ where } 1 \le r_z \le R \text{ for } 1 \le z \le e.$$

Let $dim(K_*) = dim(P_*) = t, dim(L_*) = dim(Q_*) = s, dim(H_*) = dim(R_*) = e.$

Then, with the exception of the initial level, the operation "slice" can be performed successfully at other level in the hierarchies as follows:

$$pr_{(K_*,P_*),L,H}A \oplus_{\vee} pr_{K,(L_*,Q_*),H}A \oplus_{\vee} pr_{K,L,(H_*,R_*)}A.$$

3.1.3. *Examples for Operation "slice":* The OLAP operations in this paper are realized by the data cube "Bookshops". It contains information about the books, sold in different bookshops (managed by different regional managers) in different locations. The structure of the cube "Bookshops" is presented by star schema. The fact table has the following form: Sales_Id, Bookshop_ Id, Location_ Id, Book_ Id, Number, Date. The dimensional tables are Books {Id, Title, Publisher, Genre, Price}, Bookshops {Id, Bookshop Name, Regional Manager, Owner} and Location_ Id, Town, Country}. The measures Number and Sales Count are used. The structure and the attribute relationships of the dimensions Books, Bookshops and Location are visualized on the Fig.2 and Fig.3.

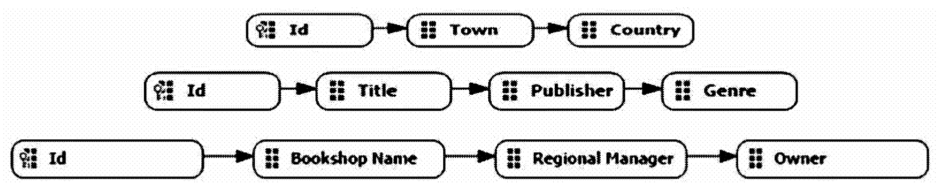

FIG. 2. Attribute relationships in dimensions Location, Books and Bookshops

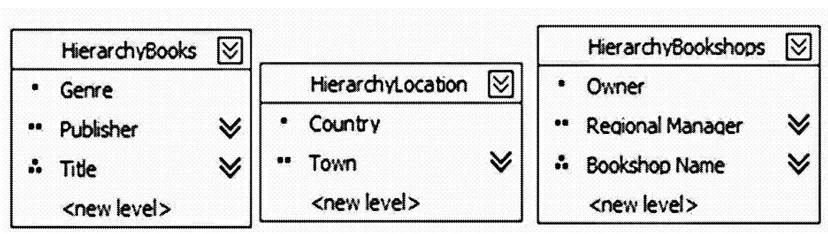

FIG. 3. Dimension structure - Books, Location, Bookshops

According to the presented structures of the dimensions, the aggregations are performed in following order: the books are grouped by publisher and after that by genre; the towns are grouped by country; the bookshops are grouped by regional manager and then by owner. We browse the dimensions to view part of their members. In dimension Bookshops, the owner has one regional manager that is responsible for the bookshops Penguins 1 and Penguins 2. The hierarchy for other owners is constructed in the same way. In dimension Books the genre "Computer books" has several publishers, which print books. The publisher Springer has printed the book "Introduction in Computer Design". In the same way the hierarchies for the genres Children Books and Cooking Books are constructed. The countries in dimension Location has several towns. The cities Burgas, Plovdiv and Sofia are located in the country Bulgaria (Fig. 4).

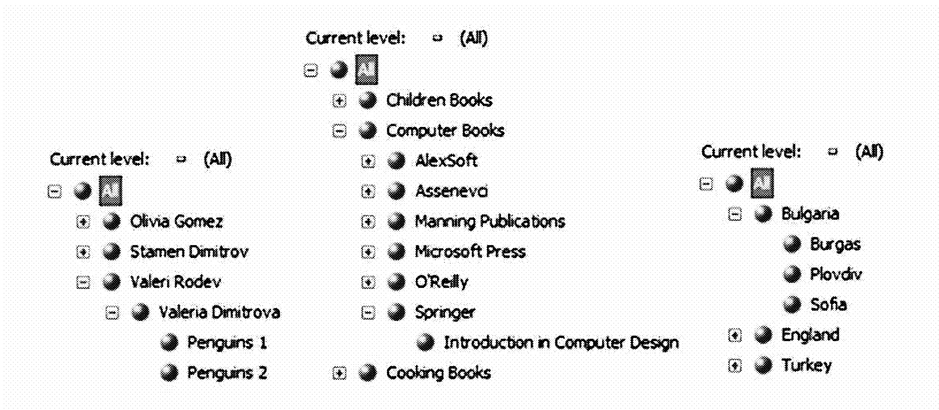

FIG. 4. Browsing the dimensions Bookshops, Books, Location

The constructed cube "Bookshops" is used to present the queries which give us practical interpretation of the operations "slice". The queries are executed by Multidimensional Expressions (MDX) [13, 19, 20, 26]. An example of the operation slice is presented on the Fig.5. The result contains the dimensions Books and Location.

