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ON THE POWER INTEGRABILITY WITH WEIGHT OF DOUBLE TRIGONOMETRIC SERIES

XHEVAT ZAHIR KRASNIQI AND LAKSHMI NARAYAN MISHRA

Abstract. In this paper we have found the necessary and sufficient conditions for the power integrability with a weight of the sum of the double sine, double cosine series and double mixed sine-cosine series whose coefficients belong to a subclass of the $\gamma DRBVS$ class.

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1. Introduction

We consider double trigonometric series of the form

$$f^{ss}(x,y)=\sum_{i=1}^{\infty}\sum_{j=1}^{\infty}a_{ij}\sin ix\sin jy, \tag{1}$$

$$f^{cc}(x,y)=\sum_{i=1}^{\infty}\sum_{j=1}^{\infty}b_{ij}\cos ix\cos jy, \tag{2}$$

$$f^{sc}(x,y)=\sum_{i=1}^{\infty}\sum_{j=1}^{\infty}c_{ij}\sin ix\cos jy, \tag{3}$$

and

$$f^{cs}(x,y)=\sum_{i=1}^{\infty}\sum_{j=1}^{\infty}d_{ij}\cos ix\sin jy, \tag{4}$$

on the positive quadrant $T^2:=(0,\pi)\times(0,\pi)$ of the two-dimensional torus.

In the accessible literature, we encounter several questions (problems), treated pertaining to the series (1)–(2). We list some of them, which mainly are not of special interest only in this paper.

(Q1) Are they point-wise (uniform) convergent?
(Q2) Are they Fourier series of their sums?
(Q3) Are they convergent in L^1-norm?
(Q4) Are their formal sums L^p-integrable with a weight?

A considerable literature treating the above questions has been accumulated. For instance, Chen [29]–[33], Guo and Yu [2], Han et al. [3], He and Zhou [4], Kórus [11], Kórus and Móricz [12], Krasniqi [5]–[9], Krasniqi and Szal [10], Leindler [16], Marzuq [17], Móricz [24]–[28], Papp [34], Ram and

Bhatia [36], Vanda [37]–[38], Yu [40], Yu et al. [39], Yu [40] are among the researchers who have contributed to the subject.

In this paper, we predominantly deal with problems (Q1) and (Q4). Solely we touch a little bit question (Q1) and we do not treat questions (Q2)–(Q3) at all.

Throughout this paper we are agree to write $\lambda := (\lambda_{ij})$ for either (a_{ij}) or (b_{ij}) or (c_{ij}) or (d_{ij}).

To my best knowledge, for the first time, the notion "*gamma rest bounded variation*" of a double sequence has been introduced in [5]. To begin with our subject, we recall the following.

Definition 1.1. *Let $\gamma := \{\gamma_{k,\ell}\}$ be a double non-negative sequence. We say that the sequence $w := \{w_{k,\ell}\}$ belongs to the class $\gamma R_0^+ BVS^2$ if*

$$w_{k,\ell} \to 0 \quad as \quad k+\ell \to \infty,$$

$$\sum_{k=m}^{\infty}\sum_{\ell=n}^{\infty} |\Delta_{1,1} w_{k,\ell}| \leq C\gamma_{m,n}$$

for all $m, n \in \mathbb{N}$,

$$\sum_{k=m}^{\infty} |\Delta_{1,0} w_{k,\ell}| \leq C\gamma_{m,\ell}$$

for each fixed ℓ, and

$$\sum_{\ell=n}^{\infty} |\Delta_{0,1} w_{k,\ell}| \leq C\gamma_{k,n}$$

for each fixed k, where $\Delta_{1,0} w_{k,\ell} = w_{k,\ell} - w_{k+1,\ell}$, $\Delta_{0,1} w_{k,\ell} = w_{k,\ell} - w_{k,\ell+1}$, and $\Delta_{1,1} w_{k,\ell} = w_{k,\ell} - w_{k+1,\ell} - w_{k,\ell+1} + w_{k+1,\ell+1}$.

Note that we have to assume that C is a positive finite constant, which may depends only on γ or just a positive finite constant, and not necessarily the same at each different occurrences.

The class $\gamma R_0^+ BVS^2$ given by Definition 1.1 is, in fact, the extension for double sequences of the $\gamma R_0^+ BVS$ class, defined earlier by Leindler (see [15]), for the single sequences. Also, in the special case, when we take $\gamma := \{w_{k,\ell}\}$ in it, we get the $R_0^+ BVS^2$ class introduced in [1].

Another reason, why we mentioned the latest cited paper, is that there are given some other useful notions and notations which, for our purposes, will be recalled here as well.

