Advanced Studies in Contemporary Mathematics
31 (2021), No. 2, pp. 271 – 277

www.jangjeon.or.kr
http://dx.doi.org/10.17777/ascm2021.31.2.271

ON r-DYNAMIC COLORINGS OF THE FRIENDSHIP GRAPH FAMILIES

G. NANDINI, M. VENKATACHALAM, T. DEEPA, AND I. N. CANGUL

Abstract. Coloring of graphs is an important area in graph theory with numerous applications including the most famous problems related to graphs. An r-dynamic coloring of a graph G is a proper coloring c of the vertices such that $|c(N(v))| \geq min\{r, d(v)\}$, for each vertex $v \in V(G)$. The r-dynamic chromatic number of a graph G is the minimum k such that G has an r-dynamic coloring with k colors. In this paper, we obtain the r-dynamic chromatic number of $P_n + F_n$, $K_n + F_n$, $\overline{L(F_n)}$ and central graphs of the friendship graph.

Keywords: coloring, r-dynamic coloring, central graph, line graph and friendship graph.

2000 Mathematics Subject Classification: 05C15.

1. Introduction

Throughout this paper, all graphs considered will be finite, undirected, connected and simple. The r-dynamic chromatic number was first introduced by Montgomery, [6], as a generalization of the dynamic chromatic number. In the special case of $r = 2$, it coincides with the classical dynamic chromatic number.

Intuitively speaking, an r-dynamic coloring of a graph G is defined as a proper coloring of G in which any multiple degree vertex is adjacent to more than one color class. Formally, an r-dynamic coloring of a graph G is a mapping c from $V(G)$ to the set of colors such that

(i) if $uv \in E(G)$, then $c(u) \neq c(v)$

and

(ii) for each vertex $v \in V(G), |c(N(v))| \geq min\{r, d(v)\}$

where $N(v)$ denotes the neighborhood of v, that is, the set of vertices adjacent to v, and $d(v)$ denotes the degree of v. Let r be a positive integer. The r-dynamic chromatic number of a graph G denoted by $\chi_r(G)$ is the minimum k such that G has an r-dynamic proper k-coloring.

The 1-dynamic chromatic number of a graph G is equal to its classical chromatic number $\chi(G)$. The 2-dynamic chromatic number of a graph has been studied under the name of dynamic chromatic number in [1, 4, 5, 3, 10]. The r-dynamic chromatic number has been

DATE SUBMITTED: 23RD SEPTEMBER, 2020

studied by many authors, for instance in [3, 10, 16, 13, 14, 15]. There are many upper bounds and lower bounds for $\chi_r(G)$ in terms of graph parameters. For example, for a graph G with $\Delta(G) \geq 3$, Lai et al., [10], proved that $\chi_r(G) \leq \Delta(G)+1$. An upper bound for the dynamic chromatic number of an r-regular graph G in terms of $\chi(G)$ and the independence number $\alpha(G)$ of G was introduced in [8]. In fact, it was proved that

$$\chi_d(G) \leq \chi(G) + 2log_2\alpha(G) + 3.$$

Taherkhani gave an upper bound for $\chi_2(G)$ in terms of the chromatic number, the maximum degree Δ and the minimum degree δ in [16]:

$$\chi_2(G) - \chi(G) \leq \left\lceil (\Delta e)/\delta log\left(2e\left(\Delta^2+1\right)\right)\right\rceil .$$

Li et al. proved that the computational complexity of $\chi_d(G)$ for a 3-regular graph is an NP-complete problem, [19]. Furthermore, Liu and Zhou showed that the problem of determining whether there exists a 3-dynamic coloring for a claw free graph with the maximum degree 3 is NP-complete, [12].

Lemma 1.1. [11] *For any graph G, $\chi_r(G) \geq min\{r, \Delta(G)\}+1$.*

In this paper, we study $\chi_r(G)$ and we calculate the the r-dynamic chromatic number of $P_n + F_n$, $K_n + F_n$, $\overline{L(F_n)}$ and central graphs of the friendship graph.