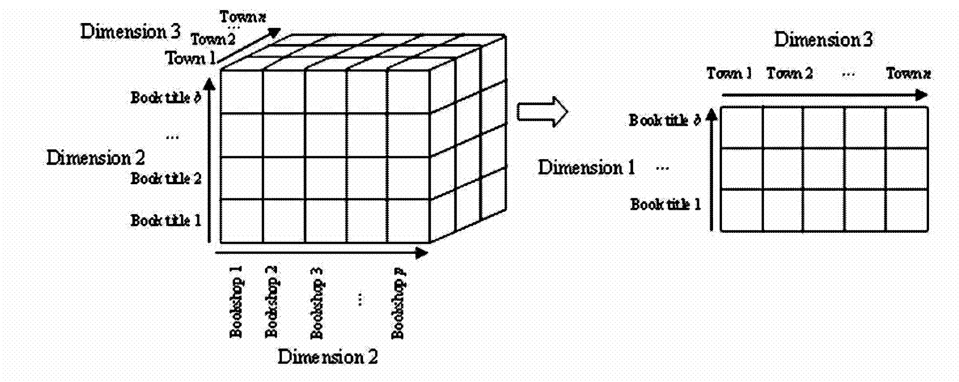

FIG. 5. Bookshops cube - operation "slice"

- *MDX query 1:*
  SELECT NONEMPTY([Location].[HierarchyLocation].[Town].Members) ON

COLUMNS,
[Books].[HierarchyBooks].[Title].[R in Action],
[Books].[HierarchyBooks].[Title].[Introduction in Computer Design],
[Books].[HierarchyBooks].[Title].[Hadoop: The Definitive Guide] ON ROWS
FROM [Bookshops2]
WHERE ([Measures].[Sales Count]);
*Result:*The MDX query extracts the information for the count of the sold titles of selected books by town. The result is presented on Fig. 8.

	Burgas	Plovdiv	Sofia	London	Mersin
R in Action	1	1	1	1	1
Introduction in Computer Design	1	1	1	1	1
Hadoop: The Definitive Guide	1	1	1	2	1

FIG. 6. Result from the "slice" query

- *MDX query 2:*
SELECT NONEMPTY([Location].[HierarchyLocation].[Town].Members) ON COLUMNS,
[Books].[HierarchyBooks].[Title].[R in Action],
[Books].[HierarchyBooks].[Title].[Introduction in Computer Design],
[Books].[HierarchyBooks].[Title].[Hadoop: The Definitive Guide] ON ROWS
FROM [Bookshops2]
WHERE ([Measures].[Sales Count]);
*Result:*The MDX query extracts the information for the count of the sold titles of selected books by town. The result is presented on Fig.7.

	Burgas	Plovdiv	Sofia	London	Mersin
R in Action	1	1	1	1	1
Introduction in Computer Design	1	1	1	1	1
Hadoop: The Definitive Guide	1	1	1	2	1

FIG. 7. Result from the "slice query

- *MDX query3:* A similar request is performed using higher hierarchy level of the cube. The query extracts the count of the sold books in the countries by genre. The result is presented on Fig. 8.
SELECT NON EMPTY([Books].[HierarchyBooks].[Genre]) ON COLUMNS,
NON EMPTY([Location].[HierarchyLocation].[Country].Members) ON ROWS
FROM [Bookshops2] WHERE ([Measures].[Sales Count]);
Result:

	Children Books	Computer Books	Cooking Books
Bulgaria	13	42	6
England	4	15	2
Turkey	4	14	2

FIG. 8. Result from the“slice query in higher level

3.2. **Operation “dice”.**

3.2.1. *Definition.* The “dice” operation (Fig. 9) performs selection of two or more dimension on an OLAP cube using criteria and returns a new sub-cube.

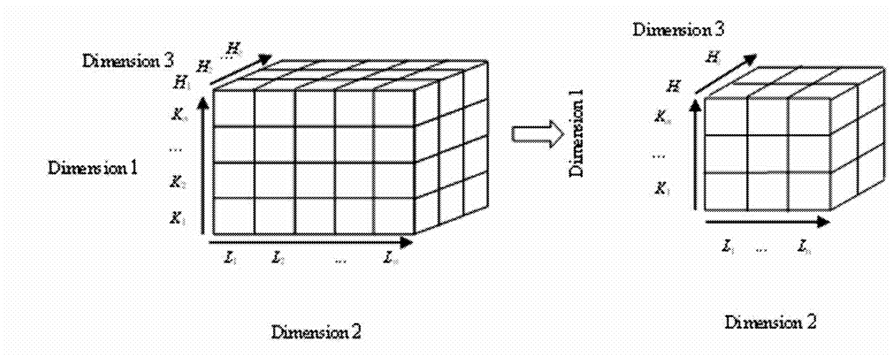

FIG. 9. Operation “dice”

3.2.2. *Presentation of the operation “dice” by index matrices.* Definition 1: In the case of 3D-EIM A: Let 3D-EIM $A = [K, L, H, \{a_{k_i,l_j,h_g}\}]$ be given. Let K_*, L_* and H_* be index sets so that:

$$K_* \subseteq K \text{ and } K_* = \{K_{v_1}, \ldots, K_{v_x}, \ldots, K_{v_t}\},$$

$$L_* \subseteq L \text{ and } L_* = \{L_{u_1}, \ldots, L_{u_y}, \ldots, L_{u_s}\},$$

$$H_* \subseteq H \text{ and } H_* = \{H_{w_1}, \ldots, H_{w_z}, \ldots, H_{w_d}\},$$

Then operation “dice” is expressed by:

$$pr_{K_*,L_*,H_*}A = [K_*, L_*, H_*, \{b_{k_e,l_o,h_s}\}],$$

where for each $k_e \in K_*$, $l_o \in L_*$ and $h_s \in H_*$ $(b_{k_e,l_o,h_s} = a_{k_e,l_o,h_s})$.

