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STRONGLY REGULAR GRAPHS OVER WEAK METRIC SCHEMES AND CHEEGER CONSTANTS OF HAMMING GRAPHS

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Abstract. Weak metric schemes are a generalized notion of metric schemes. We are interested in the characterization of the strongly regular graphs constructed from weak metric schemes with 2 classes. In this paper we show that the connected strongly regular graphs constructed from the weak metric schemes with 2 classes are complete multipartite graphs, and in fact that the weak metric schemes with 2 classes are equivalent to the complete multipartite graphs. By using our construction, we find the entire list of complete multipartite graphs up to 100 vertices. Additionally, we determine the Cheeger constants of Hamming graphs completely.

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1. Introduction

Weak metric schemes [9] are introduced as a generalization of metric schemes [3, 5, 7, 8]. They are defined as the wreath product of a finite number of symmetric association schemes satisfying certain conditions. It turns out that the weak metric schemes are the symmetric association schemes. It is known that the symmetric association schemes with 2 classes are equivalent to the strongly regular graphs. In this paper, we study the relation graphs constructed from weak metric schemes with 2 classes. There are two relation graphs constructed from weak metric schemes with 2 classes, and one of them is disconnected and the other is connected.

The strongly regular graph (abbreviated SRG) [1, 2, 8, 10, 12] is one of the most important family of regular graphs with many good properties. One of those well-known properties is the fact that the eigenvalues k, r, s of all SRGs with vertices v depend only on the parameters (v, k, λ, μ) of these graphs as follows [1, 12]:

$$r + s = \lambda - \mu, \; rs = \mu - k,$$

where λ (resp. μ) is the number of common neighbors of any two adjacent (resp. nonadjacent) vertices. In particular, if $k = \mu$ then the SRGs are called the *complete multipartite graphs* (abbreviated CMG). The CMGs have the eigenvalues $k, 0, -m$, where m is the size of the classes of the CMGs.

In this paper, we find a complete characterization of all the SRGs constructed from weak metric schemes with 2 classes. We find that the connected regular graphs from the weak metric schemes with 2 classes are the

CMGs. We show that any weak metric scheme with 2 classes gives rise to a CMG (Theorem 12) and conversely, any CMG can be constructed from a weak metric scheme with 2 classes (Theorem 20). Therefore, we conclude that the weak metric schemes with 2 classes are equivalent to the CMGs. By using our construction, we find the entire list of complete multipartite graphs up to 100 vertices.

We explain our construction very briefly as follows. We consider the weak metric schemes $\mathfrak{X} = \mathfrak{X}^{(1)} \wr \mathfrak{X}^{(2)} = (X_1 \times X_2, \{R_j\}_{j=0}^{2})$ with 2 classes as the wreath product of two metric schemes $\mathfrak{X}^{(i)}$ $(i = 1, 2)$ with 1 class. Then we show the parameters (v, k, λ, μ) of the CMGs constructed from the weak metric schemes $\mathfrak{X}$ are given as follows (Theorem 12) :

$$(v, k, \lambda, \mu) = (m_1 m_2, m_1 k^{(2)}, m_1 k^{(2)} - k^{(1)} - 1, m_1 k^{(2)}),$$

where for $i = 1, 2$, $|X_i| = m_i$ and $k^{(i)}$ is a valency of the $\mathfrak{X}^{(i)}$.

In graph theory, the Cheeger constant has an important geometric meaning. Cheeger constants are closely connected to the problem of separating the graph into two large components by making a small edge-cut. The Cheeger constant of a connected graph is strictly positive. If the Cheeger constant is "small" but positive, then there are two large sets of vertices with "few" edges between them. On the other hand, if the Cheeger constant is "large", then there are two sets of vertices with "many" edges between those two subsets. Therefore, we are interested in finding bounds of Cheeger constants of graphs simply called the Cheeger bound. The Cheeger bound h_Γ of a graph Γ with respect to λ_1 is known as follows [6, 11] :

$$h_\Gamma \leq \sqrt{2\lambda_1}, \ h_\Gamma \leq \sqrt{\lambda_1(2 - \lambda_1)}.$$

The Hamming graph is a special class of graphs used in several branches of Mathematics and Computer Science. The Hamming graphs are interested in connection with error-correcting codes and association schemes. The Hamming graph $\Gamma(n, q)$ is the graph that describes the distance-1 relation R_1 in the Hamming scheme $H(n, q)$ [1, 3, 7]. Let S be a set of q elements and n a positive integer. The Hamming graph $\Gamma(n, q)$ has q^n vertices and the set of ordered n-tuples of elements of S. Two vertices are adjacent if they differ in precisely one coordinate. The Hamming graph $\Gamma(n, q)$ is, equivalently, the Cartesian product of n complete graphs K_q. The complete graph K_q, the lattice graph and hypercube graph are all Hamming graphs.

This paper is organized as follows. In Section 2, we introduce the definitions of the weak metric scheme and some basic facts about the weak metric scheme with 2 classes. In Section 3, we show how the weak metric schemes with 2 classes give rise to the CMGs. Section 4 shows that any CMG can be constructed from a weak metric scheme with 2 classes. In Section 5, we obtain the Cheeger constants of Hamming graphs. In Section 6, we show the entire list of the CMGs for $4 \leq v \leq 100$ in Table 1.

2. Preliminaries and the weak metric schemes with 2 classes

In this section, we introduce the definitions of a metric scheme and a weak metric scheme.

Definition 1. [3, 7, 8] *Let X be a nonempty finite set and $R = \{R_0, R_1, \cdots, R_n\}$ be a family of relations defined on X. We say that the pair (X, R) is a symmetric association scheme with n classes if it satisfies the following conditions.*

(1) $R_0 = \{(x, x) \mid x \in X\}$

(2) *For every* $x, y \in X,\ (x, y) \in R_j$ *for exactly one* j.

(3) *For any triple of intgers* $i, j, k \in \{0, 1, \cdots, n\}$, *the number of* $z \in X$ *such that* $(x, z) \in R_i$ *and* $(z, y) \in R_j$ *is constant* p^k_{ij} *whenever* $(x, y) \in R_k$.

(4) $(x, y) \in R_j \Leftrightarrow (y, x) \in R_j$.

(5)$p^k_{ij} = p^k_{ji}$.

Definition 2. [3, 7, 8] *A symmetric association scheme $\mathfrak{X} = (X, \{R_j\}^n_{j=0})$ is called a metric scheme(P-polynomial scheme) with respect to the ordering $R_0, R_1, \cdots, R_n$, if there exists some complex coefficient polynomial $\psi_j(x)$ of degree j $(0 \le j \le n)$ such that $D_j = \psi_j(D_1)$, where D_j is the adjacency matrix with respect to R_j.*

Example 3. The Hamming scheme $H(n, q)$ and the Johnson scheme $J(v, d)$ are the metric schemes.

Definition 4. [4, 9] *Let $n_1, \cdots, n_t$ be positive integers with $n = n_1 + \cdots + n_t$. For each $i = 1, 2, \cdots, t$, let $\mathfrak{X}^{(i)} = (X_i, \{R^{(i)}_j\}^{n_i}_{j=0})$ be a metric scheme. Here we will always assume that the relations of $\mathfrak{X}^{(i)}$ are ordered as $R^{(i)}_0, R^{(i)}_1, \cdots, R^{(i)}_{n_i}$. Then $\mathfrak{X} = \mathfrak{X}^{(1)} \wr \cdots \wr \mathfrak{X}^{(t)} = (X = X_1 \times \cdots \times X_t, \{R_j\}^n_{j=0})$ is the wreath product of $\mathfrak{X}^{(1)}, \mathfrak{X}^{(2)}, \cdots, \mathfrak{X}^{(t)}$, so that, for $(x_1, \cdots, x_t),\ (y_1, \cdots, y_t) \in X$,*

$$(x, y) \in R_{\underline{n_{i-1}}+i_0} \Leftrightarrow x_{i+1} = y_{i+1}, \cdots, x_t = y_t \text{ and } (x_i, y_i) \in R^{(i)}_{i_0},$$

for $i = 1, \cdots, t,\ 1 \le i_0 \le n_i$, or $i = 1, i_0 = 0$, where $\underline{n_{i-1}} = n_1 + \cdots + n_{i-1}$. Here we will always assume that the relations of $\mathfrak{X}$ are ordered as $R_0, R_1, \cdots, R_n$. For each $i = 1, \cdots, t$, let $\Gamma_i = (X_i, R^{(i)}_1)$ be the graph with the distance function ∂_i. Then, for $x = (x_1, \cdots, x_t),\ y = (y_1, \cdots, y_t) \in X$, we define

$$d(x, y) = \begin{cases} 0, & \text{if } x = y, \\ \underline{n_{i-1}} + \partial_i(x_i, y_i), & \text{if } x_i \ne y_i, \text{ and,} \\ & \quad x_{i+1} = y_{i+1}, \cdots, x_t = y_t. \end{cases}$$

Then d is a distance function and we obtain a symmetric association scheme $\mathfrak{X} = \mathfrak{X}^{(1)} \wr \cdots \wr \mathfrak{X}^{(t)} = (X = X_1 \times \cdots \times X_t, \{R_j\}^n_{j=0})$.