2. Preliminaries

Let G_1 be a graph with vertex set $V(G_1)$ and edge set $E(G_1)$. Let G_2 be a graph with vertex set $V(G_2)$ and edge set $E(G_2)$. The join [9] $G = G_1 + G_2$ of graphs G_1 and G_2 with disjoint vertex sets $V(G_1) \cup V(G_2)$ and edge sets $E(G_1) \cup E(G_2)$ together with all the edges joining $V(G_1)$ and $V(G_2)$.

Let G be a graph with vertex set $V(G)$ and edge set $E(G)$. The complement [9] of a graph G is a graph $\overline{G}$ on the same vertices such that two distinct vertices of $\overline{G}$ are adjacent if and only if they are not adjacent in G.

The central graph [18] $C(G)$ of a graph G is obtained from G by adding an extra vertex on each edge of G and then joining each pair of vertices of the original graph which were previously non-adjacent.

The line graph [9] of G denoted by $L(G)$ is the graph whose vertex set is the edge set of G and two vertices of $L(G)$ are adjacent whenever the corresponding edges of G are adjacent.

For any integer $n \geq 2$, the friendship graph F_n [2] is the graph of order n obtained by joining n copies of the cycle C_3 at a common vertex.

Let $V(F_n) = \{v\} \cup \{v_i : 1 \leq i \leq 2n\}$ and $E(F_n) = \{e_i : 1 \leq i \leq 3n\}$ where $e_i = vv_i$ for $1 \leq i \leq 2n$ and $e_{2n+i} = v_{2i-1}v_{2i}$ for $1 \leq i \leq n$.

Lemma 2.1. *The lower bound of r-dynamic chromatic number of the join of path and friendship graph of order n is*

$$\chi_r(P_n+F_n) \geq \begin{cases} r+1, & 1 \leq r \leq \Delta(P_n+F_n)-1, \\ \Delta(P_n+F_n)+1, & r \geq \Delta(P_n+F_n). \end{cases}$$

Proof. Let $V(P_n+F_n) = \{v\} \cup \{u_i, v_j : 1 \leq i \leq n, 1 \leq j \leq 2n\}$.
For $1 \leq r \leq \Delta(P_n+F_n)-1$, based on Lemma 1.1, we obtain
$\chi_r(P_n+F_n) \geq min\{r, \Delta(P_n+F_n)\}+1 = r+1$. Thus $\chi_r(P_n+F_n) \geq r+1$.
For $r \geq \Delta(P_n+F_n)$, based on Lemma 1.1, we obtain $\chi_r(P_n+F_n) \geq min\{r, \Delta(P_n+F_n)\}+1 = \Delta(P_n+F_n)+1$. It concludes the proof. □

Theorem 2.2. *Let $n \geq 4$ and $\Delta(P_n+F_n) = \Delta$. The r-dynamic chromatic number of join of path and friendship graph of order n is*

$$\chi_r(P_n+F_n) = \begin{cases} 5, & 1 \leq r \leq 4, \\ r+(r-3), & 5 \leq r \leq n+2, \\ r+n-1, & n+3 \leq r \leq 2n+1, \\ \Delta+1, & r \geq 2n+2. \end{cases}$$

Proof. The maximum degrees of P_n+F_n are $\Delta(P_n+F_n) = 3n$.

Case 1: Let $1 \leq r \leq 4$. Based on the Lemma 2.1, the lower bound is $\chi_r(P_n+F_n) \geq r+1$. The upper bound is obtained as follows: The vertices $\{u_1, u_2, v, v_1, v_2\}$ induce a clique of order K_5 in P_n+F_n. Assign the color $\{c_1, c_2, c_1, c_2, \ldots\}$ to the vertices $\{u_1, u_2, \ldots, u_n\}$. Assign the color c_3 to the vertex v and $\{c_4, c_5, c_4, c_5, \ldots\}$ to the vertices $\{v_1, v_2, \ldots, v_{2n}\}$. Now the r-adjacency condition is fulfilled and hence $\chi_r(P_n+F_n) \leq 5$. Thus, $\chi_{1 \leq r \leq 4}(P_n+F_n) = 5$.

