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A GENERALIZATION OF SYMMETRIC PROPERTY BEYOND APPELL POLYNOMIALS

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Abstract. Recently, Bayad and Komatsu gave characterizations of Appell polynomials by means of symmetric property and expressed them as linear combinations of Bernoulli and Euler polynomials. Further, they presented some interesting examples as applications. The aim of this paper is to note that their method can be generalized to not necessarily Appell polynomials so as to include, for example, higher-order Genocchi polynomials. Moreover, we provide several examples that illustrate our main result.

1. Introduction

The Bernoulli polynomials $B_n(x)$ and Euler polynomials $E_n(x)$ are respectively given by the generating function $\frac{t}{e^t-1}e^{xt} = \sum_{n\geq 0} B_n(x)\frac{t^n}{n!}$ and $\frac{2}{e^t+1}e^{xt} = \sum_{n\geq 0} E_n(x)\frac{t^n}{n!}$. When $x = 0$, $B_n = B_n(0)$ and $E_n = E_n(0)$ are called the Bernoulli and Euler numbers, respectively.

Next, we would like to go over very basic facts about umbral calculus in order to explain main results obtained in [2]. The reader refers to [10] for complete treatment, where the umbral calculus has been used in numerous problems of pure and applied mathematics, for example, see [1, 3, 4, 7, 9, 10]. Let $\mathcal{J}$ be the algebra of all formal power series in the variable t with the coefficients in the field $\mathbb{C}$ of complex numbers, namely

$$\mathcal{J} = \left\{ f(t) = \sum_{k\geq 0} a_k \frac{t^k}{k!} \mid a_k \in \mathbb{C} \right\}.$$

Let $\mathbb{P}^*$ denote the vector space of all linear functionals on $\mathbb{P}$, where $\mathbb{P} = \mathbb{C}[x]$ is the ring of polynomials in x with the coefficients in $\mathbb{C}$. For $L \in \mathbb{P}^*$ and $p(x) \in \mathbb{P}$, $\langle L|p(x)\rangle$ denotes the action of the linear functional L on $p(x)$. The linear functional $\langle f(t)|\cdot\rangle$ on $\mathbb{P}$ is defined by $\langle f(t)|x^n\rangle = a_n$, $n \geq 0$, where $f(t) = \sum_{k\geq 0} a_k \frac{t^k}{k!} \in \mathcal{J}$. For $L \in \mathbb{P}^*$, let us set $f_L(t) = \sum_{k\geq 0}\langle L|x^k\rangle \frac{t^k}{k!}$. Then, we see that $\langle f_L(t)|x^n\rangle = \langle L|x^n\rangle$, and the map $L \mapsto f_L(t)$ gives a vector space isomorphism from $\mathbb{P}^*$ to $\mathcal{J}$. Thus, $\mathcal{J}$ may be viewed as the vector space of all linear functionals on $\mathbb{P}$ as well as the algebra of formal power series in t, and so an element $f(t)$ of $\mathcal{J}$ will be thought of as both a formal power series and a linear functional on $\mathbb{P}$. Here, $\mathcal{J}$ is called the *umbral algebra*, the study of which in the *umbral calculus*.

The order $o(f(t))$ of $0 \neq f(t) \in \mathcal{J}$ is the smallest integer k such that the coefficient of t^k in $f(t)$ does not vanish. In particular, $0 \neq f(t) \in \mathcal{J}$ is called an *invertible series* if $o(f(t)) = 0$ and a *delta series* if $o(f(t)) = 1$. For $f(t), g(t) \in \mathcal{J}$ with $o(g(t)) = 0$ and $o(f(t)) = 1$, there exists

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a unique sequence $s_n(x)$ with $\deg s_n(x) = n$ such that $\langle g(t)f^k(t)|x_n(x)\rangle = n!\delta_{n,k}$ for $k \geq 0$, where $\delta_{n,k}$ is the Kronecker's symbol. Such a sequence $s_n(x)$ is called the *Sheffer sequence* for $(g(t), f(t))$ which is dented by $s_n(x) \sim (g(t), f(t))$. Also, it is known that $s_n(x) \sim (g(t), f(t))$ if and only if $\frac{1}{g(\bar{f}(t))}e^{x\bar{f}(t)} = \sum_{k\geq 0} s_k(x)\frac{t^k}{k!}$, where $\bar{f}(t)$ is the compositional inverse of $f(t)$ satisfying $f(\bar{f}(t)) = \bar{f}(f(t)) = t$. Observe here that

$$s_n(x) \sim \left(\frac{1}{g(\bar{f}(t))}, \bar{f}(t)\right) \text{ if and only if } g(t)e^{xf(t)} = \sum_{k\geq 0} s_k(x)\frac{t^k}{k!}.$$

