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# CURVELET TRANSFORM ON RAPIDLY DECREASING FUNCTIONS

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Abstract. In this paper, we prove the continuity of the curvelet transform and adjoint curvelet transform between two suitable spaces of rapidly decreasing functions. Then, we extend these transforms to the context of distributions, in a natural manner, as continuous linear maps having the desired properties.

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## 1. Introduction

A new multiscale integral transform, called the curvelet transform was introduced by E. J. Candès and D. L. Donoho [1, 2] which has been used in time frequency analysis. It is a high-dimensional generalization of the wavelet transform designed to represent images at different scales and different orientations. It differs from the wavelet transform in dilations, where non-isotropic dilations are used instead of isotropic dilations. The curvelet transform overcomes the limitations of wavelet transform, in directionality and scaling.

The curvelet transform has wide range of applications in image denoising [13], image decomposition[15], image deconvolution[17], astronomical imaging [14], contrast, edge enhancement [16] and image fusion of satellites [3] etc., The continuous curvelet transform allows to resolve the singularities of an image together with their orientations[1]. Also some researchers used the applications of curvelet transform in face and color recognition problems.

On the other hand, in pure mathematical point of view, various integral transforms like, Fourier transform, Laplace transform, Hilbert transform, Mellin transform, Stieltjes transform, Hankel transform etc. have been extended to suitable generalized functions spaces. See [6, 18]. Motivated by those techniques, we have already extended the curvelet transform to a space of square integrable Boehmians in [9], periodic distributions in [10] and tempered distributions using "*kernel method*" in [11]. Adapting the technique applied for wavelet transform and ridgelet transform in [7, 8], in this paper, we prove the continuity of the curvelet transform and adjoint curvelet transform on rapidly decreasing functions and extend the curvelet transform to a space of tempered distributions using "*adjoint method*".

## 2. Preliminaries

In addition to the usual Banach space $\mathscr{L}^1(\mathbb{R}^2)$ of integrable functions on $\mathbb{R}^2$, the Hilbert space $\mathscr{L}^2(\mathbb{R}^2)$ of square integrable functions on $\mathbb{R}^2$ and the Fréchet space $\mathscr{S}(\mathbb{R}^2)$ of infinitely differentiable rapidly decreasing functions, we also use the following function space

$$\begin{aligned}\mathscr{L}^2_{a_0,b_0}(\mathbb{R}^2) &= \Big\{ f \in \mathscr{L}^2(\mathbb{R}^2) : \hat{f}(r\mathbf{e}^{i\omega}) = 0 \\ &\qquad \text{if } \frac{1}{2a_0} < r \text{ or } r < \frac{2}{b_0} \text{ or } -\pi < \omega < \sqrt{b_0} \Big\},\end{aligned}$$

where $\hat{f}$ is the Fourier transform of $f$ and $0 < 4a_0 < b_0 < \pi^2$.

Let $W(r)$ and $V(t)$ be both smooth, non-negative and real valued functions, with $W$ taking positive real arguments and supported on $r \in (\frac{1}{2}, 2)$ and $V$ taking real arguments and supported for $t \in [-1, 1]$. These functions are assumed to satisfy the following admissibility conditions.

$$\int_{\frac{1}{2}}^{2} (W(r))^2 \frac{dr}{r} = 1 \tag{1}$$

$$\int_{-1}^{1} (V(t))^2 \, dt = 1. \tag{2}$$

At scale $0 < a < b_0$, location $\mathbf{b} \in \mathbb{R}^2$ and orientation $\theta \in [0, 2\pi)$, for every $r\mathbf{e}^{i\omega} \in \mathbb{R}^2 \setminus \{\mathbf{0}\}$ with $-\pi < \omega \leq \pi$, Candès and Donoho defined

$$\hat{\gamma}_{a,\mathbf{0},0}\left(r\mathbf{e}^{i\omega}\right) = W(a\,r)V(\omega/\sqrt{a})a^{3/4}, \quad 0 < a < b_0.$$

They also defined the family of curvelets by $\gamma_{a,\mathbf{b},\theta}$, which is generated by translation and rotation of any element $\gamma_{a,\mathbf{0},0}$ such that

$$\gamma_{a,\mathbf{b},\theta}(\mathbf{x}) = \gamma_{a,\mathbf{0},0}(\mathsf{R}_\theta(\mathbf{x} - \mathbf{b})), \forall \mathbf{x} \in \mathbb{R}^2,$$

where $\mathsf{R}_\theta(\mathbf{x} - \mathbf{b})$ is simply the product of the two complex numbers $\mathbf{e}^{-i\theta}$ and $\mathbf{x} - \mathbf{b}$.

**Definition 2.1.** *The curvelet transform of $f \in \mathscr{L}^2(\mathbb{R}^2)$ is defined by*

$$(\Gamma f)(a, \mathbf{b}, \theta) = \int_{\mathbb{R}^2} \overline{\gamma_{a,\mathbf{b},\theta}(\mathbf{x})} f(\mathbf{x}) \, d\mathbf{x}, \ \forall (a, \mathbf{b}, \theta) \in \mathbb{S}_{a_0,b_0},$$

*where* $\mathbb{S}_{a_0,b_0} = [a_0, b_0] \times \mathbb{R}^2 \times [0, 2\pi] \subseteq \mathbb{R}^4$.

According to [1, 2], in the definition of the curvelet transform $(\Gamma f)(a, \mathbf{b}, \theta)$, the scaling parameter $a$ varies in $(0, a_0)$. In the above definition, we have slightly modified the range of $a$ as $[a_0, b_0]$ for the purpose of proving the continuity of the curvelet transform and its adjoint in the context of rapidly decreasing functions. One can note that this change does not make big difference in both theory and applications of the transform. Practically, even the scaling parameter is very fine, it cannot be zero and hence this modification with small $a_0$ may be considered in that case. In theoretical point of view, all results proved in [1, 2] can be obtained even after this modification. Indeed, the inversion formula and Parseval's identity are modified as follows.