Definition 5. *Let $\mathfrak{X} = \mathfrak{X}^{(1)} \wr \cdots \wr \mathfrak{X}^{(t)} = (X = X_1 \times \cdots \times X_t, \{R_j\}^n_{j=0})$ be the symmetric association scheme which is given as the wreath product of the metric schemes $\mathfrak{X}^{(i)} = (X_i, \{R^{(i)}_j\}^{n_i}_{j=0})$. Then $\mathfrak{X}$ will be called a weak metric scheme. If $\mathfrak{X}^{(i)}$ is a Hamming scheme for every i, then $\mathfrak{X}$ is called the weak Hamming scheme* [9].

Example 6. Let $H(4,2)$, $H(3,2)$ and $H(5,2)$ be the Hamming schemes, then we obtain a weak Hamming scheme $\mathfrak{X} = H(4,3,5;2) = H(4,2) \wr H(3,2) \wr H(5,2)$ with $|X| = 4096$.

As shown above, the weak metric schemes are defined as the wreath product of a finite number of symmetric association schemes satisfying certain conditions. It turns out that the weak metric schemes are the symmetric association schemes. Since the symmetric association schemes with 2 classes are equivalent to the SRGs, we consider the weak metric schemes with 2 classes for the construction of the SRGs.

Let $\mathfrak{X} = \mathfrak{X}^{(1)} \wr \mathfrak{X}^{(2)} = (X = X_1 \times X_2, \{R_j\}_{j=0}^{2})$ be the weak metric scheme with 2 classes which is given as the wreath product of the metric schemes $\mathfrak{X}^{(i)} = (X_i, \{R_j^{(i)}\}_{j=0}^{1})$. And let $\Gamma_j^{(i)}$ $(j = 0,1,2)$ (resp. Γ_j) be a graph with respect to $R_j^{(i)}$ (resp. R_j) then $\Gamma_j^{(i)}$ (resp. Γ_j) is called a j-relation graph of $\mathfrak{X}^{(i)}$ (resp. $\mathfrak{X}$).

Proposition 7. *Let $\mathfrak{X} = \mathfrak{X}^{(1)} \wr \mathfrak{X}^{(2)} = (X = X_1 \times X_2, \{R_j\}_{j=0}^{2})$ be a weak metric scheme with 2 classes, and let $|X_1| = m_1$, $|X_2| = m_2$. Then we have*

(1) *The metric schemes $\mathfrak{X}^{(1)}$ and $\mathfrak{X}^{(2)}$ have 1 class.*

(2) *$\Gamma_1^{(1)} = (X_1, R_1^{(1)})$ and $\Gamma_1^{(2)} = (X_2, R_1^{(2)})$ are distance regular graphs.*

(3) *Intersection matrices L_0, L_1, L_2 of $\mathfrak{X}$ are given as follows :*

$L_0 = I_3$,

$$L_1 = \begin{pmatrix} 0 & k^{(1)} & 0 \\ 1 & k^{(1)}-1 & 0 \\ 0 & 0 & k^{(1)} \end{pmatrix},\ L_2 = \begin{pmatrix} 0 & 0 & m_1k^{(2)} \\ 0 & 0 & m_1k^{(2)} \\ 1 & k_1^{(1)} & m_1k^{(2)} - k^{(1)} - 1 \end{pmatrix},$$

where $k^{(i)}$ $(i = 1,2)$ is a valency of $R_1^{(i)}$.

(4) *$k^{(1)} = m_1 - 1$ and $k^{(2)} = m_2 - 1$.*

Example 8. Let $\mathfrak{X} = (X_1 \times X_2, \{R_j\}_{j=0}^{2})$ be a weak metric scheme as the wreath product of two Hamming schemes $H(1,2)$ and $H(1,3)$. Then, $|X_1| = 2$, $|X_2| = 3$ and $k^{(1)} = 1$,$k^{(2)} = 2$. Also, we have

$$L_0 = I_3,\ L_1 = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix},\ L_2 = \begin{pmatrix} 0 & 0 & 4 \\ 0 & 0 & 4 \\ 1 & 1 & 2 \end{pmatrix}.$$

Now, we introduce some definitions in graph theory.

Definition 9. *Let S be a subset of set of vertices in a graph $\Gamma = (V, E)$, where V is a set of vertices of Γ and E is a set of edges of Γ.*
(a) The edge boundary of S, denoted by ∂S is defined as follows :

$$\partial S = \{\{x,y\} \in E(\Gamma) \mid x \in S \text{ and } y \in V - S\},$$

where $E(\Gamma)$ is a edge set of Γ.
(b) If $S \neq \varnothing$, then the volume of S, denoted by $\mathrm{vol}(S)$ is defined as follows :

$$\mathrm{vol}(S) = \sum_{u \in S} k_u,$$

where k_u is a valency of u in Γ. The volume of Γ is denoted by

$$\mathrm{vol}(\Gamma) = \sum_u d_u.$$

(c) The Cheeger ratio of S, denoted by h_S is defined as

$$h_S = \frac{|\partial S|}{\min\{\mathrm{vol}(S),\ \mathrm{vol}(\Gamma) - \mathrm{vol}(S)\}}.$$

(d) The Cheeger constant of Γ, denoted by h_Γ is defined as

$$h_\Gamma = \begin{matrix} min \\ S \subseteq V \end{matrix}\ h_S$$

Let Γ be a distance k-regular graph, then we have $\mathcal{L} = I - \frac{1}{k}A$, where A is the adjacency matrix of Γ. The eigenvalues of the Laplacian $\mathcal{L}$ of Γ are denoted by $0 = \lambda_0 < \lambda_1 < \cdots < \lambda_d$.

The Hamming graph $\Gamma(n, q)$ is the graph that describes the distance-1 relation R_1 in the Hamming scheme $H(n, q)$. The Hamming graph $\Gamma(n, q)$ is the graph Cartesian product of n copies of the complete graph K_q . Therefore, $\Gamma(n, q)$ has vertices q^n. Also, $\Gamma(n, q)$ is distance $n(q-1)$-regular graph with diameter n.

Let $E(A \sim B)$ be the numbers of the edges from A to B.

Lemma 10. *Let $\Gamma(n, q)$ be a graph with respect to R_1 over $H(n, q)$. Then $\Gamma(n)$ can be partitioned into q isomorphic copies S_i of $\Gamma(n-1)$ with q^{n-1} edges joining every pair of $\Gamma(n-1)$'s. And, for $s \in S_i$, $E(s \sim S_j) = 1$, $(i \neq j)$.*

3. Construction of a CMG from a weak metric scheme with 2 classes

In this section, we explain how the weak metric schemes $\mathfrak{X}$ with 2 classes give rise to CMGs. There exist the graphs Γ_0, Γ_1 and Γ_2 with respect to the relations R_0, R_1 and R_2 of $\mathfrak{X}$ respectively, where Γ_0 is a graph with only vertices. In the following lemma, we explain the graphs Γ_1 and Γ_2.

Lemma 11. *Let Γ_j $(j = 1, 2)$ be a j-relation graph of $\mathfrak{X} = \mathfrak{X}^{(1)} \wr \mathfrak{X}^{(2)} = (X = X_1 \times X_2, \{R_j\}_{j=0}^2)$. Then*

(1) Γ_1 *is a disconnected regular graph with m_1m_2 vertices and $k^{(1)}$ valency.*

(2) Γ_2 *is a connected regular graph with m_1m_2 vertices and $m_1k^{(2)}$ valency,*

where $k^{(i)}$ $(i = 1, 2)$ is a valency of $R_1^{(i)}$and $|X_1| = m_1, |X_2| = m_2$.

Proof. Let $L_1 = (p_{1h}^g)$ and $L_2 = (p_{2h}^g)$ be the intersection matrices of Γ_1 and Γ_2 respectively. Since $p_{11}^0 = k^{(1)}$ and $p_{22}^{(0)} = m_1k^{(2)}$, Γ_1 and Γ_2 are the regular graphs with valency $k^{(1)}$ and $m_1k^{(2)}$ respectively. Since a entry $p_{11}^2 = 0$. That is, every nonadjacent pair of vertices on Γ_1 have 0 common neighbors. Thus, Γ_1 is a disconnected graph. Since $p_{22}^1 = m_1k^{(2)}$ and $p_{22}^2 = m_1k^{(2)} - k^{(1)} - 1$, a graph Γ_2 is a connected regular graph. □

The following Theorem is the first main result of this paper, which shows that any weak metric scheme with 2 classes produces a CMG.