Definition 2: In the case of 3D-MLEIM $A = [K, L, H, \{a_{K^{(p)}_{i,d},L^{(q)}_{j,b},H^{(r)}_{g,c}}\}]$:

Operation “dice” is expressed by:

$pr_{(K_i^{(P)},\text{ p-layer}),(L_j^{(Q)},\text{ q-layer}),(H_g^{(R)},\text{ r-layer})}A,$

where $K_i^{(P)} \subset K, 1 \le p \le P$, $L_j^{(Q)} \subset L, 1 \le q \le Q$ and $H_g^{(R)} \subset H$, $1 \le r \le R$

or let be given index sets of index sets

$$K_* \subseteq K \text{ and } K_* = \{K^{(P)}_{v_1}, \ldots, K^{(P)}_{v_x}, \ldots, K^{(P)}_{v_t}\},$$

$$P_* = \{p_1, \ldots, p_x, \ldots, p_t\}, \text{ where } 1 \le p_x \le P \text{ for } 1 \le x \le t,$$

$$L_* \subseteq L \text{ and } L_* = \{L_{u_1}^{(Q)}, \dots, L_{u_y}^{(Q)}, \dots, L_{u_s}^{(P)}\},$$
$$Q_* = \{q_1, \dots, q_y, \dots, q_s\}, \text{ where } 1 \le q_y \le Q \text{ for } 1 \le y \le s,$$
$$H_* \subseteq H \text{ and } H_* = \{H_1^{(R)}, \dots, H_{w_z}^{(R)}, \dots, H_{w_e}^{(R)}\},$$
$$R_* = \{r_1, \dots, r_z, \dots, r_e\}, \text{ where } 1 \le r_z \le R \text{ for } 1 \le z \le e.$$

Let $dim(K_*) = dim(P_*) = t, dim(L_*) = dim(Q_*) = s, dim(H_*) = dim(R_*) = e$.

Then with exception of the initial level, the operation "slice" can be performed successfully at any other level in the hierarchies as follows:

$$pr_{(K_*,P_*),(L_*,Q_*),(H_*,R_*)}A,$$

where for each $K_{i_d}^{(p)} \in \{K_{v_x}^{(P)}, p_x - \text{ layer }\}$ for $1 \le x \le t$, $L_{j_b}^{(q)} \in \{L_{u_y}^{(Q)}, q_y - \text{ layer }\}$ for $1 \le y \le s$ and $H_{g_c}^{(r)} \in \{H_{w_z}^{(R)}, r_z - \text{ layer }\}$ for $1 \le z \le e$, $b_{K_{i,d}^{(p)}, L_{j,b}^{(q)}, H_{g,c}^{(r)}} = a_{K_{i,d}^{(p)}, L_{j,b}^{(q)}, H_{g,c}^{(r)}}$.

3.2.3. *Examples for Operation "dice":* An example of the operation "dice" is presented on Fig. 10. The result contains a part of the dimensions Books, Bookshops and Location.

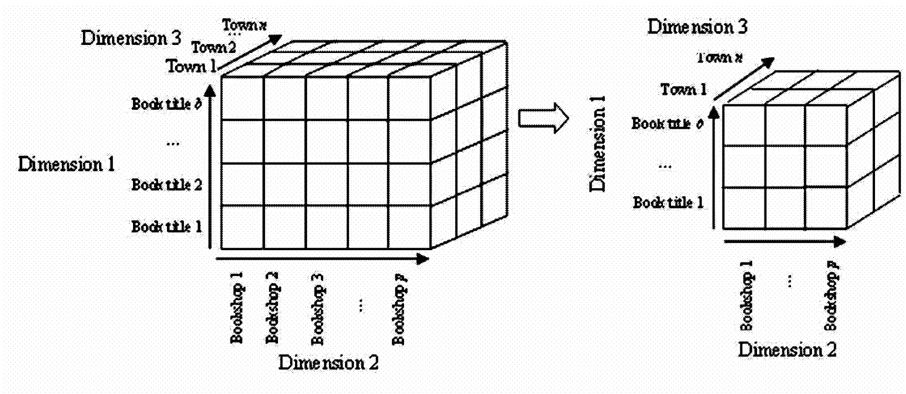

FIG. 10. Result of "dice" operation

- *MDX query:* The result of the MDX query presents (Fig. 11) the owners of bookshops Stamen Dimitrov and Valeri Rodev selling N-books (measure Sales Count) of genres Computer Books and Children Books in all cities from the specified countries - Bulgaria and Turkey.
SELECT NON EMPTY CROSSJOIN([Bookshops].[HierarchyBookshops].
[Owner].&[Stamen Dimitrov],
[Bookshops].[HierarchyBookshops].[Owner].&[Valeri Rodev],
[Books].[HierarchyBooks].[Genre].&[Computer Books],
[Books].[HierarchyBooks].[Genre].&[Children Books]) ON COLUMNS,
[Location].[HierarchyLocation].[Country].[Bulgaria],
[Location].[HierarchyLocation].[Country].[Turkey] ON ROWS FROM [Bookshops2] WHERE [Measures].[Sales Count];

