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# DISTANCE MAJORIZATION INTEGRITY OF GRAPHS

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Abstract. The concept of distance majorization integrity is introduced as a new measure of the stability of a graph $G$ and it is defined as $DMI(G) = \min\{|S| + m(G - S)\}$, where $S$ is a distance majorization set and $m(G - S)$ is the order of a maximum component of $G - S$. The distance majorization integrity of some graphs is obtained. The relations between distance majorization integrity and other parameters are determined. Also a distance majorization integrity of corona of some graphs are computed.

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## 1. Introduction

The integrity of a finite graph $G$ is $I(G) = \min\{|S| + m(G - S) : S \subseteq V(G)\}$, where $m(G - S)$ denotes the order of the largest component. The integrity can be thought of as a measurement of connectivity of a graph. $|S|$ measures the amount of work needed to damage or disconnect a graph, while $m(G - S)$ is a measure of how much of the graph is still robust. The integrity is the sum of these two quantities and was first introduced by Barefoot, Entringer, and Swart [1] inspired by the idea to measure a computer networks vulnerability. Goddard and Swart [3] investigated further the bounds and properties of the integrity of the graphs. Sultan et al. [7] introduced the concept of the hub-integrity of a graph. Hub-integrity is a useful measure of vulnerability and it is defined as follows $HI(G) = \min\{|S| + m(G - S), S \text{ } is \text{ } a \text{ } hub \text{ } set \text{ } of \text{ } G\}$, where $m(G - S)$ is the order of a maximum component of $G - S$. For more details on the hub-integrity see [8, 9, 10, 11]. R. Sundareswaran and V. Swaminathan [13] introduced the concept of the distance majorization sets in graphs. A subset $D$ of $V(G)$ is said to be a distance majorization set (or dm-set) if for every vertex $u \in V - D$, there exists a vertex $v \in D$ such that $d(u, v) \geq deg(u) + deg(v)$. The minimum cardinality of a dm-set is called the distance majorization number of $G$ (or dm-number of $G$) and is denoted by $dm(G)$. Therefore instead of considering the hub-integrity of a communication graph, depending on distance and degree of vertices in a graph, we introduce a new concept. The distance majorization integrity of a graph $G$ denoted by $DMI(G)$ is defined as $DMI(G) = \min\{|S| + m(G - S), S \text{ } is \text{ } a \text{ } distance \text{ } majorization \text{ } set \text{ } of \text{ } G\}$, where $m(G - S)$ is the order of a maximum component of $G - S$. Clearly $DMI(G) \geq I(G)$ for any graph $G$.

Throughout this paper, we consider only undirected graphs with no loops. The basic definitions and concepts used in this study are adopted from [2, 4]. Given a graph $G = (V(G), E(G))$, the cardinality $|V(G)| = p$ of the vertex set $V(G)$ is the order of $G$. The distance $d(u, v)$ between two vertices $u$ and $v$ of $G$ is the length of the shortest path joining $u$ and $v$. The degree of a vertex $v$ in a graph $G$ denoted by $degv$ is the number of edges of $G$ incident with $v$. For a vertex $v$ of $G$, the eccentricity $e(v)$ is the distance between $v$ and a vertex farthest from $v$. The minimum eccentricity among the vertices of $G$ is the radius, $rad(G)$ and the maximum eccentricity is its diameter, $diam(G)$ of $G$. The maximum (minimum) degree among the vertices of $G$ is denoted by $\Delta(G)(\delta(G))$. A vertex of degree one is called a pendant vertex. The symbols $\alpha(G)$, and $\beta(G)$ denote the vertex cover number, and the independence number of $G$, respectively.

The complement $\overline{G}$ of a graph $G$ has $V(G)$ as its vertex set, two vertices are adjacent in $\overline{G}$ if and only if they are not adjacent in $G$ [4].

The line graph $L(G)$ of $G$ has the edges of $G$ as its vertices which are adjacent in $L(G)$ if and only if the corresponding edges are adjacent in $G$ [4]. $\lceil x \rceil$ denotes the smallest integer number that is greater than or equal to $x$.

In the present work, the basic properties of distance majorization integrity and of $DMI-$sets, are explored, and bounds as well as relationship between distance majorization integrity and other graphical parameters are considered. Finally, the distance majorization integrity of corona of some graphs are determined. The following results are needed to prove the main results.

**Theorem 1.1.** [13] *If $deg(u) + deg(v) > diam(G)$ for every $u, v \in V(G)$, then $dm(G) = p$.*

**Theorem 1.2.** [1] *The integrity of the path $P_p$ is $\lceil 2\sqrt{p+1} \rceil - 2$,*

**Theorem 1.3.** [13] *Let $G$ be a self complementary graph. Then $dm(G) = p$ or $p - 1$.*

**Theorem 1.4.** [13] *For any spanning tree $T$ of $G$, $dm(T) \leq dm(G)$.*

## 2. Main results

**Definition 2.1.** *The distance majorization integrity of a graph $G$ denoted by $DMI(G)$ is defined by, $DMI(G) = \min\{|S| + m(G - S)\}$, where $S$ is a distance majorization set and $m(G - S)$ is the order of a maximum component of $G - S$.*

**Definition 2.2.** *A $DMI$-set of $G$ is any subset $S$ of $V(G)$ for which $DMI(G) = |S| + m(G - S)$.*

**Proposition 2.1.**

(a) For any complete graph $K_p$, $DMI(K_p) = p$.