Definition 1.2. *A sequence of positive numbers $\gamma := \{\gamma_{k,\ell}\}$ is said to be almost increasing (almost decreasing) if the inequality $C\gamma_{m_2,n_2} \geq \gamma_{m_1,n_1}$ ($\gamma_{m_2,n_2} \leq C\gamma_{m_1,n_1}$) holds for a certain positive constant C and for all positive integers $m_2 \geq m_1$ and $n_2 \geq n_1$.*

We will write $\gamma \in DMDS$ to indicate that the sequence γ is an almost decreasing sequence.

The function $\gamma(x, y)$ is defined by means of the sequence $\{\gamma_{k,\ell}\}$ as follows:

$$\gamma\left(\frac{\pi}{k}, \frac{\pi}{\ell}\right) = \gamma_{k,\ell} \quad \text{for all} \quad k, \ell \in \mathbb{N},$$

and there are positive constants C_1 and C_2 such that

$$C_1\gamma_{k,\ell} \le \gamma(x,y) \le C_2\gamma_{k+1,\ell+1} \quad \text{for all} \quad x \in \left(\frac{\pi}{k+1}, \frac{\pi}{k}\right), \quad y \in \left(\frac{\pi}{\ell+1}, \frac{\pi}{\ell}\right).$$

In the same paper the the following assertion is presented.

Theorem 1.1 ([1])**.** *If $(\lambda_{k,\ell}) \in R_0^+BVS^2$, then the series (1) and (2) converge in the sense of Pringsheim on* $[0,\pi)^2$.

By $L^p(T^2)$, $1 \le p < \infty$, we denote the space of all $p-$power integrable functions on T^2 with the norm

$$\|f\| := \left(\int_0^\pi \int_0^\pi |f(x,y)|^p dxdy\right)^{1/p}.$$

Now we recall the following results.

Theorem 1.2 ([1])**.** *Suppose that $(\lambda_{k,\ell}) \in R_0^+BVS^2$, $1 \le p < \infty$, and $f^{ss}(x,y)$ the sum-function of the series (1).*

(A) *If the sequence $(\gamma_{k,\ell})$ satisfies the condition that there exist some $\varepsilon_1, \varepsilon_2 > 0$ such that the sequence $\{\gamma_{k,\ell}k^{-1+\varepsilon_1}\}$ is almost decreasing for each ℓ, and the sequence $\{\gamma_{k,\ell}\ell^{-1+\varepsilon_2}\}$ is almost decreasing for each k, then the condition*

$$\sum_{k=1}^{\infty}\sum_{\ell=1}^{\infty}\gamma_{k,\ell}(k\ell)^{p-2}\lambda_{k,\ell}^p < +\infty \tag{5}$$

is sufficient for the validity of the condition

$$\gamma(x,y)|f^{ss}(x,y)|^p \in L(0,\pi)^2. \tag{6}$$

(B) *If the sequence $(\gamma_{k,\ell})$ satisfies the condition that there exist some $\varepsilon_3, \varepsilon_4 > 0$ such that the sequence $\{\gamma_{k,\ell}k^{p-1+\varepsilon_3}\}$ is almost increasing for each ℓ, and the sequence $\{\gamma_{k,\ell}\ell^{p-1+\varepsilon_4}\}$ is almost increasing for each k, then then inequality (5) is necessary for (7) to be satisfied.*

Theorem 1.3 ([1])**.** *Suppose that $(\lambda_{k,\ell}) \in R_0^+BVS^2$, $1 \le p < \infty$, and $f^{cc}(x,y)$ the sum-function of the series (2).*

(A) *If the sequence $(\gamma_{k,\ell})$ satisfies the condition that there exist some $\varepsilon_5, \varepsilon_6 > 0$ such that the sequence $\{\gamma_{k,\ell}k^{-1+\varepsilon_5}\}$ is almost decreasing for each ℓ, and the sequence $\{\gamma_{k,\ell}\ell^{-1+\varepsilon_6}\}$ is almost decreasing for each k, then it follows from condition (5) for the coefficients of the series (2) that*

$$\gamma(x,y)|f^{cc}(x,y)|^p \in L(0,\pi)^2. \tag{7}$$

(B) *If $(\lambda_{k,\ell})$ satisfies the condition*

$$\sum_{k=m}^{\infty}\sum_{\ell=n}^{\infty}\left|\frac{\lambda_{k,\ell}}{k\ell} - \frac{\lambda_{k+1,\ell}}{(k+1)\ell}\right| \le \frac{C\lambda_{m,n}}{mn} \tag{8}$$

for any m and n, and the sequence $\{\lambda_{k,\ell}\}$ satisfies the condition that there exist some $\varepsilon_7, \varepsilon_8 > 0$ such that the sequence $\{\gamma_{k,\ell}k^{p-1+\varepsilon_7}\}$ is almost increasing for each ℓ, and the sequence $\{\gamma_{k,\ell}\ell^{p-1+\varepsilon_8}\}$ is almost increasing for each k, then the condition (5) is necessary for the validity of condition (7).