**Theorem 2.2.** *The inversion formula of the curvelet transform is obtained by* $f = (\Gamma^*\Gamma)(f)$, $\forall f \in \mathscr{L}^2_{a_0,b_0}(\mathbb{R}^2)$, *where the adjoint curvelet transform* $\Gamma^* F$ *is defined by*

$$(\Gamma^* F)(\mathbf{x}) = \int\limits_{\mathbb{S}_{a_0,b_0}} F(a,\mathbf{b},\theta)\gamma_{a,\mathbf{b},\theta}(\mathbf{x})\frac{da}{a^3}d\mathbf{b}d\theta,\ \forall \mathbf{x}\in\mathbb{R}^2,\ \forall F\in\mathfrak{L}^2(\mathbb{S}_{a_0,b_0}),$$

*where* $\mathfrak{L}^2(\mathbb{S}_{a_0,b_0})$ *is the space of all functions* $F : \mathbb{S}_{a_0,b_0} \to \mathbb{C}$ *satisfying* $|||F|||_2^2 = \int\limits_{\mathbb{S}_{a_0,b_0}} |F(a,\mathbf{b},\theta)|^2 \frac{da}{a^3} d\mathbf{b}d\theta < +\infty$. *Under the same assumption, the Parseval's identity,* $\|f\|_2 = |||\Gamma f|||_2$ *also holds.*

By the above theorem, the curvelet transform is an isometric isomorphism from $\mathscr{L}^2_{a_0,b_0}(\mathbb{R}^2)$ into $\mathfrak{L}^2(\mathbb{S}_{a_0,b_0})$. Already, the authors of this paper pointed out a small mistake in the original version of this theorem presented in [2], and corrected the condition "$\hat{f}(\xi) = 0$, if $|\xi| < \frac{2}{b_0}$." as "$\hat{f}(r\mathbf{e}^{i\omega}) = 0$, if $0 < r < \frac{2}{b_0}$ or $-\pi < \omega < \sqrt{b_0}$." For more details we refer the reader to [9].

We recall that $\mathscr{S}(\mathbb{R}^2)$ is a Fréchet space, equipped with the topology induced by the following sequence of semi-norms,

$$\|f\|_{m,\beta} = \sup_{\mathbf{x}\in\mathbb{R}^2} (1+|\mathbf{x}|)^m \left|D^\beta_{\mathbf{x}} f(\mathbf{x})\right|, \forall f\in\mathscr{S}(\mathbb{R}^2),\ \forall m\in\mathbb{N}_0,\ \forall\beta\in\mathbb{N}_0^2.$$

We denote the subspace $\mathscr{S}(\mathbb{R}^2)\cap\mathscr{L}^2_{a_0,b_0}(\mathbb{R}^2)$ of $\mathscr{S}(\mathbb{R}^2)$ by $\mathscr{S}_{a_0,b_0}(\mathbb{R}^2)$. We need another space $\mathscr{S}(\mathbb{S}_{a_0,b_0})$ of all functions $F\in C^\infty(\mathbb{S}_{a_0,b_0})$ such that

$$|||F|||_{\alpha,k,\beta,n} = \sup_{(a,\mathbf{b},\theta)\in\mathbb{S}_{a_0,b_0}} \left|\mathbf{b}^\alpha D^{k,\beta,n}_{a,\mathbf{b},\theta} F(a,\mathbf{b},\theta)\right| < \infty.$$

for all $\alpha,\beta\in\mathbb{N}_0^2$ and $k,n\in\mathbb{N}_0$. Clearly, $\mathscr{S}(\mathbb{S}_{a_0,b_0})$ is also a Fréchet space with the topology given by the countable family of semi-norms $\{|||\cdot|||_{\alpha,k,\beta,n} : \alpha,\beta\in\mathbb{N}_0^2;\ k,n\in\mathbb{N}_0\}$. We now recall the multivariate Faa di Bruno formula [4], which will be applied in the proofs of main results of this paper.

$$D^n_t f(g(t)) = \sum_{FDB(n)} \Lambda_{(\lambda_j)} f^{(\lambda)}(g(t))\,(D_t g(t))^{\lambda_1}\cdots(D^n_t g(t))^{\lambda_n},$$

where $\Lambda_{(\lambda_j)} = \frac{n!}{\lambda_1!\lambda_2!\ldots\lambda_n!1!^{\lambda_1}2!^{\lambda_2}\cdots n!^{\lambda_n}}$, $\lambda = \lambda_1+\lambda_2+\cdots+\lambda_n$ and the sum $\sum\limits_{FDB(n)}$ is over all non negative values of $\lambda_1,\lambda_2,\cdots\lambda_n$ such that $\lambda_1+2\lambda_2+\cdots+n\lambda_n = n$.

## 3. Continuity of curvelet transforms on rapidly decreasing functions

First we recall the following lemma from [9].

**Lemma 3.1.** *For* $a\in(0,b_0)$ *and* $\theta\in[0,2\pi]$,

$$\hat{\gamma}_{a,\mathbf{0},\theta}(r\mathbf{e}^{i\omega}) = \hat{\gamma}_{a,\mathbf{0},0}(r\mathbf{e}^{i(\omega-\theta)}),\ \forall r\mathbf{e}^{i\omega}\in\mathbb{R}^2\setminus\{\mathbf{0}\}.$$

**Lemma 3.2.** *For* $(a,\mathbf{b},\theta)\in\mathbb{S}_{a_0,b_0}$, $\hat{\gamma}_{a,\mathbf{b},\theta}(\mathbf{y}) = \mathbf{e}^{-i\langle\mathbf{b},\mathbf{y}\rangle}\hat{\gamma}_{a,\mathbf{0},0}\left(\mathbf{y}\mathbf{e}^{-i\theta}\right)$.