(b) For any path $P_p$ with $p \geq 3$,

$$DMI(P_p) = \begin{cases} 3, \text{ if } p = 3 \text{ ;} \\ 4, \text{ if } p = 4, 5; \\ 5, \text{ if } p = 6, 7, 8; \\ \lceil 2\sqrt{p+1} \rceil - 2, \text{ if } p \geq 9. \end{cases}$$

(c) For any cycle $C_p$,

$$DMI(C_p) = \begin{cases} p, \text{ if } p \leq 7 \text{ ;} \\ 5, \text{ if } p = 9; \\ 6, \text{ if } p = 8, 10, 11, 12; \\ \lceil 2\sqrt{p} \rceil - 1, \text{ if } p \geq 13. \end{cases}$$

(d) For the star $K_{1,p-1}$, $DMI(K_{1,p-1}) = 3$.
(e) For the double star $S_{n,m}$, $DMI(S_{n,m}) = 4$.
(f) For the complete bipartite graph $K_{n,m}$, $DMI(K_{n,m}) = n + m$.
(g) For the wheel graph $W_{1,p-1}$, $DMI(W_{1,p-1}) = p$.

The relation between the distance majorization integrity and the domination integrity [12] differ in showing the vulnerability of networks. This can be shown as follows: the graphs $G_1$ and $G_2$ considered such that $G_1 = K_{3,3}$ and $G_2 = P_6$, we note that $DI(G_1) = DI(G_2) = 4$, while the distance majorization integrity of $G_1$ and $G_2$ are different, $DMI(G_1) = 6$, and $DMI(G_2) = 5$. Thus, the distance majorization integrity is better parameter than the domination integrity.

**Observation 2.1.**

- $1 \leq DMI(G) \leq p$. The lower bound is sharp for $G = K_1$, and the upper bound is sharp for $G = K_p$.
- $DMI(G) = 2$ if and only if $G \cong K_2$ or $G \cong \overline{K_2}$.
- $DHI(G) = 3$ if and only if $G \cong K_{1,p-1}$, $G \cong 2K_2$, $G \cong P_3$, $G \cong K_1 \cup P_3$, $G \cong k_3$, or $G \cong pK_1 \cup K_2$.

**Remark 2.1.** *In general, the inequality $DMI(G') \leq DMI(G)$ is not true for a subgraph $G'$ of $G$, For example, for the graph $G = C_9$ and a subgraph $G' = C_7$, $DMI(C_9) = 5$, while $DMI(C_7) = 7$.*

**Lemma 2.2.** *Let $G$ be a connected graph. If $DMI(G) = 3$, then the diameter of $G$ is $\leq 2$.*

*Proof.* Consider $diam(G) \geq 3$, then $G$ contains a path $P_4$, and $|S| \geq 3$ and $m(G - S) \geq 1$. Thus $DMI(G) \geq 4$, a contradiction. This complete the proof. □

**Corollary 2.3.** *Let $T$ be a tree of order $p$. Then $DMI(G) = 3$ if and only if $T$ is a star $K_{1,p-1}$.*

*Proof.* If $T$ is a tree of order $p$ and $DMI(G) = 3$. Then by Lemma 2.2, $diam(T) \leq 2$, hence $T$ is a star $K_{1,p-1}$. □

**Lemma 2.4.** *Let $G$ be a connected graph with $\Delta(G) \leq 2$. Then $DMI(G) = |E(G)|$ if and only if $G = C_p, 3 \leq p \leq 7$.*

**Proposition 2.5.** *If $G$ be a $(p, q)$ graph with $\alpha(G) = 1$, then $DMI(G) = p$.*

*Proof.* Since $\alpha(G) = 1$, $L(G) \cong K_p$, hence $DMI(G) = p$. □

**Proposition 2.6.** *If $diam(L(G)) = 1$, then $DMI(G) = 3$.*

*Proof.* $diam(L(G)) = 1$ if and only if $G$ is either $K_3$ or $K_{1,p-1}$. By proposition 2.1, $DMI(G) = 3$. □

**Observation 2.2.** *(i) If $G$ is connected, then $DMI(G) \leq m(G)$.*

*(ii) If $G$ is disconnected, then we may have, $DMI(G) > m(G)$. For example, if $G = 3C_4$, then $m(G) = 4$, and $DMI(G) = 6$. There are examples where $G$ is disconnected and $DMI(G) \leq m(G)$, for example, $G = P_4 \cup K_{1,5}$, $m(G) = 6 > DMI(G) = 4$. If $G = P_3 \cup C_5$, $m(G) = 5 = DMI(G)$.*

**Proposition 2.7.** *If $G$ is a graph with empty set as the only $I - set$, then $DMI(G) > I(G)$.*

*Proof.* Suppose $DMI(G) = I(G)$. Then $DMI - set$ of $G$ which is not an empty set is also an $I-set$ of $G$, a contradiction. Then $DMI(G) > I(G)$. □