In particular, $s_n(x)$ is called the *Appell sequence* for $g(t)$ if $s_n(x) \sim (g(t), t)$ and

$$s_n(x) \sim \left(\frac{1}{g(t)}, t\right) \text{ if and only if } g(t)e^{xt} = \sum_{k\geq 0} s_k(x)\frac{t^k}{k!}.$$

In 2017, Bayad and Komatsu [2] gave characterizations of Appell polynomials by means of symmetric property and expressed them as linear combinations of Bernoulli and Euler polynomials. Further, they presented some interesting examples and applications. The main goal of this paper, see Theorem 4, is to note that their method can be generalized to not necessarily Appel polynomials so as to include, for example, higher-order Genocchi polynomials. Moreover, we provide several examples that illustrate our main result, see the next section.

2. Main result and Applications

Note that the following theorem is the first main result in [2], which states a necessary and sufficient condition for Sheffer sequences to have the described symmetric property.

Theorem 1. *(see [2, Theorem 2.1]) Let $a \in \mathbb{C}$, and let $h(t) = g(t)e^{\frac{a}{2}f(t)}$. Also, we set*

$$V(a) = \{s_n(x)|s_n(x) \sim \left(\frac{1}{g(\bar{f}(t))}, \bar{f}(t)\right), s_n(a-x) = (-1)^n s_n(x), n \geq 0\}.$$

Then $V(a) \neq \emptyset$ if and only if $f(t)$ is odd ($f(-t) = -f(t)$) and $h(t)$ is even ($h(-t) = h(t)$).

In this paper, we want to pay attention to the following result which was also shown in [2].

Theorem 2. *(see [2, Theorem 2.5]) Let $s_n(x)$ be an Appell sequence for $1/g(t)$, that is, $g(t)e^{xt} = \sum_{k\geq 0} s_k(x)\frac{t^k}{k!}$, satisfying $s_n(a-x) = (-1)^n s_n(x)$, for all $n \geq 0$ and $0 \neq a \in \mathbb{C}$. Let $(a_k)_{k\geq 0}$ be a sequence of complex numbers such that $G(t) = g(t) - \sum_{k\geq 0} a_k\frac{t^k}{k!}$ is either odd or even. Then we have*

$$s_n(x) = \begin{cases} \sum\limits_{k \text{ even}} a_k\binom{n}{k}a^{n-k}E_{n-k}(x/a), & \text{if } G(t) \text{ is odd;} \\ -2\sum\limits_{k \text{ odd}} \frac{a_k}{k}\binom{n}{k-1}a^{n+1-k}B_{n+1-k}(x/a), & \text{if } G(t) \text{ is even.} \end{cases}$$

2.1. **Main result.** As we said earlier, our intention here is to extend the above theorem to not necessarily Appell polynomials so as to include, for example, higher-order Genocchi polynomials. For this purpose, we first note the following lemma.

Lemma 3. *Let $a \in \mathbb{C}$, $\ell \in \mathbb{Z}$. Also, we let, for any nonzero power series $0 \neq g(t) \in \mathcal{J}$, $g(t)e^{xt} = \sum_{k\geq 0} s_k(x)\frac{t^k}{k!}$. Then*

$$s_n(a-x) = (-1)^{n+\ell} s_n(x), \text{ for all } n \geq 0 \tag{1}$$

if and only if $g(t)e^{at} = (-1)^\ell g(-t)$.

Proof. Note that (1) is equivalent to

$$\sum_{n\geq 0} s_n(a-x)\frac{t^n}{n!} = (-1)^\ell \sum_{n\geq 0} s_n(x)\frac{(-t)^n}{n!}.$$

Since $g(t)e^{xt} = \sum_{k\geq 0} s_k(x)\frac{t^k}{k!}$, this can be written as $g(t)e^{(a-x)t} = (-1)^\ell g(-t)e^{-xt}$, which is equivalent to $g(t)e^{at} = (-1)^\ell g(-t)$, as claimed. □

The following result is a generalization of Theorem 2 that we will give several applications.