Theorem 12. *Any weak metric scheme with 2 classes gives rise to a CMG. In detail, let* $\mathfrak{X} = (X_1 \times X_2, \{R_j\}_{j=0}^2)$ *be a weak metric scheme with 2 classes deifned as in Proposition 7. Then a* 2*-relation graph* Γ_2 *of* $\mathfrak{X}$ *is a CMG with parameters* $(v, k, \lambda, \mu) = (m_1m_2, m_1k^{(2)}, m_1k^{(2)} - k^{(1)} - 1, m_1k^{(2)})$.

Proof. Let $\mathfrak{X}_1 = (X_1, R^{(1)})$ and $\mathfrak{X}_2 = (X_2, R^{(2)})$ be the metric schemes with $|X_1| = m_1$, $|X_2| = m_2$. And let $k^{(i)}$ $(i = 1, 2)$ be a valency of $\mathfrak{X}_i$. Let $\mathfrak{X} = \mathfrak{X}_1 \wr \mathfrak{X}_2$ be a weak metric scheme withe 2 classes. Then the size of the intersection matrices L_i with (g, h)-entry p_{ih}^g $(i = 0, 1, 2)$ of $\mathfrak{X}$ is 3×3 and L_0, L_1 and L_2 are the same as in (3) of Proposition 7. Let Γ_i be a graph with respect to L_i $(i = 1, 2)$. Then Γ_i have $|X_1||X_2| = m_1m_2$ vertices. Also Γ_1 is a regular graph with valency $k^{(1)}$ and Γ_2 is a regular graph with valency $m_1k^{(2)}$. Every adjacent pair of vertices of Γ_2 have $p_{22}^2 = mk^{(2)} - k^{(1)} - 1$ common neighbors, and every nonadjacent pair of vertices of Γ_2 have $p_{22}^1 = m_1k^{(2)}$ common neighbors. Therefore Γ_2 is a connected graph with diameter 2. Thus Γ_2 is a CMG with parameters (v, k, λ, μ) =$(m_1m_2, m_1k^{(2)}, m_1k^{(2)} - k^{(1)} - 1, m_1k^{(2)})$. □

Example 13. Let $\mathfrak{X} = (X_1 \times X_2, \{R_j\}_{j=0}^2)$ be a weak metric scheme with 2 classes which is given as the wreath product of the Hamming schemes $H(1, m_1)$ and $H(1, m_2)$. Then we have

(1) $m_1 = 2,\ m_2 = 3 : \mathfrak{X} = H(1, 2) \wr H(1, 3)$

$\Gamma_1 : (6, 1, 0, 0),\ \Gamma_2 : (6, 4, 2, 4)$ (Octahedral graph).

(2) $m_1 = 3,\ m_2 = 2 : \mathfrak{X} = H(1, 3) \wr H(1, 2)$

$\Gamma_1 : (6, 2, 1, 0),\ \Gamma_2 : (6, 3, 0, 3)$ (Utility graph).

4. Equivalence of the CMGs with the weak metric schemes with 2 classes

In this section, we show the converse of Theorem 12, equivalently, any CMG can be constructed from a weak metric scheme with 2 classes.

The following lemma shows the properties of CMGs.

Lemma 14. *Let* $\mathcal{G}$ *be a CMG with parameters* (v, k, λ, μ)*. Then we have*

(1) $k = \mu$

(2) $v = 2k - \lambda$ $(\lambda = 2k - v)$

(3) $(v, k, \lambda, \mu) = (2k - \lambda, k, 2k - v, k)$

(4) $2k \geq v$.

Proof. Since the CMGs are the SRGs, $\mathcal{G}$ satisfies $k(k-\lambda-1) = \mu(v-k-1)$. Since $k = \mu$, we have $k - \lambda - 1 = v - k - 1$, and so $v = 2k - \lambda$. Therefore $\mathcal{G}$ is a graph with parameters $(2k - \lambda, k, 2k - v, k)$ and we have $2k - v \geq 0$. □

Remark 15. *If* $\mathcal{G}$ *is a SRG with parameters* (v, k, λ, μ)*. Then a complement graph* $\bar{\mathcal{G}}$ *of* $\mathcal{G}$ *is SRG with parameters* $(v, v-k-1, v-2-2k+\mu, v-2k+\lambda)$.

Lemma 16. *Let $\mathcal{G}$ be a CMG with parameters* $(2k-\lambda, k, 2k-v, k)$. *Then we have*
(1) $\bar{\mathcal{G}}$ *is a SRG with parameters* $(2k-\lambda, k-\lambda-1, k-\lambda-2, 0)$.
(2) $k-\lambda$ *is a divisor of* $2k-\lambda$.
(3) If $k-\lambda$ *is not divisor of* $2k-\lambda$, *then* $(2k-\lambda, k, 2k-v, k)$ *is not a SRG.*

Proof. (1) It follows from Remark 15.
(2) Since $\mu = 0$, $\bar{\mathcal{G}}$ is a disjoint union of $(k-\lambda)$-cliques. Thus, there is a positive integer b such that $2k-\lambda = (k-\lambda)b$.
(3) It directly follows from (1) and (2). □

Corollary 17. *Let $\mathcal{G}$ be a CMG with parameters* (v, k, λ, μ). *Then v is a composite number.*

Proof. By Lemma 14, $\mathcal{G}$ is a graph with parameters $(2k-\lambda, k, 2k-v, k)$. By Lemma 16, $\bar{\mathcal{G}}$ is a SRG with parameters $(2k-\lambda, k-\lambda-1, k-\lambda-2, 0)$ and $k-\lambda$ is a divisor of $2k-\lambda$. Assuming $2k-\lambda$ is a prime number, we have that $k-\lambda$ is 1 or $2k-\lambda$. If $k-\lambda = 2k-\lambda$, then $k=0$. If $k-\lambda=1$ then $k-\lambda-2=-1<0$. Therefore v is a composite number. □

Remark 18. *Let* $\mathfrak{X}^{(i)} = (X_i, \{R_j^{(i)}\}_{j=0}^{1})$ *be a metric scheme with 1 class. Then, there exist the metric schemes with* $|X_i| = m_i$ *for any positive integer* m_i. *For instance, the Hamming schemes* $H(1, m_i)$ *with* $|X_i| = m_i$ *and the Johnson schemes* $J(v,1)$ *with* $|X_i| = \binom{m_i}{1} = m_i$ *are metric schemes with 1 class.*

Lemma 19. *Assume that $\mathcal{G}$ and $\mathcal{H}$ are the CMGs with the same parameters* (v, k, λ, μ). *Then we have the following:*
(i) *$\mathcal{G}$ and $\mathcal{H}$ have the same intersection matrix L :*

$$L = \begin{pmatrix} 0 & 0 & k \\ 0 & 0 & k \\ 1 & v-k-1 & 2k-v \end{pmatrix}.$$

(ii) *$\mathcal{G}$ is isomorphic to $\mathcal{H}$.*

Proof. (i) Let $\mathcal{K}$ be a CMG with parameters (v, k, λ, μ). Then since by Lemma 14 $(v, k, \lambda, \mu) = (2k-\lambda, k, 2k-v, k)$ and $\mathcal{K}$ is a SRG, there exist eigenvalues $k, r, s (k > r > s)$ such that $r+s = \lambda-\mu$, $rs = -k+\mu$. Since $r+s = 2k-v-k = k-v$, $rs = -k+\mu = -k+k = 0$, we have $r=0$ and $s = k-v$.

Define the distance d on $\mathcal{K}$ as follows :

$$d = \begin{cases} 1, & \text{if (x,y) nonadjacent} \\ 2, & \text{if (x,y) adjacent.} \end{cases}$$

Since $\mathcal{K}$ is a SRG, an intersection matrix of $\mathcal{K}$ is defined on symmetric association schemes with 2 classes. Thus we obtain an intersection matrix L of $\mathcal{K}$ as follows :

$$L = \begin{pmatrix} 0 & 0 & k \\ 0 & 0 & k \\ 1 & a-1 & k-a \end{pmatrix},$$

where k is a valency of $\mathcal{K}$. Since the eigenvalues of L are k,0 and $-a$, we have $a = v-k$.