**Lemma 2.8.** *If $DMI(G) = I(G)$, which $G \neq K_p$, then every $DMI - set$ of $G$ must be an independent in $G$.*

*Proof.* Let $S$ be a $DMI - set$ of $G$ such that $DMI(G) = |S| + m(G - S)$, and assume that $S$ is not independent vertex set of $G$, then there exist at least two vertices $u, v \in S$ such that $u$ is adjacent to $v$ and hence the value of $DMI(G)$ increases. Then $DMI(G) > I(G)$, a contradiction. □

**Remark 2.2.** *There exists a graph $G$ with empty set as an $I - set$ and $DMI(G) = I(G)$ for example $G = C_4 \cup K_{1,3}$. There also exists a graph $G$ with empty set as an $I - set$ and $DMI(G) > I(G)$ for example $G = 2C_4$ and there exists a graph $G = C_3 \cup C_4$ with only non-empty sets as the $I - sets$ and $DMI(G) > I(G)$.*

**Observation 2.3.** *If $G \cong K_p, G \cong P_5, G \cong P_6, G \cong C_8, G \cong 2K_1, G \cong K_1 \cup K_2, G \cong K_1 \cup P_3, G \cong K_1 \cup C_3, G \cong K_1 \cup P_4$, or $K_1 \cup C_4$, then $DMI(G) = HI(G)$.*

**Proposition 2.9.** *If $G$ or $\overline{G}$ has isolates, then $p+2 \leq DMI(G)+DMI(\overline{G}) \leq 2p$.*

*Proof.* Since $G$ or $\overline{G}$ has isolates, then $I(G) < DMI(G)$ or $I(\overline{G}) < DMI(\overline{G})$. Therefore, $p + 1 \leq I(G) + I(\overline{G}) < DMI(G) + DMI(\overline{G})$, and so $p + 2 \leq DMI(G) + DMI(\overline{G})$. □

**Theorem 2.10.** *If $G$ is self-complementary, then $DMI(G) = p$.*

*Proof.* By Theorem 1.3, the result follows. □

**Theorem 2.11.** *For any spanning tree $T$ of $G$, $DMI(T) \leq DMI(G)$.*

*Proof.* Let $T$ be a spanning tree of $G$, and $S$ be a $DMI$-set of $T$ such that $|S| + m(T - S) = DMI(T)$. Since $deg_T(u) \leq deg_G(u)$ for any $u \in V(G)$, $diam(T) \geq diam(G)$ and by Theorem 1.4, this leads to result. □

**Theorem 2.12.** *If $deg(u)+deg(v) > diam(G)$ for every $u, v \in V(G)$. Then $DMI(G) = p$.*

*Proof.* By Theorem 1.1, the result is achieved. □

**Remark 2.3.** *The distance majorization integrity of a graph $G$ and distance majorization integrity of line graph $L(G)$ are not comparable. To illustrate this consider the graphs in the following cases:*

- $DMI(P_p) \geq DMI(L(P_p))$.
- $DMI(K_{1,p-1}) \leq DMI(L(K_{1,p-1}))$.
- $DMI(C_p) = DMI(L(C_p))$.

**Proposition 2.13.** *If a connected graph $G$ is isomorphic to its line graph, then $DMI(G) = DMI(L(G))$, the converse is not true, for example the graph $G$ is given below in Figure 1.*

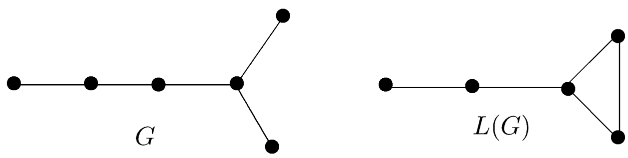


Figure 1: $G$ and $L(G)$

$DMI(G) = 5 = DMI(L(G))$, but $G$ and $L(G)$ are not isomorphic.

**Proposition 2.14.** *For any graph $G$, $DMI(G) \geq \chi(G)$, where $\chi(G)$ is the chromatic number of $G$.*

**Definition 2.3.** [5] *A firefly graph $F_{s,t,p-2s-2t-1}(s \geq 0, t \geq 0$ and $p - 2s - 2t - 1 \geq 0)$ is a graph of order $p$ that consists of $s$ triangles, $t$ pendant paths of length 2 and $n - 2s - 2t - 1$ pendant edges sharing a common vertex.*

**Theorem 2.15.**

$$DMI(F_{s,t,p-2s-2t-1}) = \begin{cases} 3, & if\ s = 0, t = 0, p - 2s - 2t - 1 > 0; \\ 4, & if\ s = 0, t = 1, p - 2s - 2t - 1 > 0; \\ 5, & if\ s = 0, t \geq 3, p - 2s - 2t - 1 > 0; \\ p, & if\ s > 0, t = 0, p - 2s - 1 > 0; \\ 5, & if\ s > 0, t > 0, p - 2s - 2t - 1 > 0; \\ p, & if\ s > 0, t = 0, p - 2s - 2t - 1 = 0. \end{cases}$$