*Proof.* Let $\mathbf{y} = r\mathbf{e}^{i\omega}$. Using previous lemma, we get

$$\begin{aligned}
\hat{\gamma}_{a,\mathbf{b},\theta}(\mathbf{y}) &= \hat{\gamma}_{a,\mathbf{b},\theta}\left(r\mathbf{e}^{i\omega}\right) \\
&= \int_{\mathbb{R}^2} \mathbf{e}^{-i\left\langle \rho\mathbf{e}^{i\zeta}, r\mathbf{e}^{i\omega}\right\rangle}\gamma_{a,\mathbf{b},\theta}(\rho\mathbf{e}^{i\zeta})d(\rho\mathbf{e}^{i\zeta}) \\
&= \int_{\mathbb{R}^2} \mathbf{e}^{-i\left\langle \rho\mathbf{e}^{i\zeta}, r\mathbf{e}^{i\omega}\right\rangle}\gamma_{a,\mathbf{0},0}(\mathbf{e}^{-i\theta}(\rho\mathbf{e}^{i\zeta} - \mathbf{b}))d(\rho\mathbf{e}^{i\zeta}) \\
&= \int_{\mathbb{R}^2} \mathbf{e}^{-i\left\langle \rho\mathbf{e}^{i\zeta}+\mathbf{b}, r\mathbf{e}^{i\omega}\right\rangle}\gamma_{a,\mathbf{0},0}(\mathbf{e}^{-i\theta}(\rho\mathbf{e}^{i\zeta}))d(\rho\mathbf{e}^{i\zeta}) \\
&= \mathbf{e}^{-i\langle \mathbf{b},\mathbf{y}\rangle}\int_{\mathbb{R}^2} \mathbf{e}^{-i\left\langle \rho\mathbf{e}^{i\zeta}, r\mathbf{e}^{i\omega}\right\rangle}\gamma_{a,\mathbf{0},0}(\mathbf{e}^{-i\theta}(\rho\mathbf{e}^{i\zeta}))d(\rho\mathbf{e}^{i\zeta}) \\
&= \mathbf{e}^{-i\langle \mathbf{b},\mathbf{y}\rangle}\int_{\mathbb{R}^2} \mathbf{e}^{-i\left\langle \rho\mathbf{e}^{i\zeta}, r\mathbf{e}^{i\omega}\right\rangle}\gamma_{a,\mathbf{0},\theta}(\rho\mathbf{e}^{i\zeta})d(\rho\mathbf{e}^{i\zeta}) \quad \text{(by Lemma 3.1)} \\
&= \mathbf{e}^{-i\langle \mathbf{b},\mathbf{y}\rangle}\hat{\gamma}_{a,\mathbf{0},\theta}\left(r\mathbf{e}^{i\omega}\right) \\
&= \mathbf{e}^{-i\langle \mathbf{b},\mathbf{y}\rangle}\hat{\gamma}_{a,\mathbf{0},0}\left(r\mathbf{e}^{i(\omega-\theta)}\right) \\
&= \mathbf{e}^{-i\langle \mathbf{b},\mathbf{y}\rangle}\hat{\gamma}_{a,\mathbf{0},0}\left(\mathbf{y}\mathbf{e}^{-i\theta}\right).
\end{aligned}$$

Hence the lemma follows. □

**Theorem 3.3.** *If $f \in \mathscr{S}(\mathbb{R}^2)$, then $\Gamma f \in \mathscr{S}(\mathbb{S}_{a_0,b_0})$ and $\Gamma : \mathscr{S}(\mathbb{R}^2) \to \mathscr{S}(\mathbb{S}_{a_0,b_0})$ is continuous.*

*Proof.* Let $\alpha$ , $\beta \in \mathbb{N}_0^2$, $k, n \in \mathbb{N}_0$, $f \in \mathscr{S}(\mathbb{R}^2)$. By applying the Parseval's identity for Fourier transform and Lemma 3.2, we first write

$$\begin{aligned}
(\Gamma f)(a, \mathbf{b}, \theta) &= (2\pi)^2 \int_{\mathbb{R}^2} \overline{\hat{\gamma}_{a,\mathbf{b},\theta}(\mathbf{y})}\hat{f}(\mathbf{y})d\mathbf{y} \\
&= (2\pi)^2 \int_{\mathbb{R}^2} \mathbf{e}^{i\langle \mathbf{b},\mathbf{y}\rangle}\overline{\hat{\gamma}_{a,\mathbf{0},0}(\mathbf{y}\mathbf{e}^{-i\theta})}\hat{f}(\mathbf{y})d\mathbf{y}.
\end{aligned}$$

Since for every $n \in \mathbb{N}_0$,

$$\begin{aligned}
&\frac{\partial^n}{\partial\theta^n}\overline{\hat{\gamma}_{a,\mathbf{0},0}}\left(\mathbf{e}^{-i\theta}\mathbf{y}\right) \\
&= \sum_{FDB(n)} \Lambda_{(\lambda_j)}\left((\overline{\hat{\gamma}_{a,\mathbf{0},0}})^{(\lambda)}(\mathbf{e}^{-i\theta}\mathbf{y})\right)\prod_{k=1}^{n}\left(D_\theta^k\left(\mathbf{e}^{-i\theta}\mathbf{y}\right)\right)^{\lambda_k} \\
&= \sum_{FDB(n)} \Lambda_{(\lambda_j)}(-i)^\lambda\left[(-\mathbf{x})^\lambda\overline{\gamma_{a,\mathbf{0},0}}(-\mathbf{x})\right]^\wedge(\mathbf{e}^{-i\theta}\mathbf{y})\prod_{k=1}^{n}\left(\mathbf{e}^{-i\theta}\mathbf{y}(-i)^k\right)^{\lambda_k} \\
&= \sum_{FDB(n)} \Lambda_{(\lambda_j)}\left[(-\mathbf{x})^\lambda\overline{\gamma_{a,\mathbf{0},0}}(-\mathbf{x})\right]^\wedge(\mathbf{e}^{-i\theta}\mathbf{y})(-i)^{\lambda+n}(\mathbf{y}\mathbf{e}^{-i\theta})^\lambda,
\end{aligned}$$