Result:

	Stamen Dimitrov	Stamen Dimitrov	Valeri Rodev	Valeri Rodev
	Computer Books	Children Books	Computer Books	Children Books
Bulgaria	42	13	(null)	(null)
Turkey	(null)	(null)	14	4

FIG. 11. Result from the "dice" query

3.3. **Operation "pivot".**

3.3.1. *Definition.* Operation "pivot" is presented in the theory of index matrices using the operation "transposition" in its different variants [5]. It rotate the dimensions on the cube (Fig. 12).

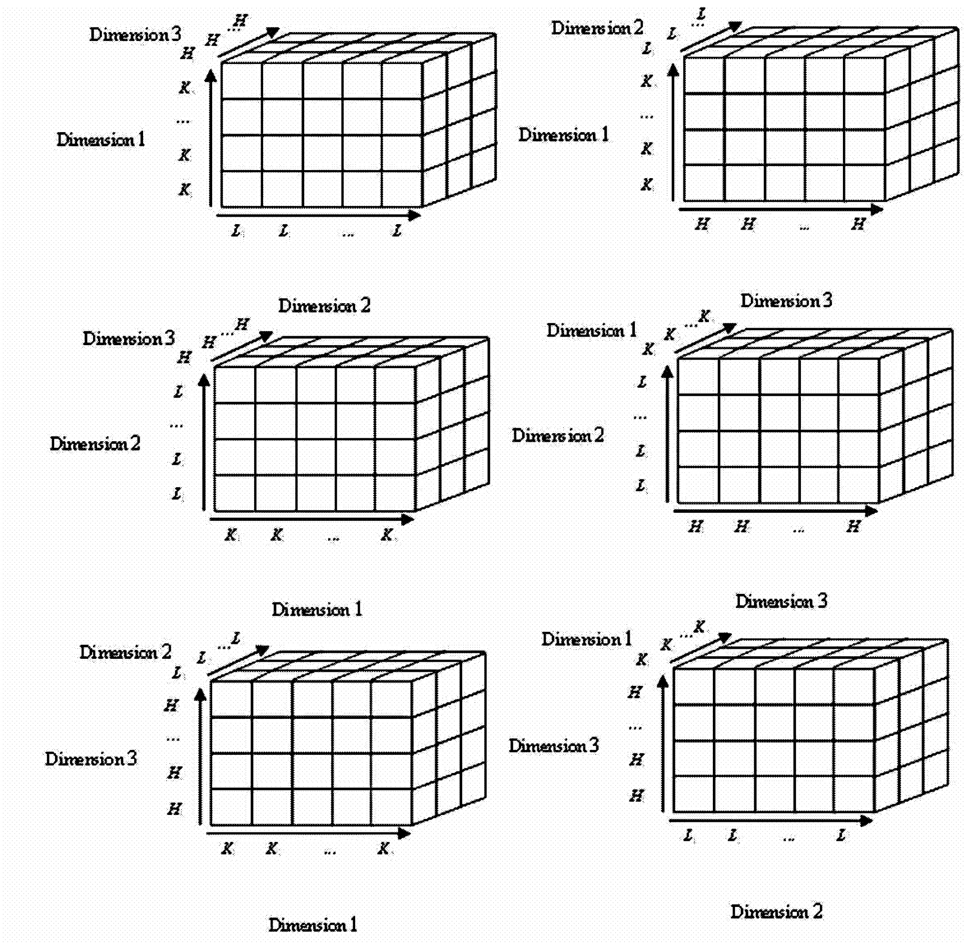

FIG. 12. Operation "pivot"

3.3.2. *Presentation of the operation "pivot" by the index matrices.* Definition 1: In the case of 3D-EIM A: Let 3D-EIM be given $A = [K, L, H, \{a_{k_i,l_j,h_g}\}]$. Then operation "pivot" is expressed by:

$$[K, L, H, \{a_{k_i,l_j,h_g}\}]^{[1,2,3]} = [K, L, H, \{a_{k_i,l_j,h_g}\}];$$
$$[K, L, H, \{a_{k_i,l_j,h_g}\}]^{[1,3,2]} = [K, H, L, \{a_{k_i,h_g,l_j}\}];$$
$$[K, L, H, \{a_{k_i,l_j,h_g}\}]^{[2,1,3]} = [L, K, H, \{a_{l_j,k_i,h_g}\}];$$
$$[K, L, H, \{a_{k_i,l_j,h_g}\}]^{[2,3,1]} = [L, H, K, \{a_{l_j,h_g,k_i}\}];$$
$$[K, L, H, \{a_{k_i,l_j,h_g}\}]^{[3,1,2]} = [H, K, L, \{a_{h_g,k_i,l_j}\}];$$
$$[K, L, H, \{a_{k_i,l_j,h_g}\}]^{[3,2,1]} = [H, L, K, \{a_{h_g,l_j,k_i}\}].$$

