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DEGENERATE ORDERED BELL NUMBERS AND POLYNOMIALS

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Abstract. In this paper, we study degenerate ordered Bell numbers and polynomials. In addition, we give some identities of these numbers and polynomials which are derived from the generating function.

1. Introduction

For $w(\neq 1) \in \mathbb{C}$, Apostol-Bernoulli polynomials are defined by the generating function to be

$$\frac{t}{we^t - 1} e^{xt} = \sum_{n=0}^{\infty} B_{n,w}(x) \frac{t^n}{n!}, \quad \text{(see [4, 9, 12])}. \tag{1.1}$$

When $x = 0$, $B_{n,w} = B_{n,w}(0)$ are called Apostol-Bernoulli numbers. Note that $B_{0,w} = 0$.

In [1,5,6], L. Carlitz considered degenerate Bernoulli polynomials which are given by the generating function to be

$$\frac{t}{(1+\lambda t)^{1/\lambda} - 1}(1+\lambda t)^{\frac{x}{\lambda}} = \sum_{n=0}^{\infty} \beta_{n,\lambda}(x) \frac{t^n}{n!}, \quad (\lambda \in \mathbb{R}). \tag{1.2}$$

Note that $\lim_{\lambda \to 0} \beta_{n,\lambda}(x) = B_n(x)$, $(n \geq 0)$, where $B_n(x)$ are the ordinary Bernoulli polynomials.

Now, we consider degenerate Apostol-Bernoulli polynomials which are defined by the generating function to be

$$\frac{t}{w(1+\lambda t)^{1/\lambda} - 1}(1+\lambda t)^{\frac{x}{\lambda}} = \sum_{n=0}^{\infty} \beta_{n,\lambda,w}(x) \frac{t^n}{n!}. \tag{1.3}$$

When $x = 0$, $\beta_{n,\lambda,w} = \beta_{n,\lambda,w}(0)$ are called the degenerate Apostol-Bernoulli numbers. Note that $\lim_{\lambda \to 0} \beta_{n,\lambda,w}(x) = B_{n,w}(x)$, $(n \geq 0)$.

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As is known, ordered Bell numbers are given by the generating function to be

$$\frac{1}{2-e^t} = \sum_{n=0}^{\infty} b_n \frac{t^n}{n!}, \quad \text{(see [2,3,8])}. \tag{1.4}$$

Thus, by (1.4), we get

$$b_n = \sum_{m=0}^{n} m! S_2(n,m), \ (n \geq 0), \quad \text{(see [2,3,8])}, \tag{1.5}$$

where $S_2(n,m)$ is the stirling number of the second kind.

The ordered Bell polynomials are defined by the generating function to be

$$\frac{1}{2-e^t} e^{xt} = \sum_{n=0}^{\infty} b_n(x) \frac{t^n}{n!}, \quad \text{(see [8])}. \tag{1.6}$$

From (1.4) and (1.6), we have

$$b_n(x) = \sum_{l=0}^{n} \binom{n}{l} b_l x^{n-l}, \ (n \geq 0). \tag{1.7}$$

Thus by (1.7), we see that $b_n(x)$ are Appell sequences for $(2-e^t, t)$.

In this paper, we consider the degenerate ordered Bell numbers and polynomials which are given by the generating functions. In addition, we derive some explicit identities of those numbers and polynomials which are derived from the generating functions.

2. Degenerate ordered Bell numbers and polynomials

For $\lambda \in \mathbb{R}$, we consider the degenerate ordered Bell polynomials which are given by the generating function to be

$$\frac{1}{2-(1+\lambda t)^{1/\lambda}} (1+\lambda t)^{\frac{x}{\lambda}} = \sum_{n=0}^{\infty} b_{n,\lambda}(x) \frac{t^n}{n!}. \tag{2.1}$$

When $x = 0$, $b_{n,\lambda} = b_{n,\lambda}(0)$ are called the degenerate ordered Bell numbers. It is not difficult to show that $\lim_{\lambda \to 0} b_{n,\lambda} = b_n(x)$, $(n \geq 0)$. From (2.1), we note that

$$\begin{aligned}\frac{1}{2-(1+\lambda t)^{1/\lambda}}(1+\lambda t)^{\frac{x}{\lambda}} &= \left(\sum_{l=0}^{\infty} b_{l,\lambda}\frac{t^l}{l!}\right)\left(\sum_{m=0}^{\infty}\binom{\frac{x}{\lambda}}{m}\lambda^m t^m\right) \\ &= \left(\sum_{l=0}^{\infty} b_{l,\lambda}\frac{t^l}{l!}\right)\left(\sum_{m=0}^{\infty}(x|\lambda)_m\frac{t^m}{m!}\right) \\ &= \sum_{n=0}^{\infty}\left(\sum_{l=0}^{n}\binom{n}{l}b_{l,\lambda}(x|\lambda)_{n-l}\right)\frac{t^n}{n!},\end{aligned} \tag{2.2}$$

where $(x|\lambda)_n = x(x-\lambda)(x-2\lambda)\cdots(x-(n-1)\lambda)$, $(n \geq 1)$, $(x|\lambda)_0 = 1$. Therefore, by (2.1) and (2.2), we obtain the following theorem.