we have

$$\begin{aligned}
&\left|\mathbf{b}^\alpha D_{a,\mathbf{b},\theta}^{k,\beta,n}(\Gamma f)(a, \mathbf{b}, \theta)\right| \\
&= (2\pi)^2\left|\mathbf{b}^\alpha D_{a,\mathbf{b}}^{k,\beta}\int_{\mathbb{R}^2}\mathbf{e}^{i\langle \mathbf{b},\mathbf{y}\rangle}\sum_{FDB(n)}\Lambda_{(\lambda_j)}\left[\mathbf{x}^\lambda\overline{\gamma_{a,\mathbf{0},0}}(-\mathbf{x})\right]^\wedge(\mathbf{e}^{-i\theta}\mathbf{y})\mathbf{y}^\lambda\hat{f}(\mathbf{y})d\mathbf{y}\right| \\
&= (2\pi)^2\left|\sum_{FDB(n)}\Lambda_{(\lambda_j)}\frac{\partial^k}{\partial a^k}\mathbf{b}^\alpha\int_{\mathbb{R}^2}\mathbf{e}^{i\langle \mathbf{b},\mathbf{y}\rangle}\mathbf{y}^{\beta+\lambda}\left(\mathbf{x}^\lambda\overline{\gamma_{a,\mathbf{0},0}}(-\mathbf{x})\right)^\wedge(\mathbf{e}^{-i\theta}\mathbf{y})\hat{f}(\mathbf{y})d\mathbf{y}\right|
\end{aligned}$$

$$= (2\pi)^2 \left| \sum_{FDB(n)} \Lambda_{(\lambda_j)} \frac{\partial^k}{\partial a^k} \int_{\mathbb{R}^2} \mathbf{e}^{i\langle \mathbf{b},\mathbf{y}\rangle} D^\alpha_{\mathbf{y}} \left[ \mathbf{y}^{\beta+\lambda} \hat{f}(\mathbf{y}) \left( \mathbf{x}^\lambda \overline{\gamma_{a,\mathbf{0},0}}(-\mathbf{x}) \right)^\wedge (\mathbf{e}^{-i\theta}\mathbf{y}) \right] d\mathbf{y} \right|$$

$$= (2\pi)^2 \left| \frac{\partial^k}{\partial a^k} \sum_{FDB(n)} \Lambda_{(\lambda_j)} \int_{\mathbb{R}^2} \mathbf{e}^{i\langle \mathbf{b},\mathbf{y}\rangle} \sum_{\delta\le\alpha} \binom{\alpha}{\delta} D^{\alpha-\delta}_{\mathbf{y}} (\mathbf{y}^{\beta+\lambda} \hat{f}(\mathbf{y})) \right.$$
$$\left. D^\delta_{\mathbf{y}} \left[ \mathbf{x}^\lambda \overline{\gamma_{a,\mathbf{0},0}}(-\mathbf{x}) \right]^\wedge (\mathbf{y}\mathbf{e}^{-i\theta}) d\mathbf{y} \right|. \tag{3}$$

Next we note that

$$\begin{aligned} & D^\delta_{\mathbf{y}} [\mathbf{x}^\lambda \overline{\gamma_{a,\mathbf{0},0}}(-\mathbf{x})]^\wedge (\mathbf{e}^{-i\theta}\mathbf{y}) \\ = & \frac{\partial^{\delta_1}}{\partial y_1^{\delta_1}} \frac{\partial^{\delta_2}}{\partial y_2^{\delta_2}} [\mathbf{x}^\lambda \overline{\gamma_{a,\mathbf{0},0}}(-\mathbf{x})]^\wedge (\mathbf{e}^{-i\theta}(y_1 + iy_2)) \\ = & \left[ (\mathbf{x}^\lambda \overline{\gamma_{a,\mathbf{0},0}}(-\mathbf{x}))^{(\delta)} \right]^\wedge (\mathbf{e}^{-i\theta}\mathbf{y}) (\sin\theta + i\cos\theta)^{\delta_2} (\cos\theta - i\sin\theta)^{\delta_1} \\ = & (-i)^{|\delta|} (\sin\theta + i\cos\theta)^{\delta_2} (\cos\theta - i\sin\theta)^{\delta_1} [\mathbf{x}^{\delta+\lambda} \overline{\gamma_{a,\mathbf{0},0}}(-\mathbf{x})]^\wedge (\mathbf{e}^{-i\theta}\mathbf{y}). \end{aligned}$$

