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5 REGULAR PARTITIONS WITH DISTINCT ODD PARTS

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ABSTRACT. In this article, we prove infinite families of congruences for $pod_5(n)$ (the number of 5-regular partitions of n with distinct odd parts (and even parts are unrestricted)) using the theory of Hecke eigenforms. We also study the divisibility properties of $pod_5(n)$ using the arithmetic properties of modular forms.

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1. INTRODUCTION

A partition of a non-negative integer n is a non-increasing sequence of positive integers whose sum is n. Let $pod(n)$ denote the number of partitions of n in which odd parts are distinct (and even parts are unrestricted). It is significant to note that Hirschhorn and Sellers [5] appear to be the first to consider $pod(n)$ from an arithmetic viewpoint. The generating function of $pod(n)$ is given by [5],

$$\sum_{n=0}^{\infty} pod(n)q^n = \frac{1}{\psi(-q)}, \tag{1.1}$$

where $\psi(q)$ is defined in (2.4) and for any positive integer l, we denote f_l as

$$f_l := \prod_{n=1}^{\infty}(1-q^{ln}), \quad |q|<1.$$

Using modular forms, Radu and Sellers [9] established several congruence properties for $pod(n)$. For more details, see [2, 5].
In this article, we consider a restricted version of $pod(n)$ in which none of the parts is divisible by 5. Let $pod_5(n)$ denote the number of 5-regular partitions with distinct odd parts of n (and even parts are unrestricted). For example $pod_5(5) = 3$, where the relevant partitions being 4+1, 3+2, 2+2+1. The generating function of $pod_5(n)$ is given by

$$\sum_{n=0}^{\infty} pod_5(n)q^n = \frac{\psi(-q^5)}{\psi(-q)}. \tag{1.2}$$

In this article, we establish several infinite families of congruences as a consequence of our main results, such as

$$pod_5\left(49n+7j+24\right) \equiv 0 \pmod 5 \quad where \quad j \not\equiv 0 \pmod 7, \tag{1.3}$$

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$$(1.4)\qquad pod_5\Big(500n+25j+62\Big)\equiv 0 \pmod 2 \quad \textit{where} \quad j\not\equiv 0 \pmod 5,$$

$$(1.5)\qquad pod_5\Big(100n+20j+12\Big)\equiv 0 \pmod 2 \quad \textit{where} \quad j\not\equiv 0 \pmod 5.$$

The following are our main results.

Theorem 1.1. *Let k, n be non negative integers and* $i\in\{1,2,\cdots,k+1\}$. *Then for every odd prime and any integer* $j\not\equiv 0 \pmod{p_{k+1}}$, *we have*

$$(1.6)\qquad pod_5\left(p_1^2...p_{k+1}^2n+\frac{p_1^2...p_k^2p_{k+1}(p_{k+1}+2j)-1}{2}\right)\equiv 0 \pmod 5.$$

Theorem 1.2. *Let k, n be non negative integers and* $i\in\{1,2,\cdots,k+1\}$. *Then for every prime* $p_i\geq 3$ *such that* $p_i\not\equiv 1 \pmod 8$ *and any integer* $j\not\equiv 0 \pmod{p_{k+1}}$, *we have*

$$(1.7)\qquad pod_5\left(20p_1^2...p_{k+1}^2n+\frac{5p_1^2...p_k^2p_{k+1}(p_{k+1}+2j)-1}{2}\right)\equiv 0 \pmod 2.$$

Theorem 1.3. *Let k, n be non negative integers and* $i\in\{1,2,\cdots,k+1\}$. *Then for every prime* $p_i\geq 3$ *such that* $p_i\not\equiv 1 \pmod 8$ *and any integer* $j\not\equiv 0 \pmod{p_{k+1}}$, *we have*

$$(1.8)\qquad pod_5\left(4p_1^2...p_{k+1}^2n+\frac{p_1^2...p_k^2p_{k+1}(p_{k+1}+8j)-1}{2}\right)\equiv 0 \pmod 2.$$

This article is organized as follows. In Section 2, we recall some basic definitions related to modular forms and some properties of Ramanujan general theta function. We also found a new dissection identity, which aids the proofs of the main theorems. Section 3 is devoted to the proofs of the Theorem 1.1-1.3 using the theory of Hecke eigenforms. In Section 4, we introduce an internal congruence by the technique of manipulating q-series. We conclude the paper by characterizing the divisibility properties of $pod_5(n)$ using the arithmetic properties of modular forms.