(*ii*) By Lemma 16, $\bar{\mathcal{G}}$ and $\bar{\mathcal{H}}$ are the SRGs with parameters $(v, k-\lambda-1, k-\lambda-2, 0)$. Thus $\bar{\mathcal{G}}$ and $\bar{\mathcal{H}}$ are the disjoint unions of $(k-\lambda)$-cliques, and the classes of $\bar{\mathcal{G}}$ and $\bar{\mathcal{H}}$ are the complete graphs with vertices $k-\lambda$. Therefore $\bar{\mathcal{G}}$ and $\bar{\mathcal{H}}$ have the same adjacency matrix which is the $(k-\lambda)$-block diagonal matrix :

$$A = \begin{pmatrix} J_{k-\lambda} - I_{k-\lambda} & & \\ & \ddots & \\ & & J_{k-\lambda} - I_{k-\lambda} \end{pmatrix},$$

where $J_{k-\lambda}$ is all one matrix and $I_{k-\lambda}$ is an identity matrix. Since a matrix $J_v - I_v - A$ is an adjacency matrix of both $\mathcal{G}$ and $\mathcal{H}$, $\mathcal{G}$ is isomorphic to $\mathcal{H}$. □

The following Theorem is the second main result of this paper which is the converse of Theorem 12, and it shows that any CMG can be constructed from a weak metric scheme with 2 classes. Therefore, we obtain the following.

Theorem 20. *The weak metric schemes with* 2 *classes are equivalent to the CMGs.*

Proof. By Theorem 12, a graph Γ_2 constructed from the weak metric schemes with 2 classes is a CMG. It is thus enough to show that any CMG can be constructed from a weak metric scheme with 2 classes.

Let $\mathcal{G}$ be a CMG with parameters (v, k, λ, μ) and let $k^{(1)}$ be a valency of a complement graph $\bar{\mathcal{G}}$ of $\mathcal{G}$, then $v = 1 + k^{(1)} + k$. By Lemma 19, an intersection matrix L of $\mathcal{G}$ is

$$\begin{pmatrix} 0 & 0 & k \\ 0 & 0 & k \\ 1 & k^{(1)} & k - k^{(1)} - 1 \end{pmatrix}.$$

By (2) of Lemma 14 and (2) of Lemma 16, $v - k = k^{(1)} + 1$ is a divisor of $2k - \lambda$. Thus $v = 2k - \lambda = (k^{(1)} + 1)b$ for some integer b. Combining $v - k = k^{(1)} + 1$ and $v = (k^{(1)} + 1)b$ yields $k = (k^{(1)} + 1)(b - 1) \quad (b > 1)$. Let $k^{(2)} = b - 1$. Then $v = (k^{(1)} + 1)(k^{(2)} + 1)$ and $k = (k^{(1)} + 1)k^{(2)}$ (Then $v \geq 4, \ \ k \geq 2$).

Let the weak metric scheme $\mathfrak{X} = (X_1 \times X_2, \{R_j\}_{j=0}^2)$ with 2 classes have $|X_1| = k^{(1)} + 1 = v - k$ and $|X_2| = k^{(2)} + 1 = \frac{v}{v-k}$, and let Γ_2 be a 2-relation graph over the weak metric scheme $\mathfrak{X}$. Then by Proposition 7, the intersection matrix L_2 of $\mathfrak{X}$ is the same as $L_{\mathcal{G}}$. Therefore, the graph $\mathcal{G}$ is isomorphic to Γ_2, which shows our result. □

The following corollary is obtained from the proof of Theorem 20.

Corollary 21. *Let* $\mathcal{G}$ *be a CMG with parameters* (v, k, λ, μ). *Then we have*

(1) *Eigenvlaues of* $\mathcal{G}$ *are* k, 0 *and* $k - v$.

(2) $\mathcal{G}$ *is a complete* $t-$*partite graph, where* $t = \dfrac{v}{v-k}$.

Proof. In proof of Theorem 20, $\mathcal{G}$ have the eigenvalues k,0 and $k - v$. Thus the size of the classes of $\mathcal{G}$ is $v-k$ and $\mathcal{G}$ is a complete $\frac{v}{v-k}-$ partite graph. □

The following corollary is obtained directly from Remark 18 and Theorem 20.

Corollary 22. *Let* $\mathfrak{X}^{(i)}$ $(i = 1, 2)$ *be a Hamming scheme or a Johnson scheme with* 1 *class, and* $\mathfrak{X} = \mathfrak{X}^{(1)} \wr \mathfrak{X}^{(2)} = (X_1 \times X_2, \{R_j\}_{j=0}^2)$ *be a weak metric scheme with 2 classes and* $|X_i| = m_i$ $(i = 1, 2)$*. Let* S *be a collection of all weak metric scheme with 2 classes* $\mathfrak{X}$ *made by a Hamming scheme or a Johnson scheme with* 1 *class as above, and let* $\mathfrak{C}$ *be a set of all the CMGs. Then any CMG in* $\mathfrak{C}$ *is equivalent to a weak metric scheme* $\mathfrak{X}$ *in* S*.*

Corollary 23. *Let* v *be any composite number and let* r *be the number of divisors of* v*. Then, there exist exactly* $r - 2$ *CMGs with vertices* v*.*

Proof. Let $\mathfrak{X} = \mathfrak{X}^{(1)} \wr \mathfrak{X}^{(2)}$ be a weak Hamming scheme with 2 classes and parameters defined as in Proposition 7. Then the graphs Γ_2 is the CMG with parameters $(m_1m_2, m_1m_2 - m_1, m_1m_2 - 2m_1, m_1m_2 - m_1)$. Since $v = m_1m_2$ is a composite integer and $m_i \geq 2$ $(i = 1, 2)$, the number of the pairs (m_1, m_2) is $r - 2$. Since $(m_1m_2, m_1m_2 - m_1, m_1m_2 - 2m_1, m_1m_2 - m_1) = (v, v - m_1, v - 2m_1, v - m_1)$, if $v = m_1m_2 = m_1'm_2'$ and $m_1 \neq m_1'$ then $v - m_1 \neq v - m_1'$. Thus there exist exactly $r - 2$ distinct CMGs with vertices v. □

5. Cheeger constants of Hamming graphs

In this section, we obtain the actual Cheeger constants of Hamming graphs.

Let $\Gamma(n, q)$ be a graph with respect to R_1 over $H(n, q)$. Then $\Gamma(n, q)$ be a distance regular graph with order q^n. Let V be a set of vertices and E be a set of edges of $\Gamma(n, q)$. There exist q distinct subgraphs of $\Gamma(n, q)$ as the isomorphic copies of $\Gamma(n - 1, q)$. Let S_i $(i = 0, 1, \cdots, q - 1)$ be a set of vertices with first position value i. Then $|S_i| = q^{n-1}$ and $V = \bigcup S_i$. We choose a subset S of V by

(1) $S = S_0 \cup S_1 \cup \cdots \cup S_{q/2-1}$, *q is even*,
(2) $S = S_0 \cup S_1 \cup \cdots \cup S_{t-1} \cup S_t/2$, $t = \lfloor q/2 \rfloor$, *q is odd*,

where $S_t/2$ is a subset od S_t and $|S_t/2| = \lfloor \frac{q^{n-1}}{2} \rfloor$. Then, by Lemma 10, $E(S_i \sim S_j) = q^{n-1}$ $(i \neq j)$ and $E(S_i \sim S_t/2) = \lfloor \frac{q^{n-1}}{2} \rfloor$ (q is odd and $i = 0, 1, \cdots, t - 1$).

In Hamming graphs, for obtaining their Cheeger constants, we need to compute the number of boundary edges which connect a vertex set S and its complement set in Hamming graphs.

In the following theorem, we determine the actual Cheeger constant of Hamming graphs $\Gamma(n, q)$ explicitly.

Theorem 24. *Let $\Gamma(n,q)$ be a Hamming graph over $H(n,q)$ and $h_{\Gamma(n,q)}$ be a Cheeger constant of $\Gamma(n,q)$. Then we have as follows :*

$$h_{\Gamma(n)} = \frac{\lfloor \frac{q+1}{2} \rfloor}{n(q-1)}.$$

That is, we have

$$h_{\Gamma(n,q)} = \begin{cases} \frac{\lambda_1}{2}, & q \text{ is even,} \\ \frac{\lambda_1}{2} + \frac{1}{2n(q-1)}, & q \text{ is odd.} \end{cases}$$

where λ_1 is the smallest positive eigenvalue of the Laplacian of $\Gamma(n,q)$.