Using this expression in (3), we get

$$\begin{aligned} & \left| \mathbf{b}^\alpha D^{k,\beta,n}_{a,\mathbf{b},\theta} (\Gamma f)(a,\mathbf{b},\theta) \right| \\ \le & (2\pi)^2 \sum_{FDB(n)} \Lambda_{(\lambda_j)} \sum_{\delta\le\alpha} \binom{\alpha}{\delta} \\ & \int_{\mathbb{R}^2} \left| D^{\alpha-\delta}_{\mathbf{y}} (\mathbf{y}^{\beta+\lambda} \hat{f}(\mathbf{y})) \frac{\partial^k}{\partial a^k} [\mathbf{x}^{\delta+\lambda} \overline{\gamma_{a,\mathbf{0},0}}(-\mathbf{x})]^\wedge (\mathbf{e}^{-i\theta}\mathbf{y}) \right| d\mathbf{y} \\ = & (2\pi)^2 \sum_{FDB(n)} \Lambda_{(\lambda_j)} \sum_{\delta\le\alpha} \binom{\alpha}{\delta} \sum_{\rho\le\alpha-\delta} \binom{\alpha-\delta}{\rho} \\ & \int_{\mathbb{R}^2} \left| D^{\alpha-\delta-\rho}_{\mathbf{y}} \hat{f}(\mathbf{y}) \right| \left| \frac{\partial^k}{\partial a^k} D^\rho_{\mathbf{y}} \mathbf{y}^{\beta+\lambda} \left[ \mathbf{x}^{\delta+\lambda} \overline{\gamma_{a,\mathbf{0},0}(-\mathbf{x})} \right]^\wedge (\mathbf{e}^{-i\theta}\mathbf{y}) \right| d\mathbf{y} \\ = & (2\pi)^2 \sum_{FDB(n)} \Lambda_{(\lambda_j)} \sum_{\delta\le\alpha} \binom{\alpha}{\delta} \sum_{\rho\le\min(\alpha-\delta,\beta+\lambda)} \binom{\alpha-\delta}{\rho} \\ & \int_{\mathbb{R}^2} \left| D^{\alpha-\delta-\rho}_{\mathbf{y}} \hat{f}(\mathbf{y}) \right| \left| D^\rho_{\mathbf{y}} \mathbf{y}^{\beta+\lambda} \frac{\partial^k}{\partial a^k} \left[ \mathbf{x}^{\delta+\lambda} \overline{\gamma_{a,\mathbf{0},0}(-\mathbf{x})} \right]^\wedge (\mathbf{e}^{-i\theta}\mathbf{y}) \right| d\mathbf{y} \\ & \left( \text{using } D^\rho_{\mathbf{y}} \mathbf{y}^{\beta+\lambda} = \begin{cases} \frac{(\beta+\lambda)!}{(\beta+\lambda-\rho)!} \mathbf{y}^{\beta+\lambda-\rho}, & \text{if } \rho \le \beta+\lambda \\ 0, & \text{otherwise} \end{cases} \right) \\ \le & (2\pi)^2 \sum_{FDB(n)} \Lambda_{(\lambda_j)} \sum_{\delta\le\alpha} \binom{\alpha}{\delta} \sum_{\rho\le\min(\alpha-\delta,\beta+\lambda)} \binom{\alpha-\delta}{\rho} \frac{(\beta+\lambda)!}{(\beta+\lambda-\rho)!} \\ & \int_{\mathbb{R}^2} |\mathbf{y}|^{|\beta+\lambda-\rho|} \left| D^{\alpha-\delta-\rho}_{\mathbf{y}} \hat{f}(\mathbf{y}) \right| d\mathbf{y} \int_{\mathbb{R}^2} |\mathbf{x}|^{|\delta+\lambda|} \left| \frac{\partial^k}{\partial a^k} \bar{\check{\gamma}}_{a,\mathbf{0},0}(\mathbf{x}) \right| d\mathbf{x} \\ \le & (2\pi)^2 \sum_{FDB(n)} \Lambda_{(\lambda_j)} \sum_{\delta\le\alpha} \binom{\alpha}{\delta} \sum_{\rho\le\min(\alpha-\delta,\beta+\lambda)} \binom{\alpha-\delta}{\rho} \frac{(\beta+\lambda)!}{(\beta+\lambda-\rho)!} \\ & \int_{\mathbb{R}^2} (1+|\mathbf{y}|)^{|\beta|+|\lambda|-|\rho|+2} \left| D^{\alpha-\delta-\rho}_{\mathbf{y}} \hat{f}(\mathbf{y}) \right| \frac{d\mathbf{y}}{(1+|\mathbf{y}|)^2} \\ & \int_{\mathbb{R}^2} (1+|\mathbf{x}|)^{|\delta|+|\lambda|+2} \left| \frac{\partial^k}{\partial a^k} \bar{\check{\gamma}}_{a,\mathbf{0},0}(\mathbf{x}) \right| \frac{d\mathbf{x}}{(1+|\mathbf{x}|)^2} \\ \le & (2\pi)^2 \sum_{FDB(n)} \Lambda_{(\lambda_j)} \sum_{\delta\le\alpha} \binom{\alpha}{\delta} \sum_{\rho\le\min(\alpha-\delta,\beta+\lambda)} \binom{\alpha-\delta}{\rho} \frac{(\beta+\lambda)!}{(\beta+\lambda-\rho)!} \\ & \left\| \hat{f} \right\|_{|\beta|+|\lambda|-|\rho|+2,\alpha-\delta-\rho} \left\| \frac{\partial^k}{\partial a^k} \bar{\check{\gamma}}_{a,\mathbf{0},0} \right\|_{|\delta|+|\lambda|+2,0} \left( \int_{\mathbb{R}^2} \frac{d\mathbf{y}}{1+|\mathbf{y}|^2} \right)^2. \end{aligned}$$

Then

$$|||\Gamma f|||_{\alpha,k,\beta,n} = \sup_{(a,\mathbf{b},\theta)\in\mathbb{S}_{a_0,b_0}} \left| \mathbf{b}^\alpha D^{k,\beta,n}_{a,\mathbf{b},\theta} (\Gamma f)(a,\mathbf{b},\theta) \right|$$

$$\leq (2\pi)^2 \sum_{FDB(n)} \Lambda_{(\lambda_j)} \sum_{\delta\leq\alpha} \binom{\alpha}{\delta} \sum_{\rho\leq\min(\alpha-\delta,\beta+\lambda)} \binom{\alpha-\delta}{\rho} \frac{(\beta+\lambda)!}{(\beta+\lambda-\rho)!}$$

$$\left\|\hat{f}\right\|_{|\beta|+|\lambda|-|\rho|+2,\alpha-\delta-\rho} \sup_{a\in[a_0,b_0]} \left\|\frac{\partial^k}{\partial a^k}\bar{\hat{\gamma}}_{a,\mathbf{0},0}\right\|_{|\delta|+|\lambda|+2,0} \left(\int_{\mathbb{R}^2} \frac{d\mathbf{y}}{1+|\mathbf{y}|^2}\right)^2. \tag{4}$$

Next we show that $\sup_{(a,\mathbf{b},\theta)\in\mathbb{S}_{a_0,b_0}} \left\|\frac{\partial^k}{\partial a^k}\bar{\hat{\gamma}}_{a,\mathbf{0},0}\right\|_{N,0} < \infty$ for all $N \in \mathbb{N}$.