Definition 2: In the case of 3D-MLEIM $A = [K, L, H, \{a_{K^{(p)}_{i,d},L^{(q)}_{j,b},H^{(r)}_{g,c}}\}]$:
Operation "pivot" has the following form:

$$[K, L, H, \{a_{K^{(p)}_{i,d},L^{(q)}_{j,b},H^{(r)}_{g,c}}\}]^{[1,2,3]} = [K, L, H, \{a_{K^{(p)}_{i,d},L^{(q)}_{j,b},H^{(r)}_{g,c}}\}];$$
$$[K, L, H, \{a_{K^{(p)}_{i,d},L^{(q)}_{j,b},H^{(r)}_{g,c}}\}]^{[1,3,2]} = [K, H, L, \{a_{K^{(p)}_{i,d},H^{(r)}_{g,c},L^{(q)}_{j,b}}\}];$$
$$[K, L, H, \{a_{K^{(p)}_{i,d},L^{(q)}_{j,b},H^{(r)}_{g,c}}\}]^{[2,1,3]} = [L, K, H, \{a_{L^{(q)}_{j,b},K^{(p)}_{i,d},H^{(r)}_{g,c}}\}];$$
$$[K, L, H, \{a_{K^{(p)}_{i,d},L^{(q)}_{j,b},H^{(r)}_{g,c}}\}]^{[2,3,1]} = [L, H, K, \{a_{L^{(q)}_{j,b},H^{(r)}_{g,c},K^{(p)}_{i,d}}\}];$$
$$[K, L, H, \{a_{K^{(p)}_{i,d},L^{(q)}_{j,b},H^{(r)}_{g,c}}\}]^{[3,1,2]} = [H, K, L, \{a_{H^{(r)}_{g,c},K^{(p)}_{i,d},L^{(q)}_{j,b}}\}];$$
$$[K, L, H, \{a_{K^{(p)}_{i,d},L^{(q)}_{j,b},H^{(r)}_{g,c}}\}]^{[3,2,1]} = [H, L, K, \{a_{H^{(r)}_{g,c},L^{(q)}_{j,b},K^{(p)}_{i,d}}\}].$$

3.3.3. *Examples for Operation "pivot":* The constructed cube "Bookshops" is used to present the queries which give us practical interpretation of the operations "pivot". The queries are executed by Multidimensional Expressions (MDX) [13, 19, 20, 26]. The result of the operation "pivot" contains the dimensions Books, Bookshops and Location located on different axes. The six view of the operation "pivot" are presented on the figures from Fig. 13 to Fig. 23.

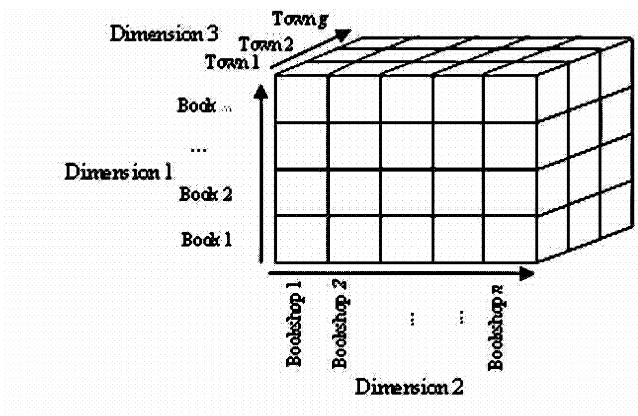

FIG. 13. Rotate the cube - 1

- *MDX query1:* In the first view of the cube, the dimension Bookshops is located on the axis x and respectively the dimension Books and the dimension Town are mapped on the axes y and z. The MDX query is presented below.
  SELECT NON EMPTY([Location].[HierarchyLocation].[Country],
  [Bookshops].[HierarchyBookshops].[Owner]) ON COLUMNS,
  NON EMPTY ([Books].[HierarchyBooks].[Genre]) ON ROWS
  FROM [Bookshops2] WHERE [Measures].[Sales Count];
  Result: The result of the query is a 2D-table which visualizes the third dimension one level up.

	Bulgaria	England	Turkey
	Stamen Dimitrov	Olivia Gomez	Valeri Rodev
Children Books	13	4	4
Computer Books	42	15	14
Cooking Books	6	2	2

FIG. 14. Result of the query "pivot-1"

- *MDX query2:* In the second view of the cube, the dimension Books is located on the axis x and respectively the dimensions Bookshops and Town are mapped on the axes y and z. The MDX query is presented below.