Proof. We consider the two cases : (i) q is even and (ii) q is odd.
(i) q is even : Let $\Gamma(n-1,q)$ be a distance regular graph over $H(n-1,q)$. Then, a valency of $\Gamma(n-1,q)$ is $(n-1)(q-1)$ and $\mathrm{vol}(S_i) = |S_i|(n-1)(q-1) = q^{n-1}(n-1)(q-1)$. Let S be a subset of V as $S_0 \cup S_1 \cup \cdots \cup S_{q/2-1}$. Then $|S| = |V|/2$. Since $E(S_i \sim S_j) = q^{n-1}$ $(i \neq j)$,

$$\begin{aligned} |\partial S| &= \mathrm{vol}(S) - \frac{q}{2}q^{n-1}(n-1)(q-1) - \frac{q}{2}(\frac{q}{2}-1)q^{n-1} \\ &= \frac{q^n}{2}n(q-1) - \frac{q^n}{2}(n-1)(q-1) - \frac{q}{2}(\frac{q}{2}-1)q^{n-1} \\ &= \frac{q^n}{2}(q-1) - \frac{q}{2}(\frac{q}{2}-1)q^{n-1} = \frac{q^{n+1}}{4}. \end{aligned}$$

Thus, we have

$$h_{\Gamma(n,q)} \leq \frac{|\partial S|}{\mathrm{vol}(S)} = \frac{q^{n+1}/2}{q^n n(q-1)} = \frac{q}{2n(q-1)}.$$

Since, $\Gamma(n,q)$ has $n(q-1)$ valency and $\lambda_1 = 1 - \frac{p_1(1)}{n(q-1)}$, where $p_1(1)$ is an eigenvalue of an adjacency matrix of $\Gamma(n,q)$. Since $p_1(1) = nq-n-q$, and $\lambda_1 = \frac{q}{n(q-1)}$. Thus, we have

$$h_{\Gamma(n,q)} \leq \frac{\lambda_1}{2} = \frac{q}{2n(q-1)}.$$

Since, for any graph Γ, $h_\Gamma \geq \frac{\lambda_1}{2}$.

$$h_{\Gamma(n,q)} = \frac{\lambda_1}{2}.$$

(ii) q is odd : Let S be a subset of V as $S_0 \cup S_1 \cup \cdots \cup S_{t-1} \cup S_t/2$ $(t = \lfloor q/2 \rfloor)$. Let $S^{(1)} = S_0 \cup S_1 \cup \cdots \cup S_{t-1}$ and $S^{(2)} = S_{t+1} \cup \cdots \cup S_q$. Then

$$\begin{aligned} E(S^{(1)} \sim S^{(2)}) &= q^{n-1}\Big\lfloor \frac{q}{2} \Big\rfloor^2, \\ E(S^{(1)} \sim (S_t - S_t/2)) &= \Big\lfloor \frac{q}{2} \Big\rfloor \Big(\frac{q^{n-1}+1}{2}\Big), \\ E(S_t/2 \sim (S^{(2)})) &= \Big\lfloor \frac{q}{2} \Big\rfloor \Big(\frac{q^{n-1}-1}{2}\Big). \end{aligned}$$

Thus, $E(S^{(1)} \sim S^{(2)}) + E(S^{(1)} \sim (S_t - S_t/2)) + E(S_t/2 \sim (S^{(2)})$ is

$$q^{n-1}\Big\lfloor \frac{q}{2} \Big\rfloor^2 + q^{n-1}\Big\lfloor \frac{q}{2} \Big\rfloor = \frac{q^{n-1}}{4}(q^2-1).$$

Since $S_t/2$ is a subset of S_t and S_t is isomorphic to $\mathbb{F}_q^{n-1}$. Thus, we obtain $|\partial S|$ by repeating the above course.

$$\sum_{i=1}^{n} \frac{q^{n-i}}{4}(q^2-1) = \frac{q^2-1}{4}\frac{q^n-1}{q-1} = \frac{(q+1)(q^n-1)}{4}.$$

Thus, we have

$$h_{\Gamma(n,q)} \leq \frac{|\partial S|}{\mathrm{vol}(S)} = \frac{\frac{(q+1)(q^n-1)}{4}}{n(q-1)\frac{q^n-1}{2}} = \frac{q+1}{2n(q-1)}.$$

Let $S^{(n-1)}$ be a set as S_i in Lemma 10, and let $h_{\Gamma(n-1,q)}$ be a Cheeger constant on $S^{(n-1)}$ of $\Gamma(n-1,q)$. Then we can choose a vertex v_i in S_i with $v_i = (i, x_2, \cdots, x_n)$. Let $T_i = \{v_i \ : \ v_i = (i, x_2, \cdots, x_n) \in S_i \ , i = 0, 1, \cdots \frac{q-3}{2}\}$ and $T_j = \{v_j \ : \ v_j = (j, x_2, \cdots, x_n) \in S_j \ , j = \frac{q-1}{2}, \cdots, q-1\}$. Then, we have $E(T_i \sim T_j) = \frac{(q-1)(q+1)}{4}$. Thus, by Lemma 10, we have

$$\begin{aligned}|\partial S^{(n)}| &= q\ h_{\Gamma(n-1,q)}\mathrm{vol}(S^{(n-1)}) + E(T_i \sim T_j)\\ &= q\ h_{\Gamma(n-1,q)}\mathrm{vol}(S^{(n-1)}) + \frac{(q-1)(q+1)}{4}\\ &= q\ |\partial S^{(n-1)}| + \frac{(q-1)(q+1)}{4}.\end{aligned}$$

Since $\Gamma(1,q)$ is a graph with valency $q-1$, $h_{\Gamma(1,q)} = \frac{q+1}{2(q-1)}$ and $|\partial S^{(1)}|$ is $\frac{(q+1)(q-1)}{4}$. Thus, $|\partial S^{(2)}| = \frac{(q+1)(q^2-1)}{4}$. Since, there exists $S^{(2)}$ with $|\partial S^{(2)}| = \frac{(q+1)(q^2-1)}{4}$ and maximum volume. Thus, we have

$$h_{\Gamma(2,q)} = \frac{|\partial S^{(2)}|}{\mathrm{vol}(S^{(2)})} = \frac{\frac{(q+1)(q^2-1)}{4}}{2(q-1)(q^2-1)/2} = \frac{q+1}{4(q-1)}.$$

By induction on n,

$$|\partial S^{(n+1)}| = \frac{(q+1)(q^{n+1}-1)}{4}.$$

Thus, we have

$$h_{\Gamma(n,q)} = \frac{q+1}{2n(q-1)}.$$

□

Corollary 25. *Let $\Gamma(n,q)$ be a Hamming graph over $H(n,q)$, and let $h_{\Gamma(n,q)} = \frac{q+1}{2n(q-1)}$. Then*

$$(1)\ \lim_{n\to\infty}(h_{\Gamma(n,q)} - \frac{\lambda_1}{2}) = 0,\ \ (2)\ \lim_{q\to\infty}(h_{\Gamma(n,q)} - \frac{\lambda_1}{2}) = 0.$$

Proof. Since $\frac{\lambda_1}{2} = \frac{q}{2n(q-1)}$ and $h_{\Gamma(n,q)} - \frac{\lambda_1}{2} = \frac{1}{2n(q-1)}$. Thus, it is clear. □

6. List of complete mulipartitie graphs

By Theorem 12, Theorem 20, Corollary 21 and Corollary 23, Table 1 in Appendix shows the complete list of the CMGs for $4 \leq v \leq 100$.

Appendix.