Let $N \in \mathbb{N}_0$. Using the continuity of Fourier transform on Schwartz space, we can find $M \in \mathbb{N}_0$, $P \in \mathbb{N}_0^2$ and $C > 0$ such that

$$\begin{aligned}
&\sup_{a\in[a_0,b_0]} \left\|\frac{\partial^k}{\partial a^k}\bar{\hat{\gamma}}_{a,\mathbf{0},0}\right\|_{N,0} \\
&\leq C \sup_{a\in[a_0,b_0]} \left\|\frac{\partial^k}{\partial a^k}\hat{\bar{\hat{\gamma}}}_{a,\mathbf{0},0}\right\|_{M,P} = C \sup_{a\in[a_0,b_0]} \left\|\frac{\partial^k}{\partial a^k}\bar{\hat{\gamma}}_{a,\mathbf{0},0}\right\|_{M,P} \\
&\leq C \sup_{a\in[a_0,b_0]} \sup_{\mathbf{x}\in\mathbb{R}^2} (1+|\mathbf{x}|)^M \left|D_{\mathbf{x}}^P \frac{\partial^k}{\partial a^k}\bar{\hat{\gamma}}_{a,\mathbf{0},0}(\mathbf{x})\right| < +\infty,
\end{aligned}$$

since $(a,\mathbf{x}) \mapsto D_{\mathbf{x}}^P \frac{\partial^k}{\partial a^k}\bar{\hat{\gamma}}_{a,\mathbf{0},0}(\mathbf{x})$ is a smooth function with compact support.

Now using the continuity of Fourier transform on $\mathscr{S}(\mathbb{R}^2)$ and (4), we get that the curvelet transform $\Gamma : \mathscr{S}(\mathbb{R}^2) \to \mathscr{S}(\mathbb{S}_{a_0,b_0})$ is continuous. □

**Theorem 3.4.** *If $F \in \mathscr{S}(\mathbb{S}_{a_0,b_0})$ then $\Gamma^*F \in \mathscr{S}(\mathbb{R}^2)$ and $\Gamma^* : \mathscr{S}(\mathbb{S}_{a_0,b_0}) \to \mathscr{S}(\mathbb{R}^2)$ is continuous.*

*Proof.* Let $\alpha, \beta \in \mathbb{N}_0^2$ and $F \in \mathscr{S}(\mathbb{S}_{a_0,b_0})$.

$$\begin{aligned}
&\left|\mathbf{x}^\alpha D_{\mathbf{x}}^\beta (\Gamma^*F)(\mathbf{x})\right| = \left|\mathbf{x}^\alpha \int_{\mathbb{S}_{a_0,b_0}} F(a,\mathbf{b},\theta) D_{\mathbf{x}}^\beta \gamma_{a,\mathbf{b},\theta}(\mathbf{x}) \frac{da}{a^3} d\mathbf{b} d\theta\right| \\
&\leq \int_{\mathbb{S}_{a_0,b_0}} |F(a,\mathbf{b},\theta)| \left|\mathbf{x}^\alpha \gamma_{a,\mathbf{0},0}^{(\beta)}(\mathbf{e}^{-i\theta}(\mathbf{x}-\mathbf{b}))\right| \frac{da}{a^3} d\mathbf{b} d\theta \\
&\leq \int_{\mathbb{S}_{a_0,b_0}} |F(a,\mathbf{b},\theta)| (|\mathbf{x}-\mathbf{b}|+|\mathbf{b}|)^{|\alpha|} \left|\gamma_{a,\mathbf{0},0}^{(\beta)}(\mathbf{e}^{-i\theta}(\mathbf{x}-\mathbf{b}))\right| \frac{da}{a^3} d\mathbf{b} d\theta \\
&\leq 2^{|\alpha|-1} \int_{\mathbb{S}_{a_0,b_0}} |F(a,\mathbf{b},\theta)| \left(|\mathbf{x}-\mathbf{b}|^{|\alpha|}+|\mathbf{b}|^{|\alpha|}\right) \left|\gamma_{a,\mathbf{0},0}^{(\beta)}(\mathbf{e}^{-i\theta}(\mathbf{x}-\mathbf{b}))\right| \frac{da}{a^3} d\mathbf{b} d\theta \\
&\leq 2^{|\alpha|-1} \int_{\mathbb{S}_{a_0,b_0}} |F(a,\mathbf{b},\theta)| \left(1+\left|\mathbf{e}^{-i\theta}(\mathbf{x}-\mathbf{b})\right|\right)^{|\alpha|} \left|\gamma_{a,\mathbf{0},0}^{(\beta)}(\mathbf{e}^{-i\theta}(\mathbf{x}-\mathbf{b}))\right| \frac{da}{a^3} d\mathbf{b} d\theta \\
&\quad +2^{|\alpha|-1} \int_{\mathbb{S}_{a_0,b_0}} |F(a,\mathbf{b},\theta)| (1+|\mathbf{b}|)^{|\alpha|} \left|\gamma_{a,\mathbf{0},0}^{(\beta)}(\mathbf{e}^{-i\theta}(\mathbf{x}-\mathbf{b}))\right| \frac{da}{a^3} d\mathbf{b} d\theta \\
&\leq 2^{|\alpha|-1} \int_{\mathbb{S}_{a_0,b_0}} |||F|||_{2,0,0,0} \|\gamma_{a,\mathbf{0},0}\|_{N,\beta} \frac{da}{a^3} \frac{d\mathbf{b}}{(1+|\mathbf{b}|)^2} d\theta \\
&\quad +2^{|\alpha|-1} \int_{\mathbb{S}_{a_0,b_0}} |||F|||_{2+\alpha,0,0,0} \|\gamma_{a,\mathbf{0},0}\|_{0,\beta} \frac{da}{a^3} \frac{d\mathbf{b}}{(1+|\mathbf{b}|)^2} d\theta \\
&\leq \left(|||F|||_{2,0,0,0} \|\gamma_{a,\mathbf{0},0}\|_{N,\beta} + |||F|||_{2+\alpha,0,0,0} \|\gamma_{a,\mathbf{0},0}\|_{0,\beta}\right) \\
&\quad 2^{|\alpha|-1} \int_{a_0}^{b_0} \int_{\mathbb{R}^2} \int_0^{2\pi} \frac{da}{a^3} \frac{d\mathbf{b}}{(1+|\mathbf{b}|)^2} d\theta < +\infty
\end{aligned}$$

Thus, $\Gamma^*F \in \mathscr{S}(\mathbb{R}^2)$ and $\Gamma^* : \mathscr{S}(\mathbb{S}_{a_0,b_0}) \to \mathscr{S}(\mathbb{R}^2)$ is continuous. □

In the next section, we shall prove that $\Gamma u \in \mathscr{S}'(\mathbb{S}_{a_0,b_0})$, $\forall u \in \mathscr{S}'(\mathbb{R}^2)$ using the continuity of $\Gamma^* : \mathscr{S}(\mathbb{S}_{a_0,b_0}) \to \mathscr{S}(\mathbb{R}^2)$, and that $\Gamma^*\Lambda \in \mathscr{S}'(\mathbb{R}^2)$, $\forall \Lambda \in \mathscr{S}'(\mathbb{S}_{a_0,b_0})$ using the continuity of $\Gamma : \mathscr{S}(\mathbb{R}^2) \to \mathscr{S}(\mathbb{S}_{a_0,b_0})$.