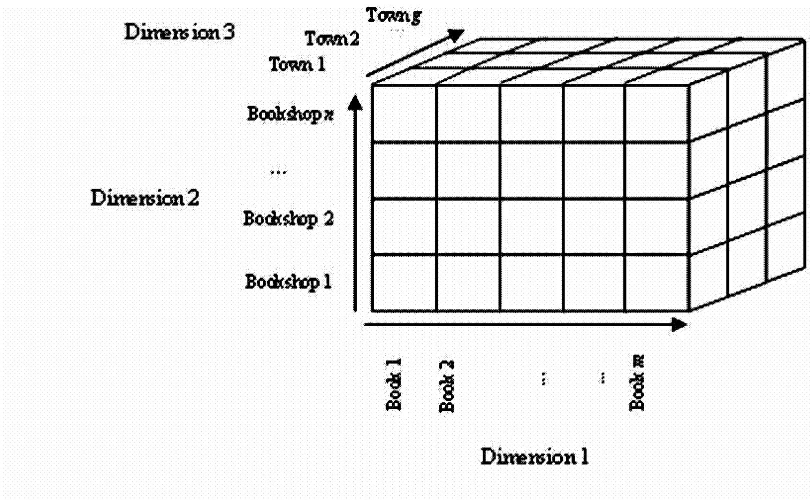

FIG. 15. Rotate the cube - 2

SELECT NON EMPTY([Location].[HierarchyLocation].[Country],
[Books].[HierarchyBooks].[Genre]) ON COLUMNS,
NON EMPTY ([Bookshops].[HierarchyBookshops].[Owner]) ON ROWS
FROM [Bookshops2] WHERE [Measures].[Sales Count].

Result: The result of the query is 2D-table which visualize the third dimension one level upper.

	Bulgaria	Bulgaria	Bulgaria	England	England	England	Turkey
	Children Books	Computer Books	Cooking Books	Children Books	Computer Books	Cooking Books	Children Books
Olivia Gomez	(null)	(null)	(null)	4	15	2	(null)
Stamen Dimitrov	13	42	6	(null)	(null)	(null)	(null)
Valeri Rodev	(null)	(null)	(null)	(null)	(null)	(null)	4

FIG. 16. Result of the query "pivot-2"

- *MDX query3:* In the third view of the cube, the dimension Bookshops is located on the axis x and respectively the dimensions Town and Books are mapped on the axes y and z. The MDX query is presented below.

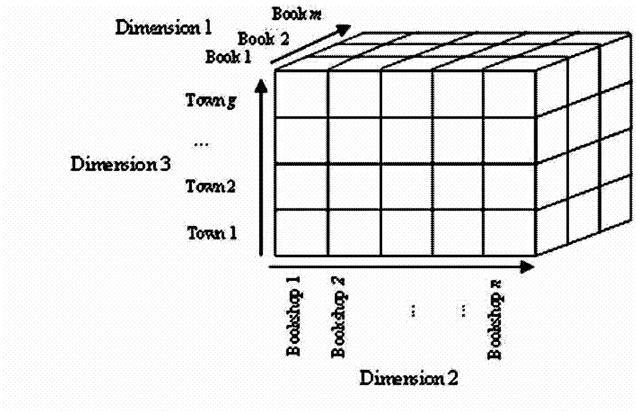

FIG. 17. Rotate the cube - 3

SELECT NON EMPTY([Books].[HierarchyBooks].[Genre], [Bookshops].[HierarchyBookshops].[Owner]) ON COLUMNS, NON EMPTY ([Location].[HierarchyLocation].[Country]) ON ROWS FROM [Bookshops2] WHERE [Measures].[Sales Count].

Result: The result of the query is a $2D$-table which visualizes the third dimension one level up.

	Children Books	Children Books	Children Books	Computer Books	Computer Books	Computer Books	Cooking Books
	Olivia Gomez	Stamen Dimitrov	Valeri Rodev	Olivia Gomez	Stamen Dimitrov	Valeri Rodev	Olivia Gomez
Bulgaria	(null)	13	(null)	(null)	42	(null)	(null)
England	4	(null)	(null)	15	(null)	(null)	2
Turkey	(null)	(null)	4	(null)	(null)	14	(null)

FIG. 18. Result of the query "pivot-3"

- *MDX query4:* In the fourth view of the cube, the dimension Bookshops is located on the axis x and respectively the dimensions Town and Books are mapped on the axes y and z. The MDX query is presented below.

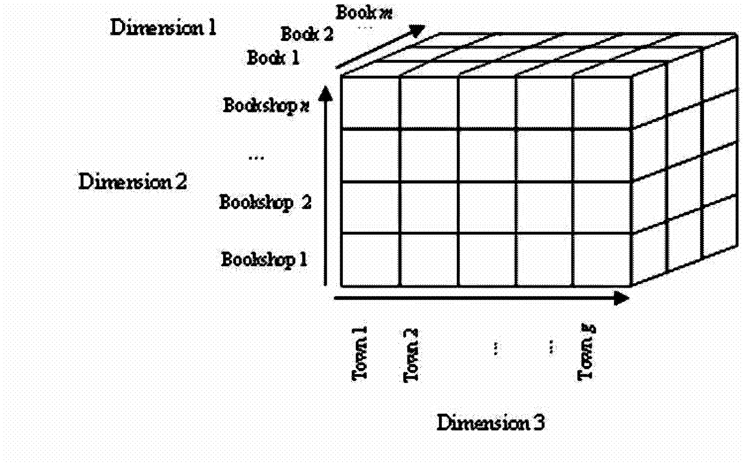

FIG. 19. Rotate the cube - 4

SELECT NON EMPTY ([Location].[HierarchyLocation].[Country]) ON COLUMNS,
NON EMPTY([Books].[HierarchyBooks].[Genre],
[Bookshops].[HierarchyBookshops].[Owner])ON ROWS
FROM [Bookshops2] WHERE [Measures].[Sales Count].