Theorem 4. *For $0 \neq g(t) \in \mathcal{J}$, assume that $g(t)e^{xt} = \sum_{k\geq 0} s_k(x)\frac{t^k}{k!}$. Also, assume that, for $0 \neq a \in \mathbb{C}$ and $\ell \in \mathbb{Z}$, we have $s_n(a-x) = (-1)^{n+\ell} s_n(x)$ for all $n \geq 0$. Let $G(t) = g(t) - \sum_{k\geq 0} a_k\frac{t^k}{k!}$ ($a_k \in \mathbb{C}$, for all k). Then we have*

(a) *If $G(t)$ is even, then $g(t) = \left(\sum_{k \text{ odd}} a_k\frac{t^k}{k!}\right)\frac{2(-1)^{\ell-1}}{e^{at}+(-1)^{\ell-1}}$. Further,*

$$s_n(x) = \begin{cases} -2\sum_{k=0}^{[n/2]} \frac{a_{2k+1}}{2k+1}\binom{n}{2k} a^{n-1-2k} B_{n-2k}(x/a), & \text{if } \ell \text{ is even;} \\ \sum_{k=0}^{[(n-1)/2]} a_{2k+1}\binom{n}{2k+1} a^{n-2k-1} E_{n-2k-1}(x/a), & \text{if } \ell \text{ is odd,} \end{cases}$$

with the understanding $s_0(x) = 0$, for ℓ odd.

(b) *If $G(t)$ is odd, then $g(t) = \left(\sum_{k \text{ even}} a_k\frac{t^k}{k!}\right)\frac{2(-1)^{\ell}}{e^{at}+(-1)^{\ell}}$. Further,*

$$s_n(x) = \begin{cases} \sum_{k=0}^{[n/2]} a_{2k}\binom{n}{2k} a^{n-2k} E_{n-2k}(x/a), & \text{if } \ell \text{ is even;} \\ -2\sum_{k=0}^{[(n-1)/2]} \frac{a_{2k+2}}{2k+2}\binom{n}{2k+1} a^{n-2k-2} B_{n-2k-1}(x/a), & \text{if } \ell \text{ is odd and } g(0) = 0, \end{cases}$$

with the understanding $s_0(x) = 0$, for ℓ odd and $g(0) = 0$.

Proof. Since the similarity between Cases (a) and (b), we will show Case (b) only. In view of our assumption and Lemma 3, $g(t)e^{at} = (-1)^\ell g(-t)$. Thus,

$$\left(G(t) + \sum_{k\geq 0} a_k\frac{t^k}{k!}\right)e^{at} = (-1)^\ell\left(G(-t) + \sum_{k\geq 0} a_k\frac{(-t)^k}{k!}\right) = (-1)^{\ell-1}G(t) + \sum_{k\geq 0} a_k(-1)^{\ell+k}\frac{t^k}{k!}.$$

From this, we obtain

$$G(t) = \frac{-e^{at} + (-1)^{\ell+k}}{e^{at} + (-1)^\ell}\sum_{k\geq 0} a_k\frac{t^k}{k!},$$

which in turn gives

$$g(t) = \frac{(-1)^\ell + (-1)^{\ell+k}}{e^{at} + (-1)^\ell} \sum_{k\geq 0} a_k \frac{t^k}{k!} = \frac{2(-1)^\ell}{e^{at} + (-1)^\ell} \sum_{k \text{ even}} a_k \frac{t^k}{k!}.$$

Again, the expression for $s_n(x)$ in case of ℓ even is left to the reader. Assume that ℓ is odd and $g(0) = 0$. Then, as $G(t)$ is odd and $g(0) = 0$, we have that $a_0 = 0$. Now, we have

$$\begin{aligned} g(t)e^{xt} &= \frac{-2}{a} \frac{at}{e^{at} - 1} e^{xt} \sum_{0 \neq k \text{ even}} a_k \frac{t^{k-1}}{k!} \\ &= \frac{-2}{a} \sum_{k\geq 0} \frac{a_{2k+2}}{2k+2} \frac{t^{2k+1}}{(2k+1)!} \sum_{m\geq 0} B_m(x/a) a^m \frac{t^m}{m!} \\ &= -2 \sum_{n\geq 1} \left(\sum_{k=0}^{[(n-1)/2]} \binom{n}{2k+1} \frac{a_{2k+2}}{2k+2} a^{n-2k-2} B_{n-2k-1}(x/a) \right) \frac{t^n}{n!}, \end{aligned}$$

as required. □

2.2. **Applications.** On the next subsections, we will present several examples that illustrate Theorem 4.