Theorem 2.1. *For $n \geq 0$, we have*

$$\begin{aligned}b_{n,\lambda}(x) &= \sum_{l=0}^{n}\binom{n}{l}b_{l,\lambda}(x|\lambda)_{n-l} \\ &= \sum_{l=0}^{n}\binom{n}{l}(x|\lambda)_l b_{n-l,\lambda}\end{aligned}$$

where $(x|\lambda)_0 = 1$, $(x|\lambda)_n = x(x-\lambda)(x-2\lambda)\cdots(x-(n-1)\lambda)$, $(n \geq 1)$.

Now, we observe that

$$\begin{aligned}\frac{1}{2-(1+\lambda t)^{1/\lambda}} &= \frac{1}{1-\Big((1+\lambda t)^{1/\lambda}-1\Big)} = \sum_{n=0}^{\infty}\Big((1+\lambda t)^{1/\lambda}-1\Big)^n \\ &= \sum_{n=0}^{\infty} n!\sum_{m=n}^{\infty} S_2(m,n)\lambda^{-m}\Big(\log(1+\lambda t)\Big)^m\frac{1}{m!} \\ &= \sum_{m=0}^{\infty}\left(\sum_{n=0}^{m} n!S_2(m,n)\lambda^{-m}\frac{1}{m!}\Big(\log(1+\lambda t)\Big)^m\right) \\ &= \sum_{m=0}^{\infty}\left(\sum_{n=0}^{m} n!S_2(m,n)\lambda^{-m}\frac{1}{m!}m!\sum_{k=m}^{\infty}S_1(k,m)\frac{\lambda^k t^k}{k!}\right) \\ &= \sum_{k=0}^{\infty}\left(\sum_{m=0}^{k}\sum_{n=0}^{m} n!S_2(m,n)S_1(k,m)\lambda^{k-m}\right)\frac{t^k}{k!},\end{aligned} \tag{2.3}$$

and

$$\frac{1}{2-(1+\lambda t)^{1/\lambda}} = \sum_{n=0}^{\infty} b_{n,\lambda}\frac{t^n}{n!}, \tag{2.4}$$

Therefore, by (2.3) and (2.4), we obtain the following theorem.

Theorem 2.2. *For $k \geq 0$, we have*

$$b_{k,\lambda} = \sum_{m=0}^{k} \sum_{n=0}^{m} n! S_2(m,n) S_1(k,m) \lambda^{k-m}.$$

Note that

$$\begin{aligned} \lim_{\lambda \to 0} b_{k,\lambda} &= \sum_{n=0}^{k} n! S_2(k,n) \\ &= b_n, \ (n \geq 0). \end{aligned}$$

By (2.4), we easily get

$$\begin{aligned} 1 &= \left(\sum_{l=0}^{\infty} b_{l,\lambda} \frac{t^l}{l!} \right) \left(2 - (1+\lambda t)^{\frac{1}{\lambda}} \right) \\ &= \left(\sum_{l=0}^{\infty} b_{l,\lambda} \frac{t^l}{l!} \right) \left(2 - \sum_{m=0}^{\infty} (1|\lambda)_m \frac{t^m}{m!} \right) \\ &= \sum_{n=0}^{\infty} 2 b_{n,\lambda} \frac{t^n}{n!} - \sum_{n=0}^{\infty} \left(\sum_{l=0}^{n} \binom{n}{l} (1|\lambda)_{n-l} b_{l,\lambda} \right) \frac{t^n}{n!} \\ &= \sum_{n=0}^{\infty} \left\{ 2 b_{n,\lambda} - \sum_{l=0}^{n} \binom{n}{l} (1|\lambda)_{n-l} b_{l,\lambda} \right\} \frac{t^n}{n!}. \end{aligned} \tag{2.5}$$

By comparing the coefficients on the both sides of (2.5), we get

$$2 b_{n,\lambda} - \sum_{l=0}^{n} \binom{n}{l} (1|\lambda)_{n-l} b_{l,\lambda} = \begin{cases} 1, & \text{if } n = 0, \\ 0, & \text{if } n > 0. \end{cases} \tag{2.6}$$

Therefore, by (2.6), we obtain the following theorem.