Result: The result of the query is a 2D-table which visualizes the third dimension one level to the left.

		Bulgaria	England	Turkey
Children Books	Olivia Gomez	(null)	4	(null)
Children Books	Stamen Dimitrov	13	(null)	(null)
Children Books	Valeri Rodev	(null)	(null)	4
Computer Books	Olivia Gomez	(null)	15	(null)
Computer Books	Stamen Dimitrov	42	(null)	(null)
Computer Books	Valeri Rodev	(null)	(null)	14
Cooking Books	Olivia Gomez	(null)	2	(null)
Cooking Books	Stamen Dimitrov	6	(null)	(null)
Cooking Books	Valeri Rodev	(null)	(null)	2

FIG. 20. Result of the query "pivot-4"

- *MDX query 5:* In the fifth view of the cube, the dimension Books is located on the axis x and respectively the dimensions Town and Bookshops are mapped on the axes y and z. The MDX query is presented below.

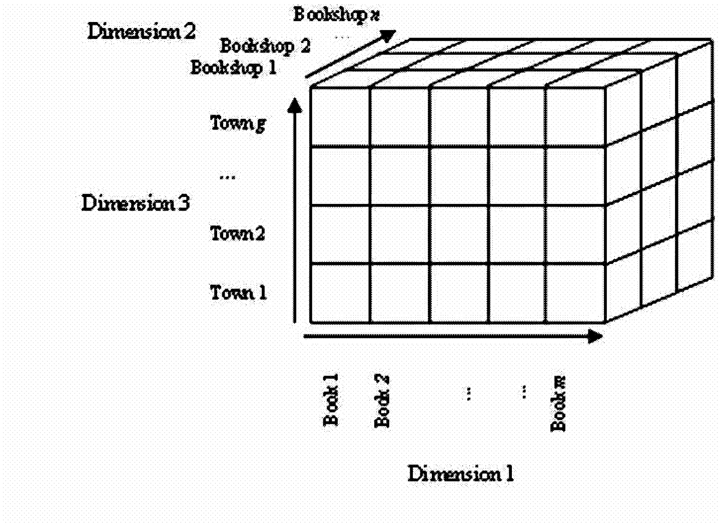

FIG. 21. Rotate the cube - 5

SELECT NON EMPTY ([Books].[HierarchyBooks].[Genre]) ON COLUMNS,
NON EMPTY([Location].[HierarchyLocation].[Country],
[Bookshops].[HierarchyBookshops].[Owner])ON ROWS
FROM [Bookshops2] WHERE [Measures].[Sales Count].

Result: The result of the query is a 2D-table which visualizes the third dimension one level to the left.

		Children Books	Computer Books	Cooking Books
Bulgaria	Stamen Dimitrov	13	42	6
England	Olivia Gomez	4	15	2
Turkey	Valeri Rodev	4	14	2

FIG. 22. Result of the query "pivot-5"

- *MDX query 6:* In the sixth view of the cube, the dimension Town is located on the axis x and respectively the dimensions Books and Bookshops are mapped on the axis y and z. The MDX query is presented below.

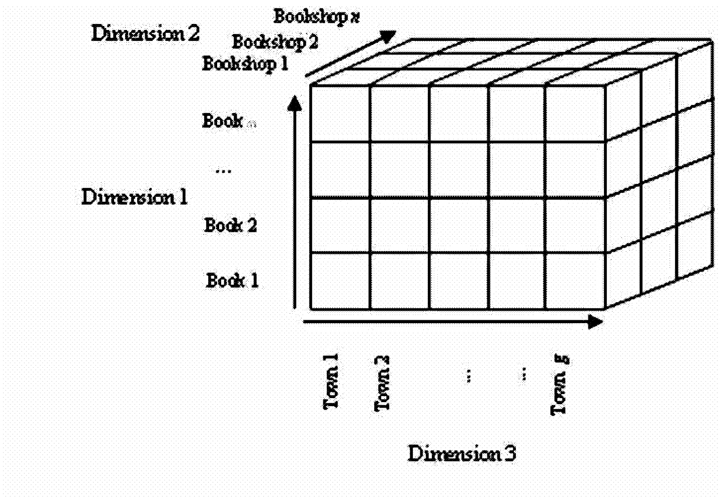

FIG. 23. Rotate the cube - 6

SELECT NON EMPTY ([Bookshops].[HierarchyBookshops].[Owner]) ON COLUMNS,
NON EMPTY([Location].[HierarchyLocation].[Country], [Books].[HierarchyBooks].[Genre])ON ROWS
FROM [Bookshops2] WHERE [Measures].[Sales Count].

Result: The result of the query is a 2D-table which visualizes the third dimension one level to the left.