2. Preliminaries

In this section, we recollect some definitions, theorems, and identities to prove our main results.

Definition 2.1. [8, Definition 1.15] If χ is a Dirichlet character modulo N, then a form $f(z)\in M_k(\Gamma_1(N))$ (resp. $S_k(\Gamma_1(N))$ has Nebentypus characterχ if

$$f\left(\frac{az+b}{cz+d}\right)=\chi(d)(cz+d)^kf(z)$$

for all $z\in\mathbb{H}$ and all $\begin{bmatrix} a & b\\ c & d\end{bmatrix}\in\Gamma_0(N)$. The space of such modular forms (resp. cusps forms) is denoted by $M_k(\Gamma_0(N),\chi)$ (resp. $S_k(\Gamma_0(N),\chi)$). If χ is trivial character then we write $M_k(\Gamma_0(N))$ and $S_k(\Gamma_0(N))$ for short.

For $z\in\mathbb{H}$, the Dedekind eta function $\eta:\mathbb{H}\to\mathbb{C}$ is defined by

$$\eta(z):=q^{\frac{1}{24}}\prod_{n=1}^{\infty}(1-q^n) \text{ where } q:=e^{2\pi iz}.$$

A function f(z) is called an eta-quotient if it is of the form

$$f(z) = \prod_{\delta | N} \eta(\delta z)^{r_\delta},$$

where N is a positive integer and $r_\delta \in \mathbb{Z}$.

Theorem 2.1. [8, Theorem 1.64 and Theorem 1.65] *If $f(z) = \prod_{\delta|N} \eta(\delta z)^{r_\delta}$ is an eta-quotient with $k = \frac{1}{2}\sum_{\delta|N} r_\delta \in \mathbb{Z}$,*

$$\sum_{\delta|N} \delta r_\delta \equiv 0 \pmod{24}$$

and

$$\sum_{\delta|N} \frac{N}{\delta} r_\delta \equiv 0 \pmod{24},$$

then $f(z)$ satisfies

$$f\left(\frac{az+b}{cz+d}\right) = \chi(d)(cz+d)^k f(z)$$

for every $\begin{bmatrix} a & b \\ c & d \end{bmatrix} \in \Gamma_0(N)$. Here the character χ is defined by $\chi(d) := \left(\frac{(-1)^k \prod_{\delta|N} \delta^{r_\delta}}{d}\right)$. In addition if c,d and N are positive integers with $d \mid N$ and $gcd(c,d) = 1$, then the order of vanishing of $f(z)$ at the cusp $\frac{c}{d}$ is $\frac{N}{24}\sum_{\delta|N} \frac{gcd(d,\delta)^2 r_\delta}{gcd(d,\frac{N}{d})d\delta}$.

Suppose that $f(z)$ is an eta-quotient satisfying conditions of the above theorem. If $f(z)$ is holomorphic at all cusps of $\Gamma_0(N)$, then $f(z) \in M_k(\Gamma_0(N), \chi)$.
Next, we recall the definition of Hecke operators.
If $f(z) = \sum_{n=0}^{\infty} a(n)q^n \in M_k(\Gamma_0(N), \chi)$ and let m be a positive integer, then the action of Hecke operator T_m on $f(z)$ is defined by

$$f(z) \mid T_m := \sum_{n=0}^{\infty} \left(\sum_{d|gcd(n,m)} \chi(d) d^{k-1} a\left(\frac{nm}{d^2}\right) \right) q^n.$$

In particular, if $m = p$ is a prime, we have

$$f(z) \mid T_p := \sum_{n=0}^{\infty} \left(a(pn) + \chi(p)p^{k-1} a\left(\frac{n}{p}\right) \right) q^n. \tag{2.1}$$

We follow the convention that $a(n) = 0$ unless n is a nonnegative integer.