TABLE 1. The complete list of the CMGs for $4 \le v \le 100$

v	$\lvert X_1 \rvert$	$\lvert X_2 \rvert$	(v,k,λ,μ)	*eigenvalues*	*graph*
4	2	2	$(4,2,0,2)$	$2,0,-2$	*square graph*
6	2	3	$(6,4,2,4)$	$4,0,-2$	*octahedral graph*
	3	2	$(6,3,0,3)$	$3,0,-3$	*utility graph*
8	2	4	$(8,6,4,6)$	$6,0,-2$	$16-$ *cell graph*
	4	2	$(8,4,0,4)$	$4,0,-4$	*complete bipartite graph*
9	3	3	$(9,6,3,6)$	$6,0,-3$	*complete tripartite graph*
10	2	5	$(10,8,6,8)$	$8,0,-2$	$5-$ *cocktail party graph*
	5	2	$(10,5,0,5)$	$5,0,-5$	*complete bipartite graph*
12	2	6	$(12,10,8,10)$	$10,0,-2$	$6-$ *cocktail party graph*
	3	4	$(12,9,6,9)$	$9,0,-3$	*complete* $4-$ *partite graph*
	4	3	$(12,8,4,8)$	$8,0,-4$	*complete tripartite graph*
	6	2	$(12,6,0,6)$	$6,0,-6$	*complete bipartite graph*
14	2	7	$(14,12,10,12)$	$12,0,-2$	$7-$ *cocktail party graph*
	7	2	$(14,7,0,7)$	$7,0,-7$	*complete bipartite graph*
15	3	5	$(15,12,9,12)$	$12,0,-3$	*complete* $5-$ *partite graph*
	5	3	$(15,10,5,10)$	$10,0,-5$	*complete tripartite graph*
16	2	8	$(16,14,12,14)$	$14,0,-2$	$8-$ *cocktail party graph*
	4	4	$(16,12,8,12)$	$12,0,-4$	*complete* $4-$ *partite graph*
	8	2	$(16,8,0,8)$	$8,0,-8$	*complete bipartite graph*
18	2	9	$(18,16,14,16)$	$16,0,-2$	$9-$ *cocktail party graph*
	3	6	$(18,15,12,15)$	$15,0,-3$	*complete* $6-$ *partite graph*
	6	3	$(18,12,6,12)$	$12,0,-6$	*complete tripartite graph*
	9	2	$(18,9,0,9)$	$9,0,-9$	*complete bipartite graph*
20	2	10	$(20,18,16,18)$	$18,0,-2$	$10-$ *cocktail party graph*
	4	5	$(20,16,12,16)$	$16,0,-4$	*complete* $5-$ *partite graph*
	5	4	$(20,15,10,15)$	$15,0,-5$	*complete* $4-$ *partite graph*
	10	2	$(20,10,0,10)$	$10,0,-10$	*complete bipartite graph*
21	3	7	$(21,18,15,18)$	$18,0,-3$	*complete* $7-$ *partite graph*
	7	3	$(21,14,7,14)$	$14,0,-7$	*complete tripartite graph*
22	2	11	$(22,20,18,20)$	$20,0,-2$	$11-$ *cocktail party graph*
	11	2	$(22,11,0,11)$	$11,0,-11$	*complete bipartite graph*
24	2	12	$(24,22,20,22)$	$22,0,-2$	$12-$ *cocktail party graph*
	3	8	$(24,21,18,21)$	$21,0,-3$	*complete* $8-$ *partite graph*
	4	6	$(24,20,16,20)$	$20,0,-4$	*complete* $6-$ *partite graph*
	6	4	$(24,18,12,18)$	$18,0,-6$	*complete* $4-$ *partite graph*
	8	3	$(24,16,8,16)$	$16,0,-8$	*complete tripartite graph*
	12	2	$(24,12,0,12)$	$12,0,-12$	*complete bipartite graph*
25	5	5	$(25,20,15,20)$	$20,0,-5$	*complete* $5-$ *partite graph*
26	2	13	$(26,24,22,24)$	$24,0,-2$	$13-$ *cocktail party graph*
	13	2	$(26,13,0,13)$	$13,0,-13$	*complete bipartite graph*
27	3	9	$(27,24,21,24)$	$24,0,-3$	*complete* $9-$ *partite graph*
	9	3	$(27,18,9,18)$	$18,0,-9$	*complete tripartite graph*
28	2	14	$(28,26,24,26)$	$26,0,-2$	$14-$ *cocktail party graph*
	4	7	$(28,24,20,24)$	$24,0,-4$	*complete* $7-$ *partite graph*
	7	4	$(28,21,14,21)$	$21,0,-7$	*complete* $4-$ *partite graph*
	14	2	$(28,14,0,14)$	$14,0,-14$	*complete bipartite graph*
30	2	15	$(30,28,24,28)$	$28,0,-2$	$15-$ *cocktail party graph*
	3	10	$(30,27,21,27)$	$27,0,-3$	*complete* $10-$ *partite graph*
	5	6	$(30,25,20,25)$	$25,0,-5$	*complete* $6-$ *partite graph*
	6	5	$(30,24,18,24)$	$24,0,-6$	*complete* $5-$ *partite graph*
	10	3	$(30,20,10,20)$	$20,0,-10$	*complete tripartite graph*
	15	2	$(30,15,0,15)$	$15,0,-15$	*complete bipartite graph*
32	2	16	$(32,30,28,30)$	$30,0,-2$	$16-$ *cocktail party graph*
	4	8	$(32,28,24,28)$	$28,0,-4$	*complete* $8-$ *partite graph*
	8	4	$(32,24,16,24)$	$24,0,-8$	*complete* $4-$ *partite graph*
	16	2	$(32,16,0,16)$	$16,0,-16$	*complete bipartite graph*
34	2	17	$(34,32,30,32)$	$32,0,-2$	$17-$ *cocktail party graph*
	17	2	$(34,17,0,17)$	$17,0,-17$	*complete bipartite graph*
35	5	7	$(35,30,25,30)$	$30,0,-5$	*complete* $7-$ *partite graph*
	7	5	$(35,28,21,28)$	$28,0,-7$	*complete* $5-$ *partite graph*
36	2	18	$(36,34,32,34)$	$34,0,-2$	$18-$ *cocktail party graph*
	3	12	$(36,33,30,33)$	$33,0,-3$	*complete* $12-$ *partite graph*
	4	9	$(36,32,28,32)$	$32,0,-4$	*complete* $9-$ *partite graph*
	6	6	$(36,30,24,30)$	$30,0,-6$	*complete* $6-$ *partite graph*
	9	4	$(36,27,18,27)$	$27,0,-9$	*complete* $4-$ *partite graph*
	12	3	$(36,24,12,24)$	$24,0,-12$	*complete tripartite graph*
	18	2	$(36,18,0,18)$	$18,0,-18$	*complete bipartite graph*
38	2	19	$(38,36,34,36)$	$36,0,-2$	$19-$ *cocktail party graph*
	19	2	$(38,19,0,19)$	$19,0,-19$	*complete bipartite graph*