Theorem 2.3. *For $n \geq 0$, we have*

$$b_{0,\lambda} = 1, \quad b_{n,\lambda} = \sum_{l=0}^{n-1} \binom{n}{l} (1|\lambda)_{n-l} b_{l,\lambda},$$

where $(1|\lambda)_n = 1 \cdot (1-\lambda)(1-2\lambda) \cdots (1-(n-1)\lambda)$, $(n \geq 1)$, $(1|\lambda)_0 = 1$.

Example

$$\begin{aligned}
b_{1,\lambda} &= b_{0,\lambda} = 1,\\
b_{2,\lambda} &= \sum_{l=0}^{1}\binom{2}{l}(1|\lambda)_{2-l}b_{l,\lambda} = (1|\lambda)_2 b_{0,\lambda} + 2(1|\lambda)_1 b_{1,\lambda}\\
&= (1-\lambda)+2 = 3-\lambda,\\
b_{3,\lambda} &= \sum_{l=0}^{2}\binom{3}{l}(1|\lambda)_{3-l}b_{l,\lambda}\\
&= (1|\lambda)_3 b_{0,\lambda} + \binom{3}{1}(1|\lambda)_2 b_{1,\lambda} + \binom{3}{2}(1|\lambda)_1 b_{2,\lambda}\\
&= (1-\lambda)(1-2\lambda)+3(1-\lambda)+3(3-\lambda)\\
&= 1-3\lambda+2\lambda^2+3-3\lambda+9-3\lambda = 13-9\lambda+2\lambda^2, \cdots .
\end{aligned}$$

From (2.1), we note that

$$\begin{aligned}
&\frac{2}{2-(1+\lambda t)^{1/\lambda}}(1+\lambda t)^{\frac{x}{\lambda}} - \frac{1}{2-(1+\lambda t)^{1/\lambda}}(1+\lambda t)^{\frac{x+1}{\lambda}}\\
=&\frac{1}{2-(1+\lambda t)^{1/\lambda}}(1+\lambda t)^{\frac{x}{\lambda}}\left(2-(1+\lambda t)^{1/\lambda}\right) = (1+\lambda t)^{\frac{x}{\lambda}}\\
=&\sum_{n=0}^{\infty}\binom{\frac{x}{\lambda}}{n}t^n = \sum_{n=0}^{\infty}(x|\lambda)_n\frac{t^n}{n!},
\end{aligned} \tag{2.7}$$

and

$$\begin{aligned}
&\frac{2}{2-(1+\lambda t)^{1/\lambda}}(1+\lambda t)^{\frac{x}{\lambda}} - \frac{1}{2-(1+\lambda t)^{1/\lambda}}(1+\lambda t)^{\frac{x+1}{\lambda}}\\
=&\sum_{n=0}^{\infty}\left\{2b_{n,\lambda}(x)-b_{n,\lambda}(x+1)\right\}\frac{t^n}{n!}.
\end{aligned} \tag{2.8}$$

Therefore, by (2.7) and (2.8), we obtain the following theorem.

Theorem 2.4. *For $n \geq 0$, we have*

$$(x|\lambda)_n = 2b_{n,\lambda}(x) - b_{n,\lambda}(x+1).$$

From (1.3), we have

$$\begin{aligned}\sum_{n=0}^{\infty} b_{n,-\lambda}(-x)\frac{t^n}{n!} &= \frac{1}{2-(1-\lambda t)^{-1/\lambda}}(1-\lambda t)^{\frac{x}{\lambda}} \\ &= \frac{1}{2(1-\lambda t)^{1/\lambda}-1}(1-\lambda t)^{\frac{x+1}{\lambda}} \\ &= -\frac{1}{t}\frac{-t}{2(1-\lambda t)^{1/\lambda}-1}(1-\lambda t)^{\frac{x+1}{\lambda}} \\ &= \frac{1}{t}\sum_{n=0}^{\infty}\beta_{n,\lambda,2}(x+1)(-1)^{n+1}\frac{t^n}{n!} \\ &= \frac{1}{t}\sum_{n=1}^{\infty}\beta_{n,\lambda,2}(x+1)(-1)^{n+1}\frac{t^n}{n!} = \sum_{n=0}^{\infty}(-1)^n\frac{\beta_{n+1,\lambda,2}(x+1)}{n+1}\frac{t^n}{n!}.\end{aligned} \tag{2.9}$$

Therefore, by (2.9), we obtain the following theorem.