*Proof.* Let $\zeta_p$ be the set of all firefly graphs $F_{s,t,p-2s-2t-1}(s \geq 0, t \geq 0$ *and* $p - 2s - 2t - 1 \geq 0)$ which shown in Figure 2 below. Let $u$ be the common vertex of $F_{s,t,p-2s-2t-1}$, we have the following cases:
**Case 1**: For $s = 0, t = 0$. Then $F_{0,0,p-1} \cong K_{1,p-1}$, so by Proposition 2.1, $DMI(F_{0,0,p-1}) = 3$.
**Case 2**: For $s = 0, t > 0$. Then $\zeta_p \cong F_{0,t,p-2t-1}$. If $t = 1$, consider $S = \{u, r_1, r_{11}\}$, a distance majorization set of $F_{0,1,p-3}$, then $m((F_{0,1,p-3}) - S) = 1$. This leads to $DMI(F_{0,1,p-3}) = |S| + m((F_{0,1,p-3}) - S) = 4$. If $t = 2$, consider $S = \{u, r_{11}, r_{22}\}$, a distance majorization set of $F_{0,2,p-5}$, then $m((F_{0,2,p-5}) - S) = 1$. Then $DMI(F_{0,2,p-5}) = |S| + m((F_{0,2,p-5}) - S) = 4$. If $t \geq 3$, then we consider $S = \{u, r_{11}, r_{22}\}$, a distance majorization set of $F_{0,t,p-2t-1}$, $m((F_{0,t,p-2t-1}) - S) = 2$. This implies that $DMI(F_{0,t,p-2t-1}) \leq |S| + m((F_{0,t,p-2t-1}) - S) = 5$. For show that the number $|S| + m((F_{0,t,p-2t-1}) - S)$ is minimum, we have to take into consideration the minimality of both $|S|$ and $m((F_{0,t,p-2t-1}) - S)$. Since $deg(u) = p - t - 1$, we can not remove it from

$S$, also if we remove $u$ from $S$, then $r_{11}, r_{22}$ can not $dm$- set. Hence $S$ is a minimum. If $m((F_{0,t,p-2t-1})-S_1)=1$, where $S_1$ is any distance majorization set other than $S$, then $|S_1|\geq 4$, so $|S_1|+m((F_{0,t,p-2t-1})-S_1)\geq 5$. If we consider $m((F_{0,t,p-2t-1})-S_1)\geq 2$, then trivially $|S_1|+m((F_{0,t,p-2t-1})-S_1)\geq 5$. Hence for any distance majorization set $S_1$, $|S_1|+m((F_{0,t,p-2t-1})-S_1)\geq |S|+m((F_{0,t,p-2t-1})-S)$. Then $DMI(F_{0,t,p-2t-1})=5$.

**Case 3**: For $t=0, s>0, p-2s-1>0$. Since $d(n_i, z_{1k})=2$ and $d(n_i, z_{2k})=2, 1\leq i\leq p-2s-1, 1\leq k\leq s$, then the distance majorization set of $F_{s,0,p-2s-1}$ consists of all vertices. Therefore, $DMI(F_{s,0,p-2s-1})=p$.

**Case 4**: For $s>0, t>0, p-2s-2t-1>0$. We have the following cases:

**Subcase 1**: For $s=1, t=1$. Then $\zeta_p\cong F_{1,1,p-5}$, consider $S=\{u, r_1, r_{11}\}$, a distance majorization set of $F_{1,1,p-5}$ and $m((F_{1,1,p-5})-S)=2$. Then $DMI(F_{1,1,p-5})\leq |S|+m((F_{1,1,p-5})-S)=5$.

**Subcase 2**: For $s=1, t\geq 2$. Then $\zeta_p\cong F_{1,t,p-2t-3}$, consider $S=\{u, r_{11}, r_{22}\}$, a distance majorization set of $F_{1,t,p-2t-3}$ and $m((F_{1,t,p-2t-3})-S)=2$. Then $DMI(F_{1,t,p-2t-3})\leq |S|+m((F_{1,t,p-2t-3})-S)=5$. In general, if $s\geq 2, t\geq 2, p-2s-2t-1>0$, consider $S=\{u, r_{11}, r_{22}\}$, a distance majorization set of $F_{s,t,p-2s-2t-1}$ and $m((F_{s,t,p-2s-2t-1})-S)=2$. This implies that $DMI((F_{s,t,p-2s-2t-1})-S)\leq |S|+m((F_{s,t,p-2s-2t-1})-S)=5$. Clearly, there does not exist any distance majorization set $S_1$ of $F_{s,t,p-2s-2t-1}$ such that $|S_1|+m((F_{s,t,p-2s-2t-1})-S_1)<|S|+m((F_{s,t,p-2s-2t-1})-S)$. Hence, $DMI(F_{s,t,p-2s-2t-1})=5$.

**Case 5**: For $t=0, p-2s-2t-1=0, s>0$. Then $\zeta_p\cong F_{s,0,0}$, since the distance between any two vertices of $F_{s,0,0}$ is $\leq 2$, then $dm-set\ =p$. Hence, $DMI(F_{s,0,0})=p$.