v	$\|X_1\|$	$\|X_2\|$	(v,k,λ,μ)	*eigenvalues*	*graph*
40	2	20	$(40,38,36,38)$	$38,0,-2$	$20-$ *cocktail party graph*
	4	10	$(40,36,32,36)$	$36,0,-4$	*complete* $10-$ *partite graph*
	5	8	$(40,35,30,35)$	$35,0,-5$	*complete* $8-$ *partite graph*
	8	5	$(40,32,24,32)$	$32,0,-8$	*complete* $5-$ *partite graph*
	10	4	$(40,30,20,30)$	$30,0,-10$	*complete* $4-$ *partite graph*
	20	2	$(40,20,0,20)$	$20,0,-20$	*complete bipartite graph*
42	2	21	$(42,40,38,40)$	$40,0,-2$	$21-$ *cocktail party graph*
	3	14	$(42,39,36,39)$	$39,0,-3$	*complete* $14-$ *partite graph*
	6	7	$(42,36,30,36)$	$36,0,-6$	*complete* $7-$ *partite graph*
	7	6	$(42,35,28,35)$	$35,0,-7$	*complete* $6-$ *partite graph*
	14	3	$(42,28,14,28)$	$28,0,-14$	*complete tripartite graph*
	21	2	$(42,21,0,21)$	$21,0,-21$	*complete bipartite graph*
44	2	22	$(44,42,40,42)$	$42,0,-2$	$22-$ *cocktail party graph*
	4	11	$(44,40,36,40)$	$40,0,-4$	*complete* $11-$ *partite graph*
	11	4	$(44,33,22,33)$	$33,0,-11$	*complete* $4-$ *partite graph*
	22	2	$(44,22,0,22)$	$22,0,-22$	*complete bipartite graph*
45	3	15	$(45,42,39,42)$	$42,0,-3$	*complete* $15-$ *partite graph*
	5	9	$(45,40,35,40)$	$40,0,-5$	*complete* $9-$ *partite graph*
	9	5	$(45,36,27,36)$	$36,0,-9$	*complete* $5-$ *partite graph*
	15	3	$(45,30,15,30)$	$30,0,-15$	*complete tripartite graph*
46	2	23	$(46,44,42,44)$	$44,0,-2$	$23-$ *cocktail party graph*
	23	2	$(46,23,0,23)$	$23,0,-23$	*complete bipartite graph*
48	2	24	$(48,46,44,46)$	$46,0,-2$	$24-$ *cocktail party graph*
	3	16	$(48,45,42,45)$	$45,0,-3$	*complete* $16-$ *partite graph*
	4	12	$(48,44,40,44)$	$44,0,-4$	*complete* $12-$ *partite graph*
	6	8	$(48,42,36,42)$	$42,0,-6$	*complete* $8-$ *partite graph*
	8	6	$(48,40,32,40)$	$40,0,-8$	*complete* $6-$ *partite graph*
	12	4	$(48,36,24,36)$	$36,0,-12$	*complete* $4-$ *partite graph*
	16	3	$(48,32,16,32)$	$32,0,-16$	*complete* $3-$ *partite graph*
	24	2	$(48,24,0,24)$	$24,0,-24$	*complete bipartite graph*
49	7	7	$(49,42,35,42)$	$42,0,-7$	*complete* $7-$ *partite graph*
50	2	25	$(50,48,46,48)$	$48,0,-2$	$25-$ *cocktail party graph*
	5	10	$(50,45,40,45)$	$45,0,-5$	*complete* $10-$ *partite graph*
	10	5	$(50,40,30,40)$	$40,0,-10$	*complete* $5-$ *partite graph*
	25	2	$(50,25,0,25)$	$25,0,-25$	*complete bipartite graph*
52	2	26	$(52,50,48,50)$	$50,0,-2$	$26-$ *cocktail party graph*
	4	13	$(52,48,44,48)$	$48,0,-4$	*complete* $13-$ *partite graph*
	13	4	$(52,39,26,39)$	$39,0,-13$	*complete* $4-$ *partite graph*
	26	3	$(52,26,0,26)$	$26,0,-26$	*complete bipartite graph*
54	2	27	$(54,52,50,52)$	$52,0,-2$	$27-$ *cocktail party graph*
	3	18	$(54,51,48,51)$	$51,0,-3$	*complete* $18-$ *partite graph*
	6	9	$(54,48,42,48)$	$48,0,-6$	*complete* $9-$ *partite graph*
	9	6	$(54,45,36,45)$	$45,0,-9$	*complete* $6-$ *partite graph*
	18	3	$(54,36,18,36)$	$36,0,-18$	*complete tripartite graph*
	27	2	$(54,27,0,27)$	$27,0,-27$	*complete bipartite graph*
55	5	11	$(55,50,45,50)$	$50,0,-5$	*complete* $11-$ *partite graph*
	11	5	$(55,44,33,44)$	$44,0,-11$	*complete* $5-$ *partite graph*
56	2	28	$(56,54,53,54)$	$54,0,-1$	$28-$ *cocktail party graph*
	4	14	$(56,52,48,52)$	$52,0,-4$	*complete* $14-$ *partite graph*
	7	8	$(56,49,42,49)$	$49,0,-7$	*complete* $8-$ *partite graph*
	8	7	$(56,48,40,48)$	$48,0,-8$	*complete* $7-$ *partite graph*
	14	4	$(56,42,28,42)$	$42,0,-14$	*complete* $4-$ *partite graph*
	28	2	$(56,28,0,28)$	$28,0,-28$	*complete bipartite graph*
57	3	19	$(57,54,51,54)$	$54,0,-3$	*complete* $19-$ *partite graph*
	19	3	$(57,38,19,38)$	$38,0,-19$	*complete tripartite graph*
58	2	29	$(58,56,54,56)$	$56,0,-2$	$29-$ *cocktail party graph*
	29	2	$(58,29,0,29)$	$29,0,-29$	*complete bipartite graph*
60	2	30	$(60,58,56,58)$	$58,0,-2$	$30-$ *cocktail party graph*
	3	20	$(60,57,54,57)$	$57,0,-3$	*complete* $20-$ *partite graph*
	4	15	$(60,56,52,56)$	$56,0,-4$	*complete* $15-$ *partite graph*
	5	12	$(60,55,50,55)$	$55,0,-5$	*complete* $12-$ *partite graph*
	6	10	$(60,54,48,54)$	$54,0,-6$	*complete* $10-$ *partite graph*
	10	6	$(60,50,40,50)$	$50,0,-10$	*complete* $6-$ *partite graph*
	12	5	$(60,48,36,48)$	$48,0,-12$	*complete* $5-$ *partite graph*
	15	4	$(60,45,30,45)$	$45,0,-15$	*complete* $4-$ *partite graph*
	20	3	$(60,40,20,40)$	$40,0,-20$	*complete tripartite graph*
	30	2	$(60,30,0,30)$	$30,0,-30$	*complete bipartite graph*
62	2	31	$(62,60,58,60)$	$60,0,-2$	$31-$ *cocktail party graph*
	31	2	$(62,31,0,31)$	$31,0,-31$	*complete bipartite graph*
63	3	21	$(63,60,57,60)$	$60,0,-3$	*complete* $21-$ *partite graph*
	7	9	$(63,56,49,56)$	$56,0,-7$	*complete* $9-$ *tripartite graph*
	9	7	$(63,54,45,54)$	$54,0,-9$	*complete* $7-$ *partite graph*
	21	3	$(63,42,21,42)$	$42,0,-21$	*complete tripartite graph*

v	$\lvert X_1\rvert$	$\lvert X_2\rvert$	(v, k, λ, μ)	*eigenvalues*	*graph*
64	2	32	(64, 62, 60, 62)	62, 0, −2	32 − *cocktail party graph*
	4	16	(64, 60, 56, 60)	60, 0, −4	*complete* 16 − *partite graph*
	8	8	(64, 56, 48, 56)	56, 0, −8	*complete* 8 − *partite graph*
	16	4	(64, 48, 32, 48)	48, 0, −16	*complete* 4 − *partite graph*
	32	2	(64, 32, 0, 32)	32, 0, −32	*complete bipartite graph*
65	5	13	(65, 60, 55, 60)	60, 0, −5	*complete* 13 − *partite graph*
	13	5	(65, 52, 39, 52)	52, 0, −13	*complete* 5 − *partite graph*
66	2	33	(66, 64, 62, 64)	64, 0, −2	33 − *cocktail party graph*
	3	22	(66, 63, 60, 63)	63, 0, −3	*complete* 22 − *partite graph*
	22	3	(66, 44, 22, 44)	44, 0, −22	*complete tripartite graph*
	33	2	(66, 33, 0, 33)	33, 0, −33	*complete bipartite graph*
68	2	34	(68, 66, 64, 66)	66, 0, −2	34 − *cocktail party graph*
	4	17	(68, 64, 60, 64)	64, 0, −4	*complete* 17 − *partite graph*
	17	4	(68, 51, 34, 51)	51, 0, −17	*complete* 4 − *partite graph*
	34	2	(68, 34, 0, 34)	34, 0, −34	*complete bipartite graph*
69	3	23	(69, 66, 63, 66)	66, 0, −3	*complete* 23 − *partite graph*
	23	3	(69, 46, 23, 46)	46, 0, −23	*complete tripartite graph*
70	2	35	(70, 68, 66, 68)	68, 0, −2	35 − *cocktail party graph*
	5	14	(70, 65, 60, 65)	65, 0, −5	*complete* 14 − *partite graph*
	7	10	(70, 63, 56, 63)	63, 0, −7	*complete* 10 − *partite graph*
	10	7	(70, 60, 50, 60)	60, 0, −10	*complete* 7 − *partite graph*
	14	5	(70, 56, 42, 56)	56, 0, −14	*complete* 5 − *partite graph*
	35	2	(70, 35, 0, 35)	35, 0, −35	*complete bipartite graph*
72	2	36	(72, 70, 68, 70)	70, 0, −2	36 − *cocktail party graph*
	3	24	(72, 69, 66, 69)	69, 0, −3	*complete* 24 − *partite graph*
	4	18	(72, 68, 64, 68)	68, 0, −4	*complete* 18 − *partite graph*
	6	12	(72, 66, 60, 66)	66, 0, −6	*complete* 12 − *partite graph*
	8	9	(72, 64, 56, 64)	64, 0, −8	*complete* 9 − *partite graph*
	9	8	(72, 63, 54, 63)	63, 0, −9	*complete* 8 − *partite graph*
	12	6	(72, 60, 48, 60)	60, 0, −12	*complete* 6 − *partite graph*
	18	4	(72, 54, 36, 54)	54, 0, −18	*complete* 4 − *partite graph*
	24	3	(72, 48, 24, 48)	48, 0, −24	*complete tripartite graph*
	36	2	(72, 36, 0, 36)	36, 0, −36	*complete bipartite graph*
74	2	37	(74, 52, 50, 52)	52, 0, −22	37 − *cocktail party graph*
	37	2	(74, 37, 0, 37)	37, 0, −37	*complete bipartite graph*
75	3	25	(75, 72, 69, 72)	72, 0, −3	*complete* 25 − *partite graph*
	5	15	(75, 70, 65, 70)	70, 0, −5	*complete* 15 − *partite graph*
	15	5	(75, 60, 45, 60)	60, 0, −15	*complete* 5 − *partite graph*
	25	3	(75, 50, 25, 50)	50, 0, −25	*complete bipartite graph*
76	2	38	(76, 74, 72, 74)	74, 0, −2	38 − *cocktail party graph*
	4	19	(76, 72, 68, 72)	72, 0, −4	*complete* 19 − *partite graph*
	19	4	(76, 57, 38, 57)	57, 0, −19	*complete* 4 − *partite graph*
	38	2	(76, 38, 0, 38)	38, 0, −38	*complete bipartite graph*
77	7	11	(77, 70, 63, 70)	70, 0, −7	*complete* 11 − *partite graph*
	11	7	(77, 66, 55, 66)	66, 0, −11	*complete* 7 − *partite graph*
78	2	39	(78, 76, 74, 76)	76, 0, −2	39 − *cocktail party graph*
	3	26	(78, 75, 72, 75)	75, 0, −3	*complete* 26 − *partite graph*
	26	3	(78, 52, 26, 52)	52, 0, −26	*complete* 3 − *partite graph*
	39	2	(78, 39, 0, 39)	39, 0, −39	*complete bipartite graph*
80	2	40	(80, 78, 76, 78)	78, 0, −2	40 − *cocktail party graph*
	4	20	(80, 76, 72, 76)	76, 0, −4	*complete* 20 − *partite graph*
	5	16	(80, 75, 70, 75)	75, 0, −5	*complete* 16 − *partite graph*
	8	10	(80, 72, 64, 72)	72, 0 − 8	*complete* 10 − *partite graph*
	10	8	(80, 70, 60, 70)	70, 0, −10	*complete* 8 − *partite graph*
	16	5	(80, 64, 48, 64)	64, 0, −16	*complete* 5 − *partite graph*
	20	4	(80, 60, 40, 60)	60, 0, −20	*complete* 4 − *partite graph*
	40	2	(80, 40, 0, 40)	40, 0, −40	*complete bipartite graph*
81	3	27	(81, 78, 75, 78)	78, 0, −3	*complete* 27 − *partite graph*
	9	9	(81, 72, 63, 72)	72, 0, −9	*complete* 9 − *partite graph*
	27	3	(81, 54, 27, 54)	54, 0, −27	*complete tripartite graph*
82	2	41	(82, 80, 78, 70)	80, 0, −2	41 − *cocktail party graph*
	41	2	(82, 41, 0, 41)	41, 0, −41	*complete bipartite graph*
84	2	42	(84, 82, 80, 82)	82, 0, −2	42 − *cocktail party graph*
	3	28	(84, 81, 78, 81)	81, 0, −3	*complete* 28 − *partite graph*
	4	21	(84, 80, 76, 80)	80, 0, −4	*complete* 21 − *partite graph*
	6	14	(84, 78, 72, 78)	78, 0, −6	*complete* 14 − *partite graph*
	14	6	(84, 70, 56, 70)	70, 0, −14	*complete* 6 − *partite graph*
	21	4	(84, 63, 42, 63)	63, 0, −21	*complete* 4 − *partite graph*
	28	3	(84, 56, 28, 56)	56, 0, −28	*complete tripartite graph*
	42	2	(84, 42, 0, 42)	42, 0, −42	*complete bipartite graph*
85	5	17	(85, 80, 75, 80)	80, 0, −5	*complete* 17 − *partite graph*
	17	5	(85, 68, 51, 68)	68, 0, −17	*complete* 5 − *partite graph*
86	2	43	(86, 84, 82, 84)	84, 0, −2	43 − *cocktail party graph*
	43	2	(86, 43, 0, 43)	43, 0, −43	*complete bipartite graph*
87	3	29	(87, 84, 81, 84)	84, 0, −3	*complete* 29 − *partite graph*
	29	3	(87, 58, 29, 58)	58, 0, −29	*complete tripartite graph*