## 4. Curvelet transform on $\mathscr{S}'(\mathbb{R}^2)$

The dual spaces of $\mathscr{S}(\mathbb{R}^2)$ and $\mathscr{S}(\mathbb{S}_{a_0,b_0})$ are respectively, denoted by $\mathscr{S}'(\mathbb{R}^2)$ and $\mathscr{S}'(\mathbb{S}_{a_0,b_0})$. Throughout this section, we use the weak*-topology on $A'$. With respect to this topology, we can write

$w_n \to w$ as $n \to \infty$ in $A'$ whenever $w_n(h) \to w(h)$ in $\mathbb{C}$ as $n \to \infty$, $\forall h \in A$,

where $A \in \{\mathscr{S}(\mathbb{R}^2)\ ,\ \mathscr{S}(\mathbb{S}_{a_0,b_0})\}$. We also say that a function $\Phi : A' \to B'$ is said to be continuous iff $\Phi(w_n) \to \Phi(w)$ in $B'$ as $n \to \infty$ whenever $w_n \to w$ as $n \to \infty$ in $A'$, where $A, B \in \{\mathscr{S}(\mathbb{R}^2), \mathscr{S}(\mathbb{S}_{a_0,b_0})\}$.

**Definition 4.1.** *We define* $\Gamma$ *on* $\mathscr{S}'(\mathbb{R}^2)$ *by* $(\Gamma u)(F) = u(\Gamma^* F)$, $F \in \mathscr{S}(\mathbb{S}_{a_0,b_0})$.

By using the linearity of $\Gamma^*$ on $\mathscr{S}(\mathbb{S}_{a_0,b_0})$ and the linearity of $u$ on $\mathscr{S}(\mathbb{R}^2)$, it follows that $\Gamma u$ is linear on $\mathscr{S}(\mathbb{S}_{a_0,b_0})$. As a consequence of Theorem 3.4, we have

$$\Gamma^* F_n \to \Gamma^* F \text{ in } \mathscr{S}(\mathbb{R}^2) \text{ whenever } F_n \to F \text{ in } \mathscr{S}(\mathbb{S}_{a_0,b_0}) \text{ as } n \to \infty.$$

Therefore, $(\Gamma u)(F_n) = u(\Gamma^* F_n) \to u(\Gamma^* F) = (\Gamma u)(F)$, whenever $F_n \to F$ in $\mathscr{S}(\mathbb{S}_{a_0,b_0})$ as $n \to \infty$. Thus $\Gamma u \in \mathscr{S}'(\mathbb{S}_{a_0,b_0})$.

**Theorem 4.2.** *The curvelet transform* $\Gamma : \mathscr{S}'(\mathbb{R}^2) \to \mathscr{S}'(\mathbb{S}_{a_0,b_0})$ *is consistent with the curvelet transform on* $\mathscr{S}(\mathbb{R}^2)$.

*Proof.* Let $f \in \mathscr{L}^2(\mathbb{R}^2)$ and $F \in \mathscr{S}(\mathbb{S}_{a_0,b_0})$. Then $f$ can be considered in a natural way, as a member of $\mathscr{S}'(\mathbb{R}^2)$ by $f(g) = \int\limits_{\mathbb{R}^2} f(\mathbf{x})g(\mathbf{x})d\mathbf{x}$, $\forall g \in \mathscr{S}(\mathbb{R}^2)$.

By applying Fubini's theorem, we get

$$\begin{aligned}
(\Gamma f)(F) &= f(\Gamma^* F) \\
&= \int\limits_{\mathbb{R}^2} f(\mathbf{x}) \int\limits_0^{2\pi} \int\limits_{\mathbb{R}^2} \int\limits_{a_0}^{b_0} \overline{\gamma_{a,\mathbf{b},\theta}(\mathbf{x})} F(a,\mathbf{b},\theta) \tfrac{da}{a^3} d\mathbf{b} d\theta d\mathbf{x} \\
&= \int\limits_{\mathbb{R}^2} f(\mathbf{x}) \int\limits_0^{2\pi} \int\limits_{\mathbb{R}^2} \int\limits_{a_0}^{b_0} \overline{\gamma_{a,\mathbf{b},\theta}}(\mathbf{x}) F(a,\mathbf{b},\theta) \tfrac{da}{a^3} d\mathbf{b} d\theta d\mathbf{x} \\
&= \int\limits_0^{2\pi} \int\limits_{\mathbb{R}^2} \int\limits_{a_0}^{b_0} \int\limits_{\mathbb{R}^2} f(\mathbf{x}) \overline{\gamma_{a,\mathbf{b},\theta}(\mathbf{x})} d\mathbf{x} F(a,\mathbf{b},\theta) \tfrac{da}{a^3} d\mathbf{b} d\theta \\
&= \int\limits_0^{2\pi} \int\limits_{\mathbb{R}^2} \int\limits_{a_0}^{b_0} (\Gamma f)(a,\mathbf{b},\theta) F(a,\mathbf{b},\theta) \tfrac{da}{a^3} d\mathbf{b} d\theta,
\end{aligned}$$

which is the identification of $\Gamma f$ in $\mathscr{S}'(\mathbb{S}_{a_0,b_0})$. Hence the distributional curvelet transform on $\mathscr{S}'(\mathbb{R}^2)$ is consistent with the classical curvelet transform on $\mathscr{S}(\mathbb{R}^2)$. □

**Definition 4.3.** *We define* $\Gamma^*$ *on* $\mathscr{S}'(\mathbb{S}_{a_0,b_0})$ *by* $(\Gamma^*\Lambda)(f) = \Lambda(\Gamma f)$, $f \in \mathscr{S}(\mathbb{R}^2)$.