Definition 2.2. A modular form $f(z) = \sum_{n=0}^{\infty} a(n)q^n \in M_k(\Gamma_0(N), \chi)$ is called a Hecke eigenform if for every $m \geq 2$ there exist a complex number $\lambda(m)$ for which

$$f(z) \mid T_m = \lambda(m) f(z). \tag{2.2}$$

Recall Ramanujan general theta function $f(a,b)$ is defined by [1, Eqn. 18.1]

$$f(a,b) = \sum_{n=-\infty}^{\infty} a^{n(n+1)/2} b^{n(n-1)/2}, \quad |ab| < 1.$$

Some special cases of $f(a,b)$ are [1, Entry 22],

$$\varphi(q) := f(q,q) = \sum_{n=-\infty}^{\infty} q^{n^2} = \frac{f_2^5}{f_1^2 f_4^2}, \tag{2.3}$$

$$\psi(q) := f(q,q^3) = \sum_{n=0}^{\infty} q^{n(n+1)/2} = \frac{f_2^2}{f_1}, \tag{2.4}$$

where the product representations arise from the famous Jacobi triple product identity [1, p.36, Entry 19].

$$f(a,b) = (-a;ab)_\infty(-b;ab)_\infty(ab;ab)_\infty. \tag{2.5}$$

We denote

$$\chi(q) := (-q;q^2)_\infty = \frac{f_2^2}{f_1 f_4}. \tag{2.6}$$

Furthermore, from [1, p.51, Example (v)] we have

$$f(q,q^5) = \psi(-q^3)\chi(q). \tag{2.7}$$

By q-series manipulation, we can see

$$\psi(-q) = \frac{f_1 f_4}{f_2}. \tag{2.8}$$

Lemma 2.1. *[1, p.49, Entry 31] The following 5-dissection holds.*

$$\psi(q) = f(q^{10};q^{15}) + qf(q^5;q^{20}) + q^3\psi(q^{25}). \tag{2.9}$$

Lemma 2.2. *[1, p.262, Entry 10] The following identity holds.*

$$\psi^2(q) - q\psi^2(q^5) = f(q,q^4)f(q^2,q^3). \tag{2.10}$$

Lemma 2.3. *The following 2-dissection holds.*

$$\varphi(q^{15}) = \varphi(q^{60}) + 2q^{15}\psi(q^{120}), \tag{2.11}$$

$$\frac{f_{15}}{f_5} = \frac{f_{20}f_{30}f_{80}f_{120}^2}{f_{10}^2 f_{40} f_{60} f_{240}} + q^5\frac{f_{30}f_{40}^2 f_{240}}{f_{10}^2 f_{80} f_{120}}. \tag{2.12}$$

Proof. We have the following 2-dissection of $\varphi(q)$ from [4, 1.9.4],

$$\varphi(q) = \varphi(q^4) + 2q\psi(q^8).$$

Replacing $q \to q^{15}$ from the above identity, we arrive at (2.11).
Again from [11], we have

$$\frac{f_3}{f_1} = \frac{f_4 f_6 f_{16} f_{24}^2}{f_2^2 f_8 f_{12} f_{48}} + q\frac{f_6 f_8^2 f_{48}}{f_2^2 f_{16} f_{24}}.$$

Changing $q \to q^5$ from the above identity, we obtain (2.12). □

Lemma 2.4. *The following 4 dissections holds.*

$$\begin{aligned}\frac{f_5}{f_1} = {}& \frac{f_8 f_{16}^5 f_{20}^2}{f_4^5 f_{32}^2 f_{40}} + 2q^2\frac{f_8^3 f_{20}^2 f_{32}^2}{f_4^5 f_{16} f_{40}} + q\frac{f_{16}^6 f_{40}^3}{f_4^4 f_8 f_{20} f_{32}^2 f_{80}} + 2q^5\frac{f_8^4 f_{32}^2 f_{80}}{f_4^5 f_{16}^2} + \\ & q^3\frac{f_8^2 f_{16}^4 f_{80}}{f_4^5 f_{32}^2} + 2q^3\frac{f_8 f_{32}^2 f_{40}^3}{f_4^4 f_{20} f_{80}}.\end{aligned} \tag{2.13}$$

$$\frac{f_2}{f_{10}} = \frac{f_4 f_{16} f_{40}^3}{f_8 f_{20}^3 f_{80}} - q^2 \frac{f_8^2 f_{80}}{f_{16} f_{20}^2}. \tag{2.14}$$