Next, we are going to give two definitions.

Definition 1.3. *A sequence $w := \{w_n\}$ of positive numbers is called almost increasing (decreasing) if there exists $C = C(w) \geq 1$ ($K(w)$ maybe depends on w) such that the inequality*

$$Kw_n \geq w_m \quad (w_n \leq Kw_m)$$

holds for any $n \geq m$.

Definition 1.4. *Let r_1, r_2 two non-negative integers and δ_1, δ_2 real numbers such that $0 < \delta_1 \leq 1$ and $0 < \delta_2 \leq 1$. We say that the sequence $w := \{w_{k,\ell}\}$ of non-negative numbers belongs to the class $RBVS_{+,2}^{r_1,r_2;\delta_1,\delta_2}$ if*

$$w_{k,\ell} \to 0 \quad as \quad k + \ell \to \infty,$$

$$\sum_{k=m}^{\infty}\sum_{\ell=n}^{\infty} |\Delta_{1,1} w_{k,\ell}| \leq \frac{C}{m^{r_1+1+\delta_1} n^{r_2+1+\delta_2}} \sum_{i=1}^{m}\sum_{j=1}^{n} i^{r_1+1} j^{r_2+1} w_{i,j}$$

for all $m, n \in \mathbb{N}$,

$$\sum_{k=m}^{\infty} |\Delta_{1,0} w_{k,\ell}| \leq \frac{C}{m^{r_1+1+\delta_1}} \sum_{i=1}^{m} i^{r_1+1} w_{i,\ell}$$

for each fixed ℓ, and

$$\sum_{\ell=n}^{\infty} |\Delta_{0,1} w_{k,\ell}| \leq \frac{C}{n^{r_2+1+\delta_2}} \sum_{j=1}^{n} j^{r_2+1} w_{k,j}$$

for each fixed k, where $\Delta_{1,0} w_{k,\ell}$, $\Delta_{0,1} w_{k,\ell}$, and $\Delta_{1,1} w_{k,\ell}$ have the same meaning as so far.

Note that $RBVS_{+,2}^{r_1,r_2;\delta_1,\delta_2}$ class is the extension of the $RBVS_{+}^{r,\delta}$ class to two- dimensional case (see [13]).

It is clear that every sequence belonging to the $R_0^+ BVS^2$ class is an almost decreasing double sequence as well, i.e.

$$R_0^+ BVS^2 \subset DMDS.$$

It is easy to see that if $r_1 > -2$, $r_2 > -2$, $0 < \delta_1 \leq 1$, $0 < \delta_2 \leq 1$, and $w \in R_0^+ BVS^2 \subset DMDS$, then $w \in DRBVS_{+}^{r_1,r_2;\delta_1,\delta_2}$ holds too. Indeed, for

$$\gamma_{m,n} = \frac{1}{m^{r_1+1+\delta_1} n^{r_2+1+\delta_2}} \sum_{i=1}^{m}\sum_{j=1}^{n} i^{r_1+1} j^{r_2+1} w_{i,j},$$

we have

$$\gamma_{m,n} \geq \frac{C w_{m,n}}{m^{r_1+1+\delta_1} n^{r_2+1+\delta_2}} \sum_{i=1}^{m}\sum_{j=1}^{n} i^{r_1+1} j^{r_2+1} \geq C m^{1-\delta_1} n^{1-\delta_2} w_{m,n} \geq C w_{m,n},$$

which means that

$$R_0^+ BVS^2 \subset DMDS \subset RBVS_{+,2}^{r_1,r_2;\delta_1,\delta_2}.$$

Motivated by last embedding relation, we aim to prove the corresponding theorems with those presented in [1], replacing mainly $R_0^+ BVS^2$ class with $RBVS_{+,2}^{r_1,r_2;\delta_1,\delta_2}$ class.

2. Lemmas

The following lemmas will be applied in the proofs of the main results.

Lemma 2.1 ([14]). *Let $f_n > 0$ and $\mu_n \geq 0$. Then*

$$\sum_{n=1}^{\infty} f_n \left(\sum_{\nu=1}^{n} \mu_\nu\right)^p \leq p^p \sum_{n=1}^{\infty} f_n^{1-p} \mu_n^p \left(\sum_{\nu=n}^{\infty} f_\nu\right)^p, \quad p \geq 1.$$

Lemma 2.2 ([35]). *Let $f_n > 0$ and $\mu_n \geq 0$. Then*

$$\sum_{n=1}^{\infty} f_n \left(\sum_{\nu=n}^{\infty} \mu_\nu\right)^p \leq p^p \sum_{n=1}^{\infty} f_n^{1-p} \mu_n^p \left(\sum_{\nu=1}^{n} f_\nu\right)^p, \quad p \geq 1.$$

Next Lemma contains summation by parts for double sequences.

Lemma 