Proceedings of the Jangjeon Mathematical Society
20 (2017), No. 2. pp. 313 – 317

www.jangjeon.or.kr
http://dx.doi.org/10.23001/pjms2017.20.2.313

CORRECTIONS TO "EXTENSIONS OF CHARACTERS OF THE NORMAL SUBGROUP OF A SEMIDIRECT PRODUCT GROUP TO ONE-DIMENSIONAL PSEUDOREPRESENTATIONS OF THE GROUP"

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Abstract. We correct the formulation of the main result of the paper "Extensions of Characters of the Normal Subgroup of a Semi-Direct Product Group to One-Dimensional Pseudorepresentations of the Group" (Proc. Jangjeon Math. Soc. **19** (2016), no. 3, 451–455), present the correct formulation of this result (which is in fact proved by the corresponding consideration in the cited paper), and state and prove the corresponding result for connected locally compact groups.

§ 1. Introduction

The following problem was treated in [1]. Let G be a group, let N be a normal subgroup of G, and let f be an (ordinary) unitary character of N. The problem, as posed in [1], was to describe the one-dimensional pseudorepresentations F of G that are extensions of a given ordinary (unitary) character f on a normal subgroup N of a given group G. However, this question was posed in an incorrect way, because there are only few groups for which the structure of one-dimensional pseudorepresentations can be described at the

2010 *Mathematics Subject Classification.* Primary 22A99, Secondary 22A25.

Key words and phrases. One-dimensional quasirepresentation, one-dimensional pseudorepresentation, character, normal subgroup.

Partially supported by the Russian Foundation for Basic Research under grant no. 14-01-00007.

level needed to study the above problem in the statement posed in [1]. In fact, the consideration in [1] proves the following assertion, in which the corrections in the statement are indicated by using the roman font.

Theorem 1. *Let G be a group, let N be a normal subgroup of G, let R be a subgroup of G such that $G = RN$ (we thus do not assume that $R \cap N = \{e_G\}$ for e_G the identity element of G), and let f be an (ordinary) unitary character of N* which is invariant with respect to the inner automorphisms of G. Suppose that there is a one-dimensional unitary pseudorepresentation Φ of R with a sufficiently small defect which ensures that Φ is a pure pseudorepresentation (for the conditions ensuring that Φ is pure, see [2]) and with

$$f(g) = \Phi(g) \quad \text{for every} \quad g \in R \cap N; \tag{1}$$

such a pseudorepresentation certainly exists (as an ordinary representation), for example, if $G \cap N = \{e\}$ or, more generally, if f is identically equal to 1 on N; then one can take the identity representation of R for Φ. In this case, *the formula*

$$F(rn) = \Phi(r)f(n), \qquad r \in R, \quad n \in N, \tag{2}$$

defines a pure one-dimensional unitary pseudorepresentation of G whose defect does not exceed that of Φ and whose restriction to N is f.

Moreover, in the case under consideration, *every one-dimensional unitary pseudorepresentation of G with small defect (in the sense of [2]) can be obtained in this way, and the correspondence (2) between the pseudorepresentations Φ (satisfying (1)) and the one-dimensional pseudorepresentations F of G that are extensions of the character f is one-to-one.*

The proof of this assertion in [1] is correct. The error in the formulation was related to the very fact that there is no general description of the one-dimensional pseudorepresentations of a given group in general. The class of groups whose one-dimensional pseudorepresentations are known is rather narrow; fortunately, it contains semisimple locally compact groups, which helps us to prove a result of desired kind for connected locally compact groups by changing the direction of investigation, namely, instead of extension problem for characters on normal subgroups, we consider the related extension problem for one-dimensional pseudorepresentations of the semisimple part of the group.

§ 2. Main theorem

In our explicit statement, we restrict ourselves to the consideration of connected Lie groups, because the passage to the general case of connected locally compact groups is purely technical.

Theorem 2. *Let G be a connected Lie group, let R be the radical of G, let S be a Levi subgroup of G (thus, $G = RN$), and let Φ be a one-dimensional unitary pseudorepresentation of S. Then there is an (ordinary) unitary character f of R which is invariant under inner automorphisms and extends the restriction of Φ to $S \cap R$, and the formula*

$$F(rn) = \Phi(r)f(n), \qquad r \in R, \quad n \in N,$$

defines a pure one-dimensional unitary pseudorepresentation of G whose defect does not exceed that of Φ and its restriction to N is f.

Moreover, every one-dimensional unitary pseudorepresentation of G with small defect (in the sense of [2]) can be obtained in this way, and the related correspondence between the unitary characters f extending the restriction of Φ to $S \cap R$ and the one-dimensional pseudorepresentations F of G that are extensions of Φ is one-to-one.

Note that every one-dimensional pseudorepresentation of S, which is a semisimple group, is automatically continuous on S and can be represented as an exponential of the Guichardet–Wigner pseudocharacter on S [2–4].

Proof. Let us prove first that the restriction Ψ of Φ to $S \cap R$ admits an extension to a character f of the normal subgroup R that is invariant with respect to the inner automorphisms of G. The one-dimensional pseudorepresentation is constant on the conjugacy classes [4]; hence so is Ψ as a restriction of the one-dimensional pseudorepresentation of S; moreover, Ψ is an ordinary representation because R (and hence $S \cap R$ as well) is amenable and Φ is pure. Therefore, Ψ takes the value 1 on the elements of the commutator subgroup $[R, S]$. Since $G = RS$, where R is a normal subgroup of G, it follows that $[R, S] = [R, G]$, and hence the group $R/[R, G]$ is Abelian. The mapping Ψ can be regarded as a character of a subgroup of $R/[R, G]$. Since a character of subgroup of an Abelian group can be always extended to a character of the group itself ([5], 24.12), the desired extension of Ψ to R exists and is automatically a character of R invariant with respect to the inner automorphisms.

The rest of the proof is aa slight variation of the calculations and considerations presented in [1], pp. 453–454, and the necessary modifications can be left to the reader. This completes the proof of Theorem 2.

§ 3. Discussion

Another (a bit more rich) class of groups for which one can list the one-dimensional pseudorepresentations is the class of almost connected groups whose identity component is a semisimple locally compact group. It seems that the situation is similar to that treated here, and therefore one can hope that the structure of one-dimensional pseudorepresentations can also be described completely for almost connected locally compact groups.

Acknowledgments

I thank Professor Taekyun Kim for the invitation to publish this paper in the Proceedings of the Jangjeon Mathematical Society.

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