One can easily observe that $\Gamma^*\Lambda$ is linear. To verify that $\Gamma^*\Lambda$ is continuous, let $f_n \to f$ in $\mathscr{S}(\mathbb{R}^2)$ as $n \to \infty$. Then, using Theorem 3.3, we get that $\Gamma f_n \to \Gamma f$ in $\mathscr{S}(\mathbb{S}_{a_0,b_0})$ as $n \to \infty$. Since $\Lambda \in \mathscr{S}'(\mathbb{S}_{a_0,b_0})$, we have

$\Lambda(\Gamma f_n) \to \Lambda(\Gamma f)$ in $\mathscr{S}(\mathbb{S}_{a_0,b_0})$ as $n \to \infty$. In other words, we have obtained that $(\Gamma^*\Lambda)(f_n) \to (\Gamma^*\Lambda)(f)$ in $\mathbb{C}$ as $n \to \infty$. Therefore, $\Gamma^*\Lambda \in \mathscr{S}'(\mathbb{R}^2)$.

**Theorem 4.4.** *The curvelet transform* $\Gamma : \mathscr{S}'(\mathbb{R}^2) \to \mathscr{S}'(\mathbb{S}_{a_0,b_0})$ *is linear.*

*Proof.* Let $u_1, u_2 \in \mathscr{S}'(\mathbb{R}^2)$ and $c_1, c_2 \in \mathbb{C}$. Then, for $F \in \mathscr{S}(\mathbb{S}_{a_0,b_0})$

$$\begin{aligned}(\Gamma(c_1u_1 + c_2u_2))(F) &= (c_1u_1 + c_2u_2)(\Gamma^*F)\\ &= c_1u_1(\Gamma^*F) + c_2u_2(\Gamma^*F)\\ &= c_1(\Gamma u_1)(F) + c_2(\Gamma u_2)(F)\\ &= (c_1\Gamma u_1 + c_2\Gamma u_2)(F).\end{aligned}$$

Hence the theorem follows. □

**Theorem 4.5.** *The curvelet transform* $\Gamma : \mathscr{S}'(\mathbb{R}^2) \to \mathscr{S}'(\mathbb{S}_{a_0,b_0})$ *is continuous.*

*Proof.* Let $u_n \to u$ as $n \to \infty$ in $\mathscr{S}'(\mathbb{R}^2)$. Then, using Theorem 3.4, we have $\Gamma^*F \in \mathscr{S}(\mathbb{R}^2)$, $\forall F \in \mathscr{S}(\mathbb{S}_{a_0,b_0})$ and hence $(\Gamma u_n)(F) - (\Gamma u)(F) = (\Gamma(u_n - u))(F) = (u_n - u)(\Gamma^*F) \to 0$ as $n \to \infty$. Hence $\Gamma$ is continuous on $\mathscr{S}'(\mathbb{R}^2)$. □

**Theorem 4.6.** *The adjoint curvelet transform* $\Gamma^* : \mathscr{S}'(\mathbb{S}_{a_0,b_0}) \to \mathscr{S}'(\mathbb{R}^2)$ *is linear.*

*Proof.* Let $\Lambda_1, \Lambda_2 \in \mathscr{S}'(\mathbb{S}_{a_0,b_0})$ and $\alpha, \beta \in \mathbb{C}$ be arbitrary. Then, for each $f \in \mathscr{S}(\mathbb{R}^2)$, we have $(\Gamma^*(\alpha\Lambda_1 + \beta\Lambda_2))(f) = (\alpha\Lambda_1 + \beta\Lambda_2)(\Gamma f) = \alpha\Lambda_1(\Gamma f) + \beta\Lambda_2(\Gamma f) = \alpha(\Gamma^*\Lambda_1)(f) + \beta(\Gamma^*\Lambda_2)(f) = (\alpha\Gamma^*(\Lambda_1) + \beta\Gamma^*(\Lambda_2))(f)$. □

**Theorem 4.7.** *The adjoint curvelet transform* $\Gamma^* : \mathscr{S}'(\mathbb{S}_{a_0,b_0}) \to \mathscr{S}'(\mathbb{R}^2)$ *is continuous.*

*Proof.* Let $\Lambda_n \to \Lambda$ in $\mathscr{S}'(\mathbb{S}_{a_0,b_0})$ as $n \to \infty$. Then, we have

$$\Lambda_n(F) \to \Lambda(F) \text{ in } \mathbb{C} \text{ as } n \to \infty,\ \forall F \in \mathscr{S}(\mathbb{S}_{a_0,b_0}). \tag{5}$$

Let $f \in \mathscr{S}(\mathbb{R}^2)$, then by Theorem 3.3, we have $\Gamma f \in \mathscr{S}(\mathbb{S}_{a_0,b_0})$. Using (5), we get $(\Gamma^*\Lambda_n)(f) = \Lambda_n(\Gamma f) \to \Lambda(\Gamma f) = (\Gamma^*\Lambda)(f)$ in $\mathbb{C}$ as $n \to \infty$. In other words, we have proved that $\Gamma^* : \mathscr{S}'(\mathbb{S}_{a_0,b_0}) \to \mathscr{S}'(\mathbb{R}^2)$ is continuous. □

**Theorem 4.8.** *The identity* $(\Gamma^* \circ \Gamma)u = u$ *holds for all* $u \in \mathscr{S}'_{a_0,b_0}(\mathbb{R}^2)$.

*Proof.* Let $u \in \mathscr{S}'_{a_0,b_0}(\mathbb{R}^2)$ and $f \in \mathscr{S}_{a_0,b_0}(\mathbb{R}^2)$ be arbitrary. By applying the inversion formula of the classical curvelet transform, we get $\langle(\Gamma^* \circ \Gamma)u, f\rangle = \langle\Gamma u, \Gamma f\rangle = \langle u, \Gamma^*\Gamma f\rangle = \langle u, f\rangle$. This completes the proof of this theorem. □

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