2.2.1. *Genocchi polynomials.* For $r \in \mathbb{Z}_{>0}$, the *Genocchi polynomials* $G_n^{(r)}(x)$ of order r are given by the generating function

$$\left(\frac{2t}{e^t + 1} \right)^r e^{xt} = \sum_{n\geq 0} G_n^{(r)}(x) \frac{t^n}{n!}.$$

Note here that the order of $g(t) = \left(\frac{2t}{e^t+1} \right)^r$ is r. For $x = 0$, $G_n^{(r)} = G_n^{(r)}(0)$ are called *Genocchi numbers* of order r. If $r = 1$, then $G_n(x) = G_n^{(1)}(x)$ and $G_n = G_n^{(1)}$ are called respectively *Genocchi polynomials* and *Genocchi numbers*. It is well known that $G_0 = 0$, $G_1 = 1$, $G_{2m+1} = 0$ for all $m \geq 1$, and $G_{2m} \neq 0$ for all $m \geq 1$. Assume that $g(t)e^{xt} = \sum_{n\geq 0} s_n(x) \frac{t^n}{n!}$, for $0 \neq g(t) \in \mathcal{J}$ and $s_n(1-x) = (-1)^{n+1} s_n(x)$. If $G(t) = g(t) - \sum_{k\geq 0} G_k \frac{t^k}{k!}$ is even, then

$$g(t)e^{xt} = \frac{2}{e^t + 1} e^{xt} \sum_{k \text{ odd}} G_k \frac{t^k}{k!} = \frac{2t}{e^t + 1} e^{xt} = \sum_{n\geq 0} G_n(x) \frac{t^n}{n!}.$$

Thus $s_n(x) = G_n(x)$ and $g(t) = \frac{2t}{e^t+1}$. In addition, by Theorem 4, we have

$$G_n(x) = \sum_{k=0}^{[(n-1)/2]} \binom{n}{2k+1} G_{2k+1} E_{n-2k-1}(x),$$

for all $n \geq 1$.

Finally, as was observed in [2], instead of assuming that $g(t) - \sum_{k\geq 0} G_k \frac{t^k}{k!}$ is even, we may assume $g(t) - \sum_{k=0}^{N} G_k \frac{t^k}{k!}$ ($N \in \mathbb{Z}_{>0}$) or $g(t) - t$ is even.

Let $g(t) = \left(\frac{2t/r}{e^{t/r}+1}\right)^r = \sum_{n\geq 0} \frac{G_n^{(r)}}{r^n}\frac{t^n}{n!}$. Then $g(t)e^t = (-1)^r g(-t)$ and $g(t)e^{xt} = \sum_{n\geq 0} \frac{G_n^{(r)}(rx)}{r^n}\frac{t^n}{n!}$. So, by Theorem 4, we obtain the following. For r even,

$$G_n^{(r)}(x) = -2\sum_{k=0}^{[n/2]} \binom{n}{2k} r^{n-2k-1} \frac{G_{2k+1}^{(r)}}{2k+1} B_{n-2k}(x/r)$$
$$= \sum_{k=0}^{[n/2]} \binom{n}{2k} r^{n-2k} G_{2k}^{(r)} E_{n-2k}(x/r);$$

for r odd and $n \geq 1$,

$$G_n^{(r)}(x) = \sum_{k=0}^{[(n-1)/2]} \binom{n}{2k+1} r^{n-2k-1} G_{2k+1}^{(r)} E_{n-2k-1}(x/r)$$
$$= -2\sum_{k=0}^{[(n-1)/2]} \binom{n}{2k+1} r^{n-2k-2} \frac{G_{2k+2}^{(r)}}{2k+2} B_{n-2k-1}(x/r).$$

2.2.2. *Bernoulli and Euler polynomials.* Let $g(t) = \frac{2}{e^t+1} = \sum_{n\geq 0} E_n \frac{t^n}{n!}$. Then $g(t)e^t = g(-t)$. Thus, by (a) in Theorem 4, we obtain

$$E_n(x) = -2\sum_{k=0}^{[n/2]} \binom{n}{2k} \frac{E_{2k+1}}{2k+1} B_{n-2k}(x).$$