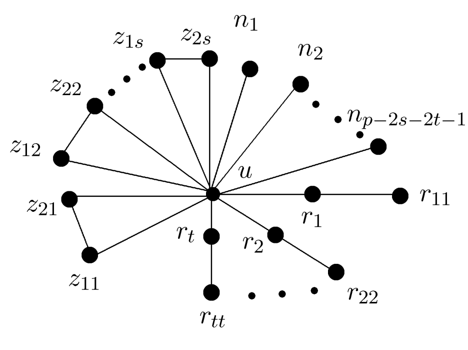


Figure 3: $F_{s,t,p-2s-2t-1}$

□

**Definition 2.4.** [6] *A broom graph $B_{p,d}$ consists of a path $P_d$, together with $(p-d)$ end vertices all adjacent to the same end vertex of $P_d$.*

**Theorem 2.16.**

$$DMI(B_{p,d})=\begin{cases} d+1, & \text{if } d=3,4\ ;\\ \lceil 2\sqrt{d-1}\rceil+1, & \text{if } d\geq 5.\end{cases}$$

*Proof.* Let $V(B_{p,d}) = \{u_1, u_2, ..., u_d, v_1, v_2, ..., v_{p-d}\}$ such that $u_1, u_2, ..., u_d$ is a path on $d$ vertices and $v_1, v_2, ..., v_{p-d}$ are end vertices that are adjacent to $u_d$. Let $S$ be a $DMI-$ set of $B_{p,d}$, the following cases are considered:
**Case 1**: $d = 3$. Consider $S = \{u_1, u_2, u_3\}$, a distance majorization set of $B_{p,d}$, so $|S| = 3$ and $m(B_{p,d} - S) = 1$, this implies that $DMI(B_{p,d}) = 4$.
**Case 2**: $d = 4$. Consider $S = \{u_1, u_3, u_4, v_1\}$, a distance majorization set of $B_{p,d}$, so $|S| = 4$ and $m(B_{p,d} - S) = 1$, this implies that $DMI(B_{p,d}) \leq 5$.
**Case 3**: $d \geq 5$. Consider $S_1 = \{u_1, v_1, u_d\}$, a distance majorization set of $B_{p,d}$, so $|S_1| = 3$ and $B_{p,d} - S_1 = P_{d-2}$, $m(B_{p,d} - S_1) = d - 2$. Let $S_2 = \{u_k : 2 \leq k \leq d-1 \text{ and } u_k \in I - set\ of\ P_{d-2}\}$. Take $V_1 = \{u_k/u_k \in I - set\ of\ P_{d-2}\}$, so $|S_2| = |V_1|$. Consider $S = S_1 \cup S_2$. Then $S$ is distance majorazation set of $B_{p,d}$, hence $|S| = |S_1| + |S_2| = |S_1| + |V_1|$ and $B_{p,d} - S = P_{d-2} - V_1$. Therefore, $m(B_{p,d} - S) = m(P_{d-2} - V_1)$. By Theorem 1.2, $|S| + m(B_{p,d} - S) = |S_1| + |V_1| + m(P_{d-2} - V_1) = |S_1| + I(P_{d-2}) = \lceil 2\sqrt{d-1} \rceil + 1$. □

**Definition 2.5.** *The corona $G_1 \circ G_2$ of two graphs $G_1$ and $G_2$ is the graph $G$ obtained by taking one copy of $G_1$ (which has $p_1$ vertices) and $p_1$ copies of $G_2$, and then joining the $i^{th}$ vertex of $G_1$ to every vertex in the $i^{th}$ copy of $G_2$.*

**Theorem 2.17.**

$$DMI(P_p \circ K_2) = \begin{cases} 6, & \text{if } p = 2\ ; \\ 8, & \text{if } p = 3, 4. \end{cases}$$

*Proof.* The graph $P_p \circ K_2$ is shown in Figure 3. We have the following cases:
**Case 1**: $p = 2$. Since $d(v_i, v_j) < deg(v_i) + deg(v_j)$, and $deg(u_1) = deg(u_2) = 3$, the distance majorization set consists of all vertices of $P_2 \circ K_2$. Then $|S| = 6$ and $m((P_2 \circ K_2) - S) = 0$, this implies that $DMI(P_p \circ K_2) = 6$.
**Case 2**: $p = 3$. The proof is similar to that of Case 1.

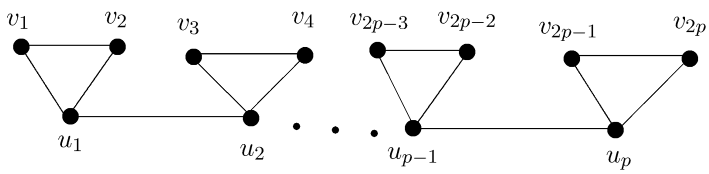


Figure 3: $P_p \circ K_2$

**Case 3**: $p = 4$. Consider $S = \{u_1, u_2, u_3, u_4, v_1, v_8\}$, a distance majorization set of $P_4 \circ K_2, |S| = 6$, and $m((P_4 \circ K_2) - S) = 2$, this implies that

$$(1) \qquad DMI(P_4 \circ K_2) \leq |S| + m((P_4 \circ K_2) - S) = 8.$$

To show that $|S| + m((P_4 \circ K_2) - S)$ is minimum, consider $S_1$, a distance majorization set of $P_4 \circ K_2$ other than $S$ with $m((P_4 \circ K_2) - S_1) = 1$, then $|S_1| \geq 2p > 6$, thus

$$(2) \qquad |S_1| + m((P_4 \circ K_2) - S_1) \geq 2p + 1 > |S| + m((P_4 \circ K_2) - S).$$