Proof. Recall the 2-dissection of $\frac{f_5}{f_1}$ due to Hirschorrn [6]

$$\frac{f_5}{f_1} = \frac{f_8 f_{20}^2}{f_2^2 f_{40}} + q \frac{f_4^3 f_{10} f_{40}}{f_2^3 f_8 f_{20}}. \tag{2.15}$$

From [1], we have

$$\frac{1}{f_1^2} = \frac{f_8^5}{f_2^5 f_{16}^2} + 2q \frac{f_4^2 f_{16}^2}{f_2^5 f_8}. \tag{2.16}$$

Multiplying (2.15) and (2.16), we obtain

$$\frac{f_5}{f_1^3} = \frac{f_8^6 f_{20}^2}{f_2^7 f_{16}^2 f_{40}} + q \frac{f_4^3 f_8^4 f_{10} f_{40}}{f_2^8 f_{16}^2 f_{20}} + 2q \frac{f_4^2 f_{16}^2 f_{20}^2}{f_2^7 f_{40}} + 2q^2 \frac{f_4^5 f_{10} f_{16}^2 f_{40}}{f_2^8 f_8^2 f_{20}}. \tag{2.17}$$

Changing $q \to q^2$ in (2.16) and (2.17) respectively, then employing the resulting identity in (2.15), we complete the proof of (2.13).

Consider the 2-dissection of $\frac{f_1}{f_5}$ from [6]

$$\frac{f_1}{f_5} = \frac{f_2 f_8 f_{20}^3}{f_4 f_{10}^3 f_{40}} - q \frac{f_4^2 f_{40}}{f_8 f_{10}^2}. \tag{2.18}$$

Changing $q \to q^2$ in (2.18), we arrive at (2.14). □

Lemma 2.5. *The following 2 dissection holds.*

$$\frac{1}{f_1 f_5} = \frac{f_{12}^2 f_{120}^5}{f_2^2 f_6 f_{10}^2 f_{60}^2 f_{240}^2} + q^6 \frac{f_4 f_6^2 f_{40}^2 f_{60} f_{240}}{f_2^3 f_{10}^2 f_{12} f_{20} f_{80} f_{120}} + q \frac{f_4 f_6^2 f_{80} f_{120}^2}{f_2^3 f_{10}^2 f_{12} f_{40} f_{240}} + 2q^{15} \frac{f_{12}^2 f_{240}^2}{f_2^2 f_6 f_{10}^2 f_{120}}. \tag{2.19}$$

Proof. Setting $\mu = 3$ and $\nu = 2$ in the following identity [1, p.69, Eqn (36.7)]

$$\psi(q^{\mu+\nu})\psi(q^{\mu-\nu}) = \varphi(q^{\mu(\mu^2-\nu^2)})\psi(q^{2\mu}) + \sum_{m=1}^{\mu-1/2} q^{\mu m^2 - \nu m} f(q^{(\mu+2m)(\mu^2-\nu^2)}, q^{(\mu-2m)(\mu^2-\nu^2)}) f(q^{2\nu m}, q^{2\mu-2\nu m}),$$

we deduce that

$$\psi(q^5)\psi(q) = \varphi(q^{15})\psi(q^6) + q f(q^{25}, q^5) f(q^4, q^2). \tag{2.20}$$

In light of (2.5), we obtain

$$f(q^2, q^4) = \frac{f_6^2 f_4}{f_2 f_{12}}. \tag{2.21}$$

Replacing $q \to q^5$ in (2.7), we obtain

$$f(q^5, q^{25}) = \psi(-q^{15})\chi(q^5) = \frac{f_{10}^2 f_{15} f_{60}}{f_5 f_{20} f_{30}}. \tag{2.22}$$