Let $g(t) = \frac{t}{e^t-1} = \sum_{n\geq 0} B_n \frac{t^n}{n!}$. Then $g(t)e^t = g(-t)$. Thus, by (b) in Theorem 4, we obtain

$$B_n(x) = \sum_{k=0}^{[n/2]} \binom{n}{2k} B_{2k} E_{n-2k}(x).$$

2.2.3. *Bessel polynomials.* The reader may refer to [5] for the details about this example. Let the polynomials $V_{k,n}(x)$, $n \geq 0$, be defined by the generating function

$$g(t)e^{xt} = \frac{\frac{k!}{(2k+1)!} t^{2k+1} e^{xt}}{t^k y_k(-2/t)e^t - (-t)^k y_k(2/t)} = \sum_{n\geq 0} V_{k,n}(x) \frac{t^n}{n!},$$

where $y_k(x)$ is the Bessel polynomial of degree k given by

$$y_k(x) = \sum_{j=0}^{k} \frac{(k+j)!}{j!(k-j)!}(x/2)^j.$$

In the special case of $k = 1$, we have $\frac{\frac{1}{6}t^3 e^{xt}}{(t-2)e^t+(t+2)} = \sum_{n\geq 0} V_{1,n}(x)\frac{t^n}{n!}$. In fact, $V_{k,n}(x) = B_n^{(k,k)}(x)$ is the n-th Bernoulli-Padé polynomial of order (k,k), see [5]. By Equation (29) of [5], we know that $V_{k,n}(1-x)=(-1)^n V_{k,n}(x)$. So, by Theorem 4, we obtain

$$\begin{aligned} V_{k,n}(x) &= \sum_{\ell=0}^{[n/2]} \binom{n}{2\ell} V_{k,2\ell} E_{n-2\ell}(x) \\ &= -2\sum_{\ell=0}^{[n/2]} \binom{n}{2\ell} \frac{V_{k,2\ell+1}}{2\ell+1} B_{n-2\ell}(x). \end{aligned}$$

2.2.4. *Hermite polynomials.* Let $g(t)=\left(\frac{2t}{e^t+1}\right)^r e^{-\nu t^2/2}$ with $r\ge 1$. Then $g(t)e^{rt}=(-1)^r g(-t)$ and $g(t)e^{xt}=\sum_{n\ge0} s_n(x)\frac{t^n}{n!}$ with $s_n(x)=\sum_{\ell=0}^{n}\binom{n}{\ell}G_\ell^{(r)}H_{n-\ell}^{(\nu)}(x)$, where $H_n^{(\nu)}(x)$ are the Hermite polynomials with the generating function given by $e^{xt-\nu t^2/2}=\sum_{n\ge0}H_n^{(\nu)}(x)\frac{t^n}{n!}$. Observe here that

$$H_n^{(\nu)} = H_n^{(\nu)}(0) = \begin{cases} (-\nu/2)^{n/2}\frac{n!}{(n/2)!}, & \text{for } n \text{ even}, \\ 0, & \text{otherwise}. \end{cases}$$

Thus,

$$\begin{aligned} s_{2m}(0) &= \sum_{\ell=0}^{m}\binom{2m}{2\ell}G_{2\ell}^{(r)}H_{2m-2\ell}^{(\nu)}(0) = \sum_{\ell=0}^{m}\binom{2m}{2\ell}\frac{(2m-2\ell)!}{(m-\ell)!}(-\nu/2)^{m-\ell}G_{2\ell}^{(r)}, \\ s_{2m+1}(0) &= \sum_{\ell=0}^{m}\binom{2m+1}{2\ell+1}G_{2\ell+1}^{(r)}H_{2m-2\ell}^{(\nu)}(0) = \sum_{\ell=0}^{m}\binom{2m+1}{2\ell+1}\frac{(2m-2\ell)!}{(m-\ell)!}(-\nu/2)^{m-\ell}G_{2\ell+1}^{(r)}. \end{aligned}$$