In case, if $m((P_4 \circ K_2) - S_1) \geq 2$, then it is easy to show that

$$(3) \qquad |S_1| + m((P_4 \circ K_2) - S_1) \geq |S| + m((P_4 \circ K_2) - S).$$

From (1), (2) and (3), $DMI(P_4 \circ K_2) = 8$. □

**Theorem 2.18.** $DMI(P_p \circ K_2) = p + 2, if\ p = 5, 6, 7.$

*Proof.* Consider $S = \{u_2, u_3, ..., u_{p-1}\} \cup \{v_1, v_{2p}\}$, a distance majorization set of $P_p \circ K_2, |S| = p$, and $m((P_p \circ K_2) - S) = 2$, this implies that $|S| + m((P_p \circ K_2) - S) = p + 2$. Consider $m((P_p \circ K_2) - S) = 1$, then $|S| \geq 2p$. Thus $|S| + m((P_p \circ K_2) - S) \geq 2p + 1 > p + 2$. Hence $DMI(P_p \circ K_2) = p + 2$. □

**Theorem 2.19.** $DMI(P_8 \circ K_2) = 9.$

*Proof.* Consider $S = \{v_1, u_2, u_4, u_5, u_7, v_{16}\}$, a distance majorization set of $P_8 \circ K_2$ such that $|S| = 6, m((P_8 \circ K_2) - S) = 3$. This implies that $DMI(P_8 \circ K_2) \leq |S| + m((P_8 \circ K_2) - S = 9$. If $m((P_8 \circ K_2) - S) = 2$, then $|S| \geq p$, thus $|S| + m((P_8 \circ K_2) - S) \geq p + 1$. Consider $m((P_8 \circ K_2) - S) = 1$, then $|S| \geq 2p$, so $|S| + m((P_8 \circ K_2) - S) \geq 2p + 1$. Finally consider $m((P_8 \circ K_2) - S) \geq 3$, then it is easy to see that $|S| + m((P_8 \circ K_2) - S) \geq p + 1$. This completes the proof. □

**Theorem 2.20.**

$$DMI(P_p \circ K_2) = \begin{cases} 10, & if\, p = 9, 10, 11, 12\ ; \\ 11, & if\, p = 14. \end{cases}$$

*Proof.* The following cases are discussed:
**Case 1**: $p = 9, 10$. Consider $S = \{u_i, 1 \leq i \leq 9\ and\ i\ is\ odd\} \cup \{v_1, v_{2p}\}$, a distance majorization set of $P_p \circ K_2, |S| = 7$, and $m((P_p \circ K_2) - S) = 3$, this implies that $DMI(P_p \circ K_2) \leq |S| + m((P_p \circ K_2) - S) = p + 1$. If $m((P_p \circ K_2) - S) = 2$, then $|S| \geq p$. Thus $|S| + m((P_p \circ K_2) - S) \geq 2p + 1 > p + 1$. Consider $m((P_p \circ K_2) - S) = 1$, then $|S| \geq 2p$, thus $|S| + m((P_p \circ K_2) - S) \geq 2p + 1$. Hence $DMI(P_p \circ K_2) = p + 1$.
**Case 2**: $p = 11$. Consider $S = \{v_1, u_1, u_3, u_5, u_7, u_9, u_{11}\}$, a distance majorization set of $P_{11} \circ K_2$ such that $|S| = 7, m((P_{11} \circ K_2) - S) = 3$. Therefore $DMI(P_{11} \circ K_2) \leq |S| + m((P_{11} \circ K_2) - S) = 10$. The remain Proof is similar to Case 1.
**Case 3**: $p = 12, 14$. Consider $S = \{u_i, 1 \leq i \leq p - 1\ and\ i\ is\ odd\} \cup \{v_{2p}\}$, a distance majorization set of $P_p \circ K_2$ such that $|S| = 7, 8$, respectively, $m((P_p \circ K_2) - S) = 3$. This implies that $DMI(P_p \circ K_2) \leq |S| + m((P_p \circ K_2) - S) = 10, 11$. The remain proof is similar to Case 1. □

**Theorem 2.21.**

$$DMI(P_p \circ K_2) = \begin{cases} \frac{p}{2} + 3, & if\, p\ is\ even\ and\ p \geq 16\ ; \\ \lceil \frac{p}{2} \rceil + 3, & if\, p\ is\ odd\ and\ p \geq 13. \end{cases}$$

*Proof.* Two cases are considered:
**Case 1**: $p\ is\ even, p \geq 16$. Consider $S = \{u_1, u_3, u_5, u_7, ..., u_{p-1}\}$, a distance majorization set of $P_p \circ K_2$ such that $|S| = \frac{p}{2}, m((P_p \circ K_2) - S) = 3$. This implies that

$$DMI(P_p \circ K_2) \leq |S| + m((P_p \circ K_2) - S) = \frac{p}{2} + 3. \tag{4}$$