Employing (2.21) and (2.22) in (2.20), we obtain

$$\psi(q^5)\psi(q) = \varphi(q^{15})\psi(q^6) + q \frac{f_{10}^2 f_{15} f_{60}}{f_5 f_{20} f_{30}} \frac{f_6^2 f_4}{f_2 f_{12}}. \tag{2.23}$$

Applying the 2-dissections (2.11) and (2.12) respectively in (2.23), then multiplying the resulting identity by $\dfrac{1}{f_2^2 f_{10}^2}$, we arrive at (2.19). □

3. Proofs of Theorem 1.1-1.3

In this section, we prove the infinite families of congruences of $pod_5(n)$ using the theory of Hecke eigenforms, which is similar to the approach done by Ray and Barman [10] for Andrews partition with even parts below odd parts.

Proof of Theorem 1.1: Invoking binomial theorem in the generating function for $pod_5(n)$ (1.2), we deduce

$$\sum_{n=0}^{\infty} pod_5(n)q^n \equiv \psi^4(-q) \pmod 5. \tag{3.1}$$

Replacing $q \to q^2$ in (3.1), then rewriting the resulting identity in terms of eta-quotients, we obtain

$$\sum_{n=0}^{\infty} pod_5(n)q^{2n+1} \equiv \frac{\eta^4(2z)\eta^4(8z)}{\eta^4(4z)} \pmod 5. \tag{3.2}$$

Denote $\dfrac{\eta^4(2z)\eta^4(8z)}{\eta^4(4z)} = \sum_{n=0}^{\infty} c(n)q^n$, then this leads to

$$pod_5(n) \equiv c(2n+1) \pmod 5 \text{ and } c(n) = 0 \text{ if } n \text{ is even.} \tag{3.3}$$

Now by Theorem 2.2, $\dfrac{\eta^4(2z)\eta^4(8z)}{\eta^4(4z)} \in M_2\big(\Gamma_0(16)\big)$. Moreover, $\dfrac{\eta^4(2z)\eta^4(8z)}{\eta^4(4z)}$ is an eigen form (for example, see [7]). Therefore

$$\frac{\eta^4(2z)\eta^4(8z)}{\eta^4(4z)} \mid T_p = \sum_{n=1}^{\infty}\left[c(pn) + pc\left(\frac{n}{p}\right)\right]q^n = \lambda(p)\sum_{n=1}^{\infty} c(n)q^n. \tag{3.4}$$

which implies

$$c(pn) + pc\left(\frac{n}{p}\right) = \lambda(p)c(n). \tag{3.5}$$

Let $n = 1$ in the above identity, we obtain $c(p) = \lambda(p)$, since $c(1) = 1$. However $c(p) = 0$ only for $p = 2$, so that $\lambda(2) = 0$. Hence for all odd primes, we have

$$c(pn) + pc\left(\frac{n}{p}\right) = 0. \tag{3.6}$$

From (3.6), we derive that for all $n \geq 0$ and $p \nmid r$,

$$c(p^2n + pr) = 0 \tag{3.7}$$

and

$$c(p^2n) \equiv 4pc(n) \pmod 5. \tag{3.8}$$

Set $n = 2n - pr + 1$ in (3.7) and together with (3.3), we deduce

$$pod_5\left(p^2n + \frac{(p^2-1)}{2} + pr\frac{(1-p^2)}{2}\right) \equiv 0 \pmod 5. \tag{3.9}$$

Again setting $n = 2n+1$ in (3.8) and combining (3.3), we obtain

$$pod_5\left(p^2n + \frac{(p^2-1)}{2}\right) \equiv 4p \ \ pod_5(n) \pmod 5. \tag{3.10}$$