Then, from Theorem 4, we have that for r even,

$$\begin{aligned} s_n(x) &= -2\sum_{k=0}^{[n/2]}\binom{n}{2k}\frac{s_{2k+1}(0)}{2k+1}r^{n-2k-1}B_{n-2k}(x/r) \\ &= \sum_{k=0}^{[n/2]}\binom{n}{2k}s_{2k}(0)r^{n-2k}E_{n-2k}(x/r); \end{aligned}$$

and for r odd and $n\ge1$,

$$\begin{aligned} s_n(x) &= -2\sum_{k=0}^{[(n-1)/2]}\binom{n}{2k+1}\frac{s_{2k+2}(0)}{2k+2}r^{n-2k-2}B_{n-2k-1}(x/r) \\ &= \sum_{k=0}^{[(n-1)/2]}\binom{n}{2k+1}s_{2k+1}(0)r^{n-2k-1}E_{n-2k-1}(x/r). \end{aligned}$$

2.2.5. *Two more examples.* Let $g(t)=\left(\frac{2t}{e^t+1}\right)^r\cos t$ with $r\ge1$. Then $g(t)e^{rt}=(-1)^r g(-t)$ and $g(t)e^{xt}=\sum_{n\ge0}s_n(x)\frac{t^n}{n!}$ with $s_n(x)=\sum_{\ell=0}^{[n/2]}\binom{n}{2\ell}(-1)^\ell G_{n-2\ell}^{(r)}(x)$. Then, from Theorem 4, we have that for r even,

$$s_n(x) = -2\sum_{k=0}^{[n/2]}\binom{n}{2k}\frac{s_{2k+1}(0)}{2k+1}r^{n-2k-1}B_{n-2k}(x/r)$$
$$= \sum_{k=0}^{[n/2]}\binom{n}{2k}s_{2k}(0)r^{n-2k}E_{n-2k}(x/r);$$

and for r odd and $n \geq 1$,

$$s_n(x) = -2\sum_{k=0}^{[(n-1)/2]}\binom{n}{2k+1}\frac{s_{2k+2}(0)}{2k+2}r^{n-2k-2}B_{n-2k-1}(x/r)$$
$$= \sum_{k=0}^{[(n-1)/2]}\binom{n}{2k+1}s_{2k+1}(0)r^{n-2k-1}E_{n-2k-1}(x/r).$$

Let $g(t) = \left(\frac{2t}{e^t+1}\right)^r \sin t$ with $r \geq 1$. Then $g(t)e^{rt} = (-1)^{r+1}g(-t)$ and $g(t)e^{xt} = \sum_{n\geq 0} s_n(x)\frac{t^n}{n!}$ with $s_0(x) = 0$ and $s_n(x) = \sum_{\ell=0}^{[(n-1)/2]}\binom{n}{2\ell+1}(-1)^\ell G_{n-2\ell-1}^{(r)}(x)$, $(n \geq 1)$. Then, from Theorem 4, we have that for r even and $n \geq 1$,

$$s_n(x) = -2\sum_{k=0}^{[(n-1)/2]}\binom{n}{2k+1}\frac{s_{2k+2}(0)}{2k+2}r^{n-2k-2}B_{n-2k-1}(x/r)$$
$$= \sum_{k=0}^{[(n-1)/2]}\binom{n}{2k+1}s_{2k+1}(0)r^{n-2k-1}E_{n-2k-1}(x/r);$$

and for r odd,

$$s_n(x) = -2\sum_{k=0}^{[n/2]}\binom{n}{2k}\frac{s_{2k+1}(0)}{2k+1}r^{n-2k-1}B_{n-2k}(x/r)$$
$$= \sum_{k=0}^{[n/2]}\binom{n}{2k}s_{2k}(0)r^{n-2k}E_{n-2k}(x/r).$$

All the polynomials $s_n(x)$ in Theorem 4 are expressed as linear combinations of Bernoulli or Euler polynomials. So we can find the Fourier series expansions of $s_n(\langle x\rangle)$ ($\langle x\rangle = x - [x]$ denotes the fractional part of x, for any real number x) from the well known Fourier series expansions for the Bernoulli function $B_n(\langle x\rangle)$ and the Euler function $E_n(\langle x\rangle)$, see Equations (2.8) and (2.9) in [2]. The reader refers to [2] for the details of this idea which was used, for example, to find the Fourier series expansions for the higher-order Bernoulli and Euler functions. For a different and more direct way of obtaining the Fourier series expansions of higher-order Bernoulli and Euler functions, the reader refers the recent papers [6] and [8].

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