v	$\lvert X_1\rvert$	$\lvert X_2\rvert$	(v,k,λ,μ)	*eigenvalues*	*graph*
88	2	44	$(88, 86, 84, 86)$	$86, 0, -2$	$44-$ *cocktail party graph*
	4	22	$(88, 84, 80, 84)$	$84, 0, -4$	*complete* $22-$ *partite graph*
	8	11	$(88, 80, 72, 80)$	$80, 0, -8$	*complete* $11-$ *partite graph*
	11	8	$(88, 77, 66, 77)$	$77, 0, -11$	*complete* $8-$ *partite graph*
	22	4	$(88, 66, 44, 66)$	$66, 0, -22$	*complete* $4-$ *partite graph*
	44	2	$(88, 44, 0, 44)$	$44, 0, -44$	*complete bipartite graph*
90	2	45	$(90, 88, 86, 88)$	$88, 0, -2$	$45-$ *cocktail party graph*
	3	30	$(90, 87, 84, 87)$	$87, 0, -3$	*complete* $30-$ *partite graph*
	5	18	$(90, 85, 80, 85)$	$85, 0, -5$	*complete* $18-$ *partite graph*
	6	15	$(90, 84, 78, 84)$	$84, 0, -6$	*complete* $15-$ *partite graph*
	9	10	$(90, 81, 72, 81)$	$81, 0, -9$	*complete* $10-$ *partite graph*
	10	9	$(90, 80, 70, 80)$	$80, 0, -10$	*complete* $9-$ *partite graph*
	15	6	$(90, 75, 60, 75)$	$75, 0, -15$	*complete* $6-$ *partite graph*
	18	5	$(90, 72, 54, 72)$	$72, 0, -18$	*complete* $5-$ *partite graph*
	30	3	$(90, 60, 30, 60)$	$60, 0, -30$	*complete tripartite graph*
	45	2	$(90, 45, 0, 45)$	$45, 0, -45$	*complete bipartite graph*
91	7	13	$(91, 84, 77, 84)$	$84, 0, -7$	*complete* $13-$ *partite graph*
	13	7	$(91, 78, 65, 78)$	$78, 0, -13$	*complete* $7-$ *partite graph*
92	2	46	$(92, 90, 88, 90)$	$90, 0, -2$	$46-$ *cocktail party graph*
	4	23	$(92, 88, 84, 88)$	$88, 0, -4$	*complete* $23-$ *partite graph*
	23	4	$(92, 69, 46, 69)$	$69, 0, -23$	*complete* $4-$ *partite graph*
	46	2	$(92, 46, 0, 46)$	$46, 0, -46$	*complete bipartite graph*
93	3	31	$(93, 90, 87, 90)$	$90, 0, -3$	*complete* $31-$ *partite graph*
	31	3	$(93, 62, 31, 62)$	$62, 0, -31$	*complete tripartite graph*
94	2	47	$(94, 92, 90, 92)$	$92, 0, -2$	$47-$ *cocktail party graph*
	47	2	$(94, 47, 0, 47)$	$47, 0, -47$	*complete bipartite graph*
95	5	19	$(95, 90, 85, 90)$	$90, 0, -5$	*complete* $19-$ *partite graph*
	19	5	$(95, 76, 57, 76)$	$76, 0, -19$	*complete* $5-$ *partite graph*
96	2	48	$(96, 94, 92, 94)$	$94, 0, -2$	$48-$ *cocktail party graph*
	3	32	$(96, 93, 90, 93)$	$93, 0, -3$	*complete* $32-$ *partite graph*
	4	42	$(96, 92, 88, 92)$	$92, 0, -4$	*complete* $24-$ *partite graph*
	6	16	$(96, 90, 84, 90)$	$90, 0, -6$	*complete* $16-$ *partite graph*
	8	12	$(96, 88, 80, 88)$	$88, 0, -8$	*complete* $12-$ *partite graph*
	12	8	$(96, 84, 72, 84)$	$84, 0, -12$	*complete* $8-$ *partite graph*
	16	6	$(96, 80, 64, 80)$	$80, 0, -16$	*complete* $6-$ *partite graph*
	24	4	$(96, 72, 48, 72)$	$72, 0, -24$	*complete* $4-$ *partite graph*
	32	3	$(96, 64, 32, 64)$	$64, 0, -32$	*complete tripartite graph*
	48	2	$(96, 48, 0, 48)$	$48, 0, -48$	*complete bipartite graph*
98	2	49	$(98, 96, 94, 96)$	$96, 0, -2$	$49-$ *cocktail party graph*
	7	14	$(98, 91, 84, 91)$	$91, 0, -7$	*complete* $14-$ *partite graph*
	14	7	$(98, 84, 70, 84)$	$84, 0, -14$	*complete* $7-$ *partite graph*
	49	2	$(98, 49, 0, 49)$	$49, 0, -49$	*complete bipartite graph*
99	3	33	$(99, 96, 93, 96)$	$96, 0, -3$	*complete* $33-$ *partite graph*
	9	11	$(99, 90, 81, 90)$	$90, 0, -9$	*complete* $11-$ *partite graph*
	11	9	$(99, 88, 77, 88)$	$88, 0, -11$	*complete* $9-$ *partite graph*
	33	3	$(99, 66, 33, 66)$	$66, 0, -33$	*complete tripartite graph*
100	2	50	$(100, 98, 96, 98)$	$98, 0, -2$	$50-$ *cocktail party graph*
	4	25	$(100, 96, 92, 96)$	$96, 0, -4$	*complete* $25-$ *partite graph*
	5	20	$(100, 95, 90, 95)$	$95, 0, -5$	*complete* $20-$ *partite graph*
	10	10	$(100, 90, 80, 90)$	$90, 0, -10$	*complete* $10-$ *partite graph*
	20	5	$(100, 80, 60, 80)$	$80, 0, -20$	*complete* $5-$ *partite graph*
	25	4	$(100, 75, 50, 75)$	$75, 0, -25$	*complete* $4-$ *partite graph*
	50	2	$(100, 50, 0, 50)$	$50, 0-50$	*complete bipartite graph*

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