Theorem 2.5. *For $n \geq 0$, we have*

$$b_{n,-\lambda}(-x) = (-1)^n\frac{\beta_{n+1,\lambda,2}(x+1)}{n+1}.$$

Let $r \in \mathbb{N}$. Then we define the degenerate ordered Bell numbers which are given by the generating function to be

$$\underbrace{\left(\frac{1}{2-(1+\lambda t)^{1/\lambda}}\right)\times\cdots\times\left(\frac{1}{2-(1+\lambda t)^{1/\lambda}}\right)}_{r-\text{times}}(1+\lambda t)^{\frac{x}{\lambda}} = \sum_{n=0}^{\infty} b_{n,\lambda}^{(r)}(x)\frac{t^n}{n!}. \tag{2.10}$$

When $x = 0$, $b_{n,\lambda}^{(r)} = b_{n,\lambda}^{(r)}(0)$ are called degenerate ordered Bell numbers. From (2.10), we have

$$b_n^{(r)}(x) = \sum_{l=0}^{n}\binom{n}{l}b_{l,\lambda}^{(r)}(x|\lambda)_{n-l}, \ (n \geq 0), \tag{2.11}$$

and

$$b_n^{(r)} = \sum_{l_1+\cdots+l_r=n}\binom{n}{l_1,\cdots,l_r}b_{l_1,\lambda}b_{l_2,\lambda}\cdots b_{l_r,\lambda}. \tag{2.12}$$

As is well know, Frobenius-Euler polynomials of order r are defined by the generating function to be

$$\left(\frac{1-u}{e^t-u}\right)^r e^{xt} = \underbrace{\left(\frac{1-u}{e^t-u}\right) \times \cdots \times \left(\frac{1-u}{e^t-u}\right)}_{r-\text{times}} e^{xt} = \sum_{n=0}^{\infty} H_n^{(r)}(x|u)\frac{t^n}{n!}. \tag{2.13}$$

Thus, by (2.13), we get

$$\begin{aligned}
&\left(\frac{1}{2-(1+\lambda t)^{1/\lambda}}\right)^r (1+\lambda t)^{\frac{x}{\lambda}} = \left(\frac{1}{2-e^{\frac{1}{\lambda}\log(1+\lambda t)}}\right)^r e^{\frac{x}{\lambda}\log(1+\lambda t)} \\
&= \sum_{m=0}^{\infty} H_m^{(r)}(x|2)\frac{1}{m!}\lambda^{-m}\Big(\log(1+\lambda t)\Big)^m \\
&= \sum_{m=0}^{\infty} H_m^{(r)}(x|2)\lambda^{-m}\frac{1}{m!}m!\sum_{n=m}^{\infty} S_1(n,m)\frac{\lambda^n t^n}{n!} \\
&= \sum_{n=0}^{\infty}\left(\sum_{m=0}^{n}\lambda^{n-m}H_m^{(r)}(x|2)S_1(n,m)\right)\frac{t^n}{n!}.
\end{aligned} \tag{2.14}$$

Therefore, by (2.10) and (2.14), we obtain the following theorem.

Theorem 2.6. *For $n \geq 0$, we have*

$$b_{n,\lambda}^{(r)}(x) = \sum_{m=0}^{n} \lambda^{n-m} H_m^{(r)}(x|2) S_1(n,m).$$

Remark. For $r \in \mathbb{N}$, we have

$$\begin{aligned}
&\sum_{n=0}^{\infty}\left\{2b_{n,\lambda}^{(r)}(x) - b_{n,\lambda}^{(r)}(x+1)\right\}\frac{t^n}{n!} \\
&= 2\left(\frac{1}{2-(1+\lambda t)^{1/\lambda}}\right)^r (1+\lambda t)^{\frac{x}{\lambda}} - \left(\frac{1}{2-(1+\lambda t)^{1/\lambda}}\right)^r (1+\lambda t)^{\frac{x+1}{\lambda}} \\
&= \left(\frac{1}{2-(1+\lambda t)^{1/\lambda}}\right)^r (1+\lambda t)^{\frac{x}{\lambda}}\Big(2-(1+\lambda t)^{1/\lambda}\Big) \\
&= \left(\frac{1}{2-(1+\lambda t)^{1/\lambda}}\right)^{r-1} (1+\lambda t)^{\frac{x}{\lambda}} = \sum_{n=0}^{\infty} b_{n,\lambda}^{(r-1)}(x)\frac{t^n}{n!}.
\end{aligned}$$

Thus, we note that

$$2b_{n,\lambda}^{(r)}(x) - b_{n,\lambda}^{(r)}(x+1) = b_{n,\lambda}^{(r-1)}(x), \ (n \geq 0).$$

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