		Olivia Gomez	Stamen Dimitrov	Valeri Rodev
Bulgaria	Children Books	(null)	13	(null)
Bulgaria	Computer Books	(null)	42	(null)
Bulgaria	Cooking Books	(null)	6	(null)
England	Children Books	4	(null)	(null)
England	Computer Books	15	(null)	(null)
England	Cooking Books	2	(null)	(null)
Turkey	Children Books	(null)	(null)	4
Turkey	Computer Books	(null)	(null)	14
Turkey	Cooking Books	(null)	(null)	2

FIG. 24. Result of the query "pivot-6"

3.4. **Combining operations "slice", "dice" and "roll-up".** In [25] the operation "roll-up" was presented in terms of index matrices by aggregating operators, which definition is given in section 2 of the paper. Operation "roll-up" can be represented as follows:

$$\alpha_{(K,K_*,\text{p-layer},\circ)}(A, W_*) \oplus_{(\vee,\wedge)} \alpha_{(L,L_*,\text{q-layer},\circ)}(A, U_*) \oplus_{(\vee,\wedge)} \alpha_{(H,H_*,\text{r-layer},\circ)}(A, V_*),$$

where

$$K_* \subseteq K \text{ and } K_* = \{K_{w_1}^{(P)}, \dots, K_{w_x}^{(P)}, \dots, K_{w_W}^{(P)}\},$$
$$W_* = \{K_{w_1,0}^{(p)}, \dots, K_{w_x,0}^{(p)}, \dots, K_{w_t,0}^{(p)}\} \notin K \text{ for } 1 \le p \le P;$$
$$L_* \subseteq L \text{ and } L_* = \{L_{u_1}^{(Q)}, \dots, L_{u_y}^{(Q)}, \dots, L_{u_U}^{(P)}\},$$
$$U_* = \{L_{u_1,0}^{(q)}, \dots, L_{u_y,0}^{(q)}, \dots, L_{u_U,0}^{(q)}\} \notin L \text{ for } 1 \le q \le Q;$$
$$H_* \subseteq H \text{ and } H_* = \{H_{v_1}^{(R)}, \dots, H_{v_z}^{(R)}, \dots, H_{v_V}^{(R)}\},$$
$$V_* = \{H_{v_1,0}^{(r)}, \dots, H_{v_z,0}^{(r)}, \dots, H_{v_V,0}^{(r)}\} \notin H \text{ for } 1 \le r \le R;$$

or

$$\alpha_{(K,K_*,P_*,\circ)}(A, W_*) \oplus_{(\vee,\wedge)} \alpha_{(L,L_*,Q_*,\circ)}(A, U_*) \oplus_{(\vee,\wedge)} \alpha_{(H,H_*,R_*,\circ)}(A, V_*),$$

where

$$K_* \subseteq K \text{ and } K_* = \{K_{w_1}^{(P)}, \dots, K_{w_x}^{(P)}, \dots, K_{w_W}^{(P)}\},$$
$$W_* = \{K_{w_1,0}^{(p_1)}, \dots, K_{w_x,0}^{(p_x)}, \dots, K_{w_W,0}^{(p_W)}\} \notin K$$

and

$$P_* = \{p_1, \dots, p_x, \dots, p_W\}, \text{ where } p_x \in \{1 \dots, P\} \text{ for } 1 \le x \le W;$$
$$L_* \subseteq L \text{ and } L_* = \{L_{u_1}^{(Q)}, \dots, L_{u_y}^{(Q)}, \dots, L_{u_U}^{(Q)}\},$$
$$U_* = \{L_{u_10}^{(q_1)}, \dots, L_{u_y,0}^{(q_y)}, \dots, L_{u_U,0}^{(q_U)}\} \notin L$$
$$Q_* = \{q_1, \dots, q_y, \dots, q_Q\}, \text{ where } q_y \in \{1 \dots, Q\} \text{ for } 1 \le y \le U;$$
$$H_* \subseteq H \text{ and } H_* = \{H_{v_1}^{(R)}, \dots, H_{v_z}^{(R)}, \dots, H_{v_V}^{(R)}\},$$
$$V_* = \{H_{v_1,0}^{(r_1)}, \dots, H_{v_y,0}^{(r_z)}, \dots, H_{v_V,0}^{(r_V)}\} \notin H$$
$$R_* = \{r_1, \dots, r_z, \dots, r_V\}, \text{ where } r_z \in \{1 \dots, P\} \text{ for } 1 \le z \le V.$$

Before using the operations "slice" and "dice" over the matrix A operation "roll-up" could be applied.

4. Conclusion

The operations described in this paper are part of an investigation aiming to present the OLAP operations in the area of the index matrices. Several researches are available submitting the solution of different problems using the apparatus of the index matrices. Nowadays the attention is focused on the tasks for presenting the OLAP concept using 3D-IMs and the use of these operations in the future development of the Intercriteria Decision Making method [6]. In the current research were presented three of the OLAP operations using index matrices. This paper is the second part of series of papers investigating the OLAP operations by index matrices. The aim is to present OLAP analysis in the area of index matrices and to improve its quality by adding the definitions of new operations. In the future the authors will complete the studies and some fields of application will be discussed.

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