Case 2: Let $5 \leq r \leq n+2$. Based on the Lemma 2.1, the lower bound is $\chi_r(P_n+F_n) \geq r+1$. The upper bound is obtained as follows: Assign the color $\{c_1, c_2, \ldots, c_{r-2}, c_1, c_2, \ldots\}$ to the vertices $\{u_1, u_2, \ldots, u_n\}$. Assign the color c_{r-1} to the vertex v and $\{c_r, c_{r+1}, \ldots, c_{r+(r-3)}\}$ to the vertices $\{v_1, v_2, \ldots, v_{2n}\}$. Now the r-adjacency condition is fulfilled and hence $\chi_r(P_n+F_n) \leq r+(r-3)$. Thus, $\chi_{5 \leq r \leq n+2}(P_n+F_n) = r+(r-3)$.

Case 3: Let $n+3 \leq r \leq 2n+1$. Based on the Lemma 2.1, the lower bound is $\chi_r(P_n+F_n) \geq r+1$. The upper bound is obtained as follows: Assign the color $\{c_1, c_2, \ldots, c_n\}$ to the vertices $\{u_1, u_2, \ldots, u_n\}$. Assign the color c_{n+1} to the vertex v and $\{c_{n+2}, c_{n+3}, \ldots, c_{r+n-1}, c_{n+2}, c_{n+3}, \ldots,\}$ to the vertices $\{v_1, v_2, \ldots, v_{2n}\}$. Now the r-adjacency condition is fulfilled and hence $\chi_r(P_n+F_n) \leq r+n-1$.
Thus, $\chi_{n+3 \leq r \leq 2n+1}(P_n+F_n) = r+n-1$.

Case 4: Let $r \geq 2n+2$. Based on the Lemma 2.1, the lower bound is $\chi_r(P_n+F_n) \geq r+1$. The upper bound is obtained as follows: Assign the color $\{c_1, c_2, \ldots, c_n\}$ to the vertices $\{u_1, u_2, \ldots, u_n\}$. Assign the color c_{n+1} to the vertex v and $\{c_{n+2}, c_{n+3}, \ldots, c_{\Delta+1}\}$ to the vertices $\{v_1, v_2, \ldots, v_{2n}\}$. Now the r-adjacency condition is fulfilled and hence $\chi_r(P_n+F_n) \leq \Delta+1$. Thus, $\chi_{r \geq 2n+2}(P_n+F_n) = \Delta+1$. □

Lemma 2.3. *The lower bound of r-dynamic chromatic number of the join of complete and friendship graph of order n is*

$$\chi_r(K_n + F_n) \geq \begin{cases} n+3, & 1 \leq r \leq n+2, \\ r+1, & n+3 \leq r \leq \Delta(K_n + F_n) - 1, \\ \Delta(K_n + F_n) + 1, & r \geq \Delta(K_n + F_n). \end{cases}$$

Proof. Let $V(K_n + F_n) = \{v\} \cup \{u_i, v_j : 1 \leq i \leq n, 1 \leq j \leq 2n\}$. For $1 \leq r \leq n+2$, the vertices $V = \{u_i, v, v_1, v_2 : 1 \leq i \leq n\}$ induce a clique of order K_{n+3} in $K_n + F_n$. Thus $\chi_r(K_n + F_n) \geq n+3$.
For $n+3 \leq r \leq \Delta(K_n + F_n) - 1$, based on Lemma 1.1, we obtain
$\chi_r(K_n + F_n) \geq min\{r, \Delta(K_n + F_n)\} + 1 = r+1$. Thus $\chi_r(K_n + F_n) \geq r+1$.
For $r \geq \Delta(K_n+F_n)$, based on Lemma 1.1, we obtain $\chi_r(K_n+F_n) \geq min\{r, \Delta(K_n + F_n)\} + 1 = \Delta(K_n + F_n) + 1$. It concludes the proof. □