To show that $|S| + m(P_p \circ K_2) - S)$ is minimum, consider $m((P_p \circ K_2) - S) = 2$, then $|S| \geq p$, so

$$|S| + m((P_p \circ K_2) - S) \geq p + 2. \tag{5}$$

In case, $m((P_p \circ K_2) - S) = 1$, $|S| \geq 2p$, then

$$|S| + m((P_p \circ K_2) - S) \geq 2p + 1. \tag{6}$$

Finally, consider $m((P_p \circ K_2) - S) \geq 3$, then

$$|S| + m((P_p \circ K_2) - S) \geq \frac{p}{2} + 3. \tag{7}$$

From (4), (5), (6) and (7), the result follows.
**Case 2**: *p is odd*, $p \geq 13$. Consider $S = \{u_1, u_3, u_5, u_7, ..., u_p\}$, a distance majorization set of $P_p \circ K_2$ such that $|S| = \lceil \frac{p}{2} \rceil, m((P_p \circ K_2) - S) = 3$. This implies that

$$DMI(P_p \circ K_2) \leq |S| + m((P_p \circ K_2) - S) = \lceil \frac{p}{2} \rceil + 3. \tag{8}$$

We will show that the number $|S| + m((P_p \circ K_2) - S)$ is minimum, for that consider $m((P_p \circ K_2) - S) = 2$. Then $|S| \geq p$, so

$$|S| + m((P_p \circ K_2) - S) \geq p + 2. \tag{9}$$

If $m((P_p \circ K_2) - S) = 1$, $|S| \geq 2p$. Then

$$|S| + m((P_p \circ K_2) - S) \geq 2p + 1. \tag{10}$$

Finally, consider $m((P_p \circ K_2) - S) \geq 3$, then

$$|S| + m((P_p \circ K_2) - S) \geq \frac{p}{2} + 3. \tag{11}$$

From (8), (9), (10) and (11), $DMI(P_p \circ K_2) = \lceil \frac{p}{2} \rceil + 3$. □

**Theorem 2.22.** $DMI(K_2 \circ P_p) = 2p + 2$.

*Proof.* Since distance between any two vertices of $K_2 \circ P_p$ is $\leq 3$, and $\delta(K_2 \circ P_p) = 2$, the distance between any two vertices is always less than summation of degree of two vertices. Then $dm(K_2 \circ P_p) = 2p+2$. Therefore, $DMI(K_2 \circ P_p) = 2p + 2$. □

**Theorem 2.23.**

$$DMI(C_p \circ P_p) = \begin{cases} 12, & \textit{if } p = 3\ ; \\ 17, & \textit{if } p = 4; \\ 22, & \textit{if } p = 5. \end{cases}$$

*Proof.* Let $S$ be a $DMI$-set of $C_p \circ P_p$ such that $DMI(C_p \circ P_p) = |S| + m((C_p \circ P_p) - S)$, and $|V(C_p \circ P_p)| = p^2 + p$, the graph $C_p \circ P_p$ is shown in Figure 4. The following cases are discussed;

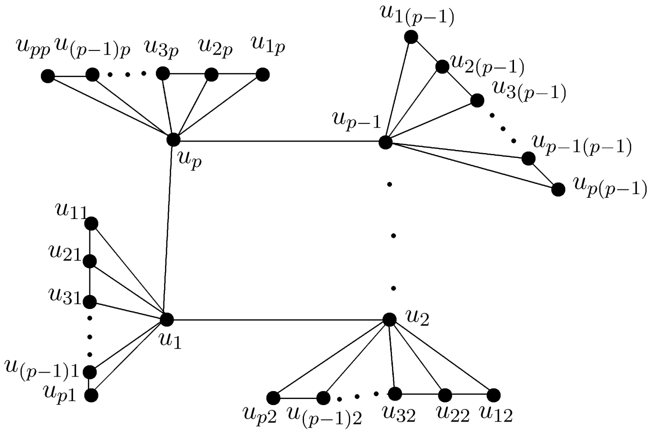

Figure 4: $C_p \circ P_p$

**Case 1**: $p = 3$. Consider $S$ consisting of all vertices of $C_3 \circ P_3$ such that $m((C_3 \circ P_3) - S) = 0$. This implies that $DMI(C_3 \circ P_3) = 12$. The set $S$ is minimum, since if any vertex of $C_3 \circ P_3$ is removed, then the distance between it and another vertex is less than summation of their degree. Hence, $DMI(C_3 \circ P_3) = 12$.
**Case 2**: $p = 4$. Consider $S = \{u_1, u_{11}, u_{21}, u_{31}, u_2, u_{12}, u_{22}, u_{32}, u_3, u_{13}, u_{23}, u_{33}, u_4, u_{14}, u_{24}, u_{34}\}$, a distance majorization set of $C_4 \circ P_4$, $|S| = 16$, and $m((C_4 \circ P_4) - S) = 1$. Then, $DMI(C_4 \circ P_4) \leq |S| + m((C_4 \circ P_4) - S) = 17$. The number $|S| + m((C_4 \circ P_4) - S)$ is minimum, because if $S_1$ is a distance majorization set of $C_4 \circ P_4$ other than $S$ and $m((C_4 \circ P_4) - S_1) = 0$, then $|S_1| = 5p$ which implies that $|S_1| + m((C_4 \circ P_4) - S_1) = 20$. Therefore, $DMI(C_4 \circ P_4) = 17$.
**Case 3**: $p = 5$.
Consider $S = \{u_1, u_{11}, u_{21}, u_{31}, u_{41}, u_2, u_{22}, u_{32}, u_{42}, u_3, u_{23}, u_{33}, u_{43}, u_4, u_{24}, u_{34}, u_{44}, u_5, u_{25}, u_{35}, u_{45}\}$, a distance majorization set of $C_5 \circ P_5$, $|S| = 21$, and $m((C_5 \circ P_5) - S) = 1$. Therefore, $DMI(C_5 \circ P_5) = 22$. The number $|S| + m((C_5 \circ P_5) - S)$ is minimum, since if $S_1$ is a distance majorization set of $C_5 \circ P_5$ other than $S$ and $m((C_5 \circ P_5) - S_1) = 0$, then $|S_1| = 6p$ which implies that $|S_1| + m((C_5 \circ P_5) - S_1) = 30$. Therefore, $DMI(C_5 \circ P_5) = 22$. □