Since p is an odd prime, $2 \mid (1-p^2)$ and $\gcd\left(\frac{1-p^2}{2}, p\right) = 1$. Therefore when r runs over a residue system excluding the multiple of p, so does $\frac{1-p^2}{2}r$. Thus (3.9) can be rewritten as

$$pod_5\left(p^2n + \frac{(p^2-1)}{2} + pj\right) \equiv 0 \pmod 5. \tag{3.11}$$

where $p \nmid j$.
For all odd primes p the following holds.

$$p_1^2...p_k^2n + \frac{p_1^2...p_k^2-1}{2} = p_1^2\left(p_2^2...p_k^2n + \frac{p_2^2...p_k^2-1}{2}\right) + \frac{p_1^2-1}{2}, \tag{3.12}$$

Employing (3.10) frequently we obtain

$$pod_5\left(p_1^2...p_k^2n + \frac{p_1^2...p_k^2-1}{2}\right) \equiv 4p \ \ pod_5(n) \pmod 5. \tag{3.13}$$

Let $j \not\equiv 0 \pmod{p_{k+1}}$. Setting $n = p_{k+1}^2n + \frac{p_{k+1}^2-1}{2} + p_{k+1}j$ in (3.13) and combining with (3.11), we conclude

$$pod_5\left(p_1^2...p_{k+1}^2n + \frac{p_1^2...p_k^2p_{k+1}(p_{k+1}+2j)-1}{2}\right) \equiv 0 \pmod 5. \tag{3.14}$$

Hence we complete the proof of Theorem.
Remark 3.1: Let p be an odd prime. By taking all the primes $p_1, p_2, ..., p_{k+1}$ to be equal to the same prime p in the above Theorem, we obtain the following infinite family of congruences

$$pod_5\left(p^{2k+2}n + p^{2k+1}j + \frac{p^{2k+2}-1}{2}\right) \equiv 0 \pmod 5. \tag{3.15}$$

In particular, setting $k = 0$, for all non negative integers n and $j \not\equiv 0 \pmod 7$, we have

$$pod_5\left(49n + 7j + 24\right) \equiv 0 \pmod 5.$$

Proof of Theorem 1.2: Employing (2.15) in (1.2), then extracting the even powers of q and replacing $q^2 \to q$ from the resulting identity, we obtain

$$\sum_{n=0}^{\infty} pod_5(2n)q^n = \frac{f_4f_{10}^3}{f_1f_2f_5f_{20}}. \tag{3.16}$$

Using (2.19) in (3.16), then extracting odd powers of q and replacing $q^2 \to q$ from the resulting identity, we obtain

$$\sum_{n=0}^{\infty} pod_5(4n+2)q^n = 2q^7\frac{f_2f_5f_6^2f_{120}^2}{f_1^3f_3f_{10}f_{60}} + \frac{f_2^2f_3^2f_5f_{40}f_{60}^2}{f_1^4f_6f_{10}f_{20}f_{120}}. \tag{3.17}$$

Invoking binomial theorem under modulo 2, we obtain

$$\sum_{n=0}^{\infty} pod_5(4n+2)q^n \equiv f_5 f_{10} \pmod 2. \tag{3.18}$$

Extracting the terms involving q^{5n} from the above identity and rewriting the resulting identity in terms of eta-quotients, we attain

$$\sum_{n=0}^{\infty} pod_5(20n+2)q^{8n+1} \equiv \eta(8z)\eta(16z) \pmod 2. \tag{3.19}$$

By Theorem 2.2, we can easily verify that $\eta(8z)\eta(16z) \in S_1\big(\Gamma_0(128), \big(\frac{-2}{\bullet}\big)\big)$. Besides $\eta(8z)\eta(16z)$ is a Hecke eigen form [see, [7]], we can complete the proof by using the same procedures done in previous theorem.