Theorem 2.4. *Let $n \geq 4$ and $\Delta(K_n + F_n) = \Delta$. The r-dynamic chromatic number of join of complete and friendship graph of order n is*

$$\chi_r(K_n + F_n) = \begin{cases} n+3, & 1 \leq r \leq n+2, \\ r+1, & n+3 \leq r \leq \Delta - 1, \\ \Delta + 1, & r \geq \Delta. \end{cases}$$

Proof. The maximum degrees of $K_n + F_n$ is $\Delta(K_n + F_n) = 3n$.

Case 1: Let $1 \leq r \leq n+2$. Based on the Lemma 2.3, the lower bound is $\chi_r(K_n+F_n) \geq n+3$. The upper bound is obtained as follows: Assign the color $\{c_1, c_2, \ldots, c_n\}$ to the vertices $\{u_1, u_2, \ldots, u_n\}$. Assign the color c_{n+1} to the vertex v and $\{c_{n+2}, c_{n+3}, c_{n+2}, c_{n+3}, \ldots\}$ to the vertices $\{v_1, v_2, \ldots, v_{2n}\}$. Now the r-adjacency condition is fulfilled and hence $\chi_r(K_n + F_n) \leq n+3$. Thus, $\chi_{1 \leq r \leq n+2}(K_n + F_n) = n+3$.

Case 2: Let $n+3 \leq r \leq \Delta - 1$. Based on the Lemma 2.3, the lower bound is $\chi_r(K_n + F_n) \geq r+1$. The upper bound is obtained as follows: Assign the color $\{c_1, c_2, \ldots, c_n\}$ to the vertices $\{u_1, u_2, \ldots, u_n\}$. Assign the color c_{n+1} to the vertex v and $\{c_{n+2}, c_{n+3}, \ldots, c_{r+1}, c_{n+2}, c_{n+3}, \ldots\}$ to the vertices $\{v_1, v_2, \ldots, v_{2n}\}$. Now the r-adjacency condition is fulfilled and hence $\chi_r(K_n + F_n) \leq r+1$. Thus, $\chi_{n+3 \leq r \leq \Delta-1}(K_n+F_n) = r+1$.

Case 3: Let $r \geq \Delta$. Based on the Lemma 2.3, the lower bound is $\chi_r(P_n + F_n) \geq \Delta + 1$. The upper bound is obtained as follows: Assign the color $\{c_1, c_2, \ldots, c_n\}$ to the vertices $\{u_1, u_2, \ldots, u_n\}$. Assign the color c_{n+1} to the vertex v and $\{c_{n+2}, c_{n+3}, \ldots, c_{\Delta+1}\}$ to the vertices $\{v_1, v_2, \ldots, v_{2n}\}$. Now the r-adjacency condition is fulfilled and hence $\chi_r(K_n + F_n) \leq \Delta + 1$. Thus, $\chi_{r \geq \Delta}(K_n + F_n) = \Delta + 1$. □

Lemma 2.5. *The lower bound of r-dynamic chromatic number of the complement of the line graph of friendship graph of order n is*

$$\chi_r(\overline{L(F_n)}) \geq \begin{cases} r+1, & 1 \leq r \leq \Delta(\overline{L(F_n)}) - 1, \\ \Delta(\overline{L(F_n)}) + 1, & r \geq \Delta(\overline{L(F_n)}). \end{cases}$$

Proof. Let $V(L(F_n)) = \{e_i : 1 \leq i \leq 3n\}$ where $e_i = vv_i$ for $1 \leq i \leq 2n$ and $e_{2n+i} = v_{2i-1}v_{2i}$ for $1 \leq i \leq n$. Let $V\left(\overline{L(F_n)}\right) = V(L(F_n))$.

For $1 \leq r \leq \Delta(\overline{L(F_n)})-1$, based on Lemma 1.1, we obtain $\chi_r(\overline{L(F_n)}) \geq min\left\{r, \Delta(\overline{L(F_n)})\right\}+1 = r+1$. Thus $\chi_r(\overline{L(F_n)}) \geq r+1$.
For $r \geq \Delta(\overline{L(F_n)})$, based on Lemma 1.1, we obtain $\chi_r(\overline{L(F_n)}) \geq min\left\{r, \Delta(\overline{L(F_n)})\right\}+1 = \Delta(\overline{L(F_n)})+1$. It concludes the proof. □

Theorem 2.6. *Let $n \geq 2$ and $\Delta(\overline{L(F_n)}) = \Delta$. The r-dynamic chromatic number of the complement of line graph of friendship graph of order n is*

$$\chi_r(\overline{L(F_n)}) = \begin{cases} n, & 1 \leq r \leq n-1, \\ r+1, & n \leq r \leq 2n-1, \\ r+3, & 2n \leq r \leq \Delta-1, \\ 3n, & r \geq \Delta. \end{cases}$$

Proof. The maximum degrees of $\overline{L(F_n)}$ is $\Delta(\overline{L(F_n)}) = 3n-3$.

Case 1: Let $1 \leq r \leq n-1$. Based on the Lemma 2.5, the lower bound is $\chi_r(\overline{L(F_n)}) \geq r+1$. The upper bound is obtained as follows:
The vertices $\{e_i : 2n+1 \leq i \leq 3n\}$ induce a clique of order K_n in $\overline{L(F_n)}$. Assign the color $\{c_1, c_2, \ldots, c_n\}$ to the vertices $\{e_i : 2n+1 \leq i \leq 3n\}$. Assign the color $\{c_1, c_1, c_2, c_2, \ldots, c_n, c_n\}$ to the vertices $\{e_1, e_2, \ldots, e_{2n}\}$. Now the r-adjacency condition is fulfilled and hence $\chi_r\left(\overline{L(F_n)}\right) \leq n$. Thus, $\chi_{1\leq r\leq n-1}(\overline{L(F_n)}) = n$.

Case 2: Let $n \leq r \leq 2n-1$. Based on the Lemma 2.5, the lower bound is $\chi_r(\overline{L(F_n)}) \geq r+1$. The upper bound is obtained as follows:
The vertices $\{e_i : 2n+1 \leq i \leq 3n\}$ induce a clique of order K_n in $\overline{L(F_n)}$. Assign the color $\{c_1, c_2, \ldots, c_n\}$ to the vertices $\{e_i : 2n+1 \leq i \leq 3n\}$. Assign the color $\{c_{n+1}, c_{n+2}, \ldots, c_{r+1}, c_{n+1}, c_{n+2}, \ldots, c_{r+1}, \ldots, \}$ to the vertices $\{e_1, e_2, \ldots, e_{2n}\}$. Now the r-adjacency condition is fulfilled and hence $\chi_r\left(\overline{L(F_n)}\right) \leq r+1$. Thus, $\chi_{n\leq r\leq 2n-1}(\overline{L(F_n)}) = r+1$.

Case 3: Let $2n \leq r \leq \Delta-1$. Based on the Lemma 2.5, the lower bound is $\chi_r(\overline{L(F_n)}) \geq r+1$. The upper bound is obtained as follows:
The vertices $\{e_i : 2n+1 \leq i \leq 3n\}$ induce a clique of order K_n in $\overline{L(F_n)}$. Assign the color $\{c_1, c_2, \ldots, c_n\}$ to the vertices $\{e_i : 2n+1 \leq i \leq 3n\}$. Assign the color $\{c_{n+1}, c_{n+2}, \ldots, c_{r+3}, c_{n+1}, c_{n+2}, \ldots, \}$ to the vertices $\{e_1, e_2, \ldots, e_{2n}\}$. Now the r-adjacency condition is fulfilled and hence $\chi_r\left(\overline{L(F_n)}\right) \leq r+3$. Thus, $\chi_{2n\leq r\leq \Delta-1}(\overline{L(F_n)}) = r+3$.

Case 4: Let $r \geq \Delta$. Based on the Lemma 2.5, the lower bound is $\chi_r(\overline{L(F_n)}) \geq \Delta+1$. The upper bound is obtained as follows:
The vertices $\{e_i : 2n+1 \leq i \leq 3n\}$ induce a clique of order K_n in $\overline{L(F_n)}$. Assign the color $\{c_1, c_2, \ldots, c_n\}$ to the vertices $\{e_i : 2n+1 \leq i \leq 3n\}$. Assign the color $\{c_{n+1}, c_{n+2}, \ldots, c_{3n}\}$ to the vertices $\{e_1, e_2, \ldots, e_{2n}\}$. Now the r-adjacency condition is fulfilled and hence $\chi_r\left(\overline{L(F_n)}\right) \leq 3n$. Thus, $\chi_{r\geq\Delta}(\overline{L(F_n)}) = 3n$.

□

Theorem 2.7. *Let $n \geq 3$ and $\Delta(C(F_n)) = \Delta$. The r-dynamic chromatic number of the central graph of a friendship graph of order n is*

$$\chi_r(C(F_n)) = \begin{cases} n, & r = 1, \\ 2n+1, & r \geq 2. \end{cases}$$

Proof. Let $V(C(F_n)) = \{v\} \cup \{v_i : 1 \leq i \leq 2n\} \cup \{x_i : 1 \leq i \leq 3n\}$ where x_i is the vertex corresponding to the edge vv_i for $1 \leq i \leq 2n$ and x_{2n+i} is the vertex corresponding to the edge $v_{2i-1}v_{2i}$ for $1 \leq i \leq n$ in F_n. Note that $deg(x_i) = 2$ for $1 \leq i \leq 3n$, $deg(v_i) = 2n$ for $1 \leq i \leq 2n$ and finally $deg(v) = 2n$.

Case 1: Let $r = 1$. Color the vertices v_{2i-1} and v_{2i} with color c_i for $1 \leq i \leq n$. Then assign color c_1 to the vertex v. For $1 \leq i \leq n$, color the vertices x_{2n+i} with the colors $c_2, c_3, \ldots, c_n, c_1$. For $1 \leq i \leq n-1$, color the vertices x_{2i-1} and x_{2i} with c_{i+1} and color the vertices x_{2n-1} and x_{2n} with c_2. Now, an easy check shows that the r-adjacency condition is fulfilled. Hence, $\chi_r(C(F_n)) = n$ for $r = 1$.

Case 2: Let $r \geq 2$. Then clearly, the graph induced by $\{v_1, v_{2i-1} : \ i = 2, \ldots, n\}$ is a complete graph. Thus, a proper coloring assign at least n colors to them. The same happens with the subgraph induced by $\{v_2, v_{2i} : \ i = 2, \ldots, n\}$. Moreover the color assigned to odd vertices should be different from the color assigned to the even vertices and that all of them should be different from the color assigned to the vertex v. Thus, $\chi_r(C(F_n)) \geq 2n + 1$.

The r-dynamic $2n+1$-coloring is as follows: For $1 \leq i \leq 2n$, assign the color c_i to v_i and assign the color c_{2n+1} to v and x_{2n+i} for $1 \leq i \leq n$. For $1 \leq i \leq 2n$, assign the color c_{2i} to x_{2i-1} and c_{2i-1} to x_{2i}. Now, the r-adjacency condition is fulfilled hence $\chi_r(C(F_n)) \leq 2n+1$. Since $2n+1 \leq \chi_r(C(F_n)) \geq 2n+1$, we have $\chi_{r \geq 2}(C(F_n)) = 2n+1$. It completes the proof. □

References

[1] A. Ahadi, S. Akbari, A. Dehghana, M. Ghanbari, On the difference between chromatic number and dynamic chromatic number of graphs, *Discrete Math.*, 312 (2012), 2579–2583.

[2] A. Albina and U. Mary, *A Study on Dominator Coloring of Friendship and Barbell Graphs,* International Journal of Mathematics and its Applications., Vol. 6(4) (2018), 99-105.

[3] M. Alishahi, Dynamic chromatic number of regular graphs, *Discrete Appl. Math.*, 160 (2012), 2098–2103.

[4] S. Akbari, M. Ghanbari, S. Jahanbekam, On the dynamic chromatic number of graphs, *in: Combinatorics and Graphs, in: Contemp. Math., Amer. Math. Soc.*, 531 (2010), 11–18.

[5] S. Akbari, M. Ghanbari, S. Jahanbekam, On the list dynamic coloring of graphs, *Discrete Appl. Math.*, 157 (2009), 3005–3007

[6] B. Montgomery, Dynamic coloring of graphs, ProQuest LLC, Ann Arbor, MI, (2001), PhD Thesis, West Virginia University.

[7] Danuta Michalak, *On Middle and Total Graphs with Coarseness number equal* 1*,* Springer Verlag Graph Theory., Lagow proceedings, Berlin Heidelberg, New York, Tokyo, Vol. 1018 (1981), 139-150.

[8] A. Dehghan, A. Ahadi, Upper bounds for the 2-hued chromatic number of graphs in terms of the independence number, *Discrete Appl. Math.*, 160(15) (2012), 2142–2146.

[9] F. Harary, *Graph Theory,* Narosa Publishing home., New Delhi 1969.

[10] H. J. Lai, B. Montgomery, H. Poon, Upper bounds of dynamic chromatic number, *Ars Combin.*, 68 (2003), 193–201.

[11] H.J. Lai, B. Montgomery, Dynamic coloring of graph, Department of Mathematics, West Virginia University, 2002.
[12] X. Li, W. Zhou, The 2-nd order conditional 3-coloring of claw-free graphs, *Theoret. Comput. Sci.*, 396 (2008), 151–157.
[13] N.Mohanapriya, G.Nandini, M.Venkatachalam , δ- Dynamic chromatic number of double wheel graph, *Int. j. Open Problems Compt. Math.*, Vol. 11, No.4 (2018).
[14] G.Nandini, M.Venkatachalam, S.Gowri, On r- Dynamic coloring of the family of Bistar graphs, *Commun. Fac. Sci. Univ. Ank. ser. A1 Math.Stat.*, Vol 68, No 1, 923-928 (2019).
[15] G.Nandini, M.Venkatachalam, S.Gowri, On r- Dynamic coloring of Splitting graphs, *American Int. Journal of Research in Science, Tech., Engg. & Mathematics.*, Special issue of 2^{nd} (ICCSPAM-2019) 119-124.
[16] A.Taherkhani, r-Dynamic chromatic number of graphs, *Discrete Appl. Math.*, 201(2016), 222–227.
[17] P.Venkata Subba Reddy and K.Viswanathan Iyer, Conditional coloring of some parameterized graphs, *Proceedings of 76th Annual Conference of the Indian Mathematical Society, Surat, India*, December, 2010.
[18] J. Vernold Vivin , Ph.D Thesis, Harmonious coloring of total graphs, $n-$leaf, central graphs and circumdetic graphs, Bharathiar University, (2007), Coimbatore, India.
[19] X. Li, X. Yao, W. Zhou, H.Broersma, Complexity of conditional colorability of graphs, *Appl. Math. Lett.*, 22 (2009), 320–324.

Department of Mathematics, SNS College of Technology, Coimbatore - 641 035, Tamil Nadu-India., Email: nandiniap2006@gmail.com

Department of Mathematics, Kongunadu Arts and Science College, Coimbatore - 641 029, Tamil Nadu-India, Email: venkatmaths@gmail.com

Department of Mathematics, Kongunadu Arts and Science College, Coimbatore - 641 029, Tamil Nadu-India, Email: deepathangavelu88@gmail.com

Department of Mathematics, Uludag University, Bursa 16059, Turkey, Email: ncangul@gmail.com, corresponding author