**Theorem 2.24.**

$$DMI(C_p \circ P_p) = \begin{cases} 19, & \textit{if } p = 6\ ; \\ 20, & \textit{if } p = 7. \end{cases}$$

*Proof.* We have two cases:
**Case 1**: $p = 6$. Consider $S = \{u_1, u_{11}, u_2, u_{12}, u_3, u_{13}, u_{63}, u_4, u_{14}, u_{34}, u_5, u_{15}, u_6, u_{16}\}$, a distance majorization set of $C_6 \circ P_6$, $|S| = 14$, and $m((C_6 \circ P_6) - S) = 5$. Then,

$$DMI(C_6 \circ P_6) \leq |S| + m((C_6 \circ P_6) - S) = 19. \tag{12}$$

We will show that the number $|S| + m((C_6 \circ P_6) - S)$ is minimum. For that we have to take into account the minimality of both $|S|$ and $m((C_6 \circ P_6) - S)$. If vertices $u_i$ or $u_{1i}, 1 \leq i \leq 6$ are removed from set $S$, then $S$ is not a distance majorization set. So $S$ is minimum set. It remains to show that if $S_1$ is any distance majorization set other than $S$ and $m((C_6 \circ P_6) - S_1) = 4$, then $|S_1| \geq 18$, hence $|S_1| + m((C_6 \circ P_6) - S_1) = 22 > 19$. Consider $m((C_6 \circ P_6) - S_1) = 3$, then $|S_1| \geq 20$, consequently, $|S_1| + m((C_6 \circ P_6) - S_1) \geq 23$, also if $m((C_6 \circ P_6) - S_1) \leq 2$. Then $|S_1| + m((C_6 \circ P_6) - S_1) > 19$. Hence for any distance majorzation set $S_1$,

$$|S_1| + m((C_6 \circ P_6) - S_1) \geq 19. \tag{13}$$

From (12) and (13), $DMI(C_6 \circ P_6) = 19$.
**Case 2**: $p = 7$. Consider $S = \{u_i, u_{1i}, 1 \leq i \leq 7\}$, a distance majorization set of $C_7 \circ P_7$, $|S| = 14$, and $m((C_7 \circ P_7) - S) = 6$. This implies that $DMI(C_7 \circ P_7) \leq |S| + m((C_7 \circ P_7) - S) = 20$. The proof of minimality of $|S| + m((C_7 \circ P_7) - S)$ is similar to that of Case 1. □

**Theorem 2.25.** $DMI(C_p \circ P_p) = 2p, p \geq 8$.

*Proof.* Consider $S = \{u_1, u_2, ..., u_{p-1}, u_p\}$, a distance majorization set of $C_p \circ P_p$, $|S| = p$, and $m((C_p \circ P_p) - S) = p$. Therefore,

$$DMI(C_p \circ P_p) \leq |S| + m((C_p \circ P_p) - S) = 2p. \tag{14}$$

For showing that the number $|S| + m((C_p \circ P_p) - S)$ is minimum, the minimality of both $|S|$ and $m((C_p \circ P_p) - S)$ is taken into consideration. We claim that set $S$ is a minimum distance majorization set. Since $deg(u_i) = p+2, 1 \leq i \leq p$, if $u_1$ is removed from set $S$, then there does not exist a vertex $u_i, 2 \leq i \leq p$ such that $d(u_1, u_i) \geq deg(u) + deg(u_i)$. Hence the number $|S|$ is minimum. It remains to show that if $S_1$ is any distance majorization set other than $S$, $|S_1| + m((C_p \circ P_p) - S_1) \geq 2p$. Consider $m((C_p \circ P_p) - S_1) \leq p-1$. Then $|S_1| \geq 2p$, hence $|S_1| + m((C_p \circ P_p) - S_1) \geq 3p - 1$. Hence for any distance majorization set $S_1$,

$$|S_1| + m((C_p \circ P_p) - S_1) \geq 2p + 4. \tag{15}$$

From (14) and (15), $DMI(C_p \circ P_p) = 2p$. □

## 3. Conclusion

In this research work, we introduced the concept of distance majorization integrity of graphs, we have obtained the distance majorization integrity of some graphs. Relations between distance majorization integrity and some parameters are established. The following are some open problems for further investigation:
(1) Characterize the graphs $G$ for which $DMI(G) = p$.
(2) Characterize the graphs $G$ for which $DMI(G) = HI(G) = I(G)$.

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