Remark 3.2: Let $p \geq 3$ be a prime such that $p \not\equiv 1 \pmod 8$ and consider any integer $j \not\equiv 0 \pmod p$. By taking all the primes $p_1, p_2, ..., p_{k+1}$ to be equal to the same prime p in the above Theorem, we obtain the following infinite family of congruences, which is similar to Remark 3.1.

$$pod_5\left(20p^{2k+2}n + 5p^{2k+1}j + \frac{5p^{2k+2}-1}{2}\right) \equiv 0 \pmod 2. \tag{3.20}$$

Let $k = 0$ and $p = 5$ in (3.20), we arrive at (1.4)

Proof of Theorem 1.3: We have

$$\sum_{n=0}^{\infty} pod_5(n)q^n = \frac{f_2 f_5 f_{20}}{f_1 f_4 f_{10}}. \tag{3.21}$$

Employing (2.13) and (2.18) in the above identity, then extracting terms involving q^{4n} and changing $q^4 \to q$ from the resulting identity, we obtain

$$\sum_{n=0}^{\infty} pod_5(4n)q^n = \frac{f_{10}^2 f_4^6}{f_1^5 f_8^2 f_{20}} - 2q\frac{f_2^2 f_5 f_8^2 f_{20}}{f_1^6 f_4 f_{10}}. \tag{3.22}$$

Under modulo 2, the above identity can be rewritten as

$$\sum_{n=0}^{\infty} pod_5(4n)q^n \equiv \frac{f_4}{f_1} \equiv f_1 f_2 \pmod 2. \tag{3.23}$$

In terms of eta-quotients, we deduce

$$\sum_{n=0}^{\infty} pod_5(4n)q^{8n+1} \equiv \eta(8z)\eta(16z) \pmod 2. \tag{3.24}$$

We omit the remaining proof as it follows the same lines of the previous theorem.

Remark 3.3 Let $p \geq 3$ be a prime such that $p \not\equiv 1 \pmod 8$ and consider any integer $j \not\equiv 0 \pmod p$. By taking all the primes $p_1, p_2, ..., p_{k+1}$ to be equal to the same prime p in the Theorem 1.3, we obtain the following infinite family of congruences.

$$pod_5\left(4p^{2k+2}n + 4p^{2k+1}j + \frac{p^{2k+2}-1}{2}\right) \equiv 0 \pmod 2. \tag{3.25}$$

Setting $k = 0$ and $p = 5$ in the above identity, we obtain (1.5).

4. Internal congruence

Theorem 4.1. *For any non negative integer k and n, we have*

$$pod_5\left(5^k n + \frac{5^k - 1}{2}\right) \equiv pod_5(n) \pmod 5. \tag{4.1}$$

Proof. From (3.1), we have

$$\sum_{n=0}^{\infty} pod_5(n) q^n \equiv \psi^4(-q) \pmod 5. \tag{4.2}$$

Replacing $q \to -q$ in (2.9) we obtain

$$\psi(-q) = f(q^{10}, -q^{15}) + q f(-q^5, q^{20}) + q^3 \psi(-q^{25}). \tag{4.3}$$

Employing (4.3) in (4.2), then extracting terms involving q^{5n+2} and changing $q^5 \to q$ from the resulting identity, we deduce

(4.4)

$$\sum_{n=0}^{\infty} pod_5(5n+2) q^n \equiv q^2 \psi^4(-q^5) + f^2(-q, q^4) f^2(q^2, -q^3) + 3q\psi^2(-q^5) f(-q, q^4) f(q^2, -q^3).$$

Changing $q \to -q$ in (2.10) and employing the resulting identity in (4.4), we obtain

$$\sum_{n=0}^{\infty} pod_5(5n+2) q^n \equiv \psi^4(-q) \equiv pod_5(n) \pmod 5. \tag{4.5}$$

By induction, we complete the proof. □

5. Divisibility properties

For a fixed positive integer k, Gordon and Ono [3] proved that the number of partitions of n into distinct parts is divisible by 2^k for almost all n. Similar studies are done by many mathematicians for certain kinds of partitions.

Theorem 5.1. *Let m be a fixed positive integer, then $pod_5(n)$ is almost always divisble by 2^m, namely*

$$\lim_{X \to \infty} \frac{\#\{n \leq X : pod_5(n) \equiv 0 \pmod{2^m}\}}{X} = 1. \tag{5.1}$$

Proof. The generating function of $pod_5(n)$ is given by

$$\sum_{n=0}^{\infty} pod_5(n) q^n = \frac{f_2 f_5 f_{20}}{f_1 f_4 f_{10}}.$$

The above identity can be rewritten in terms of eta-quotients such that

$$\sum_{n=0}^{\infty} pod_5(n) q^{24n+12} = \frac{\eta(48z)\eta(120z)\eta(480z)}{\eta(24z)\eta(96z)\eta(240z)}. \tag{5.2}$$

Let $A_p(z) = \dfrac{\eta^2(24z)}{\eta(24pz)}$, then by binomial theorem we have

$$A_p^{p^m}(z) = \frac{\eta^{p^{m+1}}(24z)}{\eta^{p^m}(24pz)} \equiv 1 \pmod{p^{m+1}}.$$

Define $B_m(z)$ by

$$B_m(z) = \frac{\eta(48z)\eta(120z)\eta(480z)}{\eta(24z)\eta(120z)\eta(240z)} A_p^{p^m}(z). \tag{5.3}$$

For $p = 2$, we have

$$B_m(z) \equiv \sum_{n=0}^{\infty} pod_5(n) q^{24n+12} \pmod{2^{m+1}}. \tag{5.4}$$

By Theorem 2.1, $B_m(z)$ is a form of weight 2^{m-1} on $\Gamma_0(960)$. The cusps of $\Gamma_0(960)$ are represented by fractions $\frac{c}{d}$ where $d \mid 96$ and $gcd(c,d) = 1$. $B_m(z)$ is holomorphic at a cusp $\frac{c}{d}$ if and only if

$$\begin{aligned} &\frac{gcd(d,24)^2}{24}(2^{k+1}-1) + \frac{gcd(d,120)^2}{120} + \frac{gcd(d,480)^2}{480} \\ &\qquad - \frac{gcd(d,48)^2}{48}(2^k-1) - \frac{gcd(d,96)^2}{96} - \frac{gcd(d,240)^2}{240} \geq 0 \end{aligned}$$

Now

$$\begin{aligned} &\frac{gcd(d,480)^2}{480}\Bigg(20\frac{gcd(d,24)^2}{gcd(d,480)^2}(2^{k+1}-1) + 4\frac{gcd(d,120)^2}{gcd(d,480)^2} + 1 \\ &- 10\frac{gcd(d,48)^2}{gcd(d,480)^2}(2^k-1) - 5\frac{gcd(d,96)^2}{gcd(d,480)^2} - 2\frac{gcd(d,240)^2}{gcd(d,480)^2}\Bigg) \\ &\geq \frac{gcd(d,480)^2}{480}\left(\frac{2^{k+1}-1}{20} + \frac{1}{4} + 1 - \frac{2^k-1}{10} - \frac{1}{5} - \frac{1}{2}\right) \\ &> 0. \end{aligned}$$

Hence by Theorem 2.2, $B_m(z) \in M_{2^{m-1}}(\Gamma_0(960), \left(\frac{5}{\bullet}\right))$. Recall the following result due to Serre [8, p. 43],

If $f(z) = \sum_{n=0}^{\infty} a(n)q^n \in M_k(\Gamma_0(N), \chi)$ has Fourier expansion

$$f(z) = \sum_{n=0}^{\infty} c(n)q^n \in \mathbb{Z}[[q]],$$

then there exist a constant $\alpha > 0$ such that

$$\#\left\{n \leq X : c(n) \not\equiv 0 \pmod{l}\right\} = \mathcal{O}\left(\frac{X}{(logX)^{\alpha}}\right).$$

Here, let $l = 2^m$, using (5.4) we can complete the proof. □

Theorem 5.2. *Let m be a fixed positive integer, then $pod_5(n)$ is almost always divisble by 5^m, namely*

$$\lim_{X\to\infty} \frac{\#\left\{n \leq X : pod_5(n) \equiv 0 \pmod{5^m}\right\}}{X} = 1. \tag{5.5}$$

Proof. Let $p = 5$ in (5.3), and employing the same arguments of the above theorem, we arrive at the desired result. □

Remark: From the above two theorems, we can easily deduce $pod_5(n)$ is almost always divisible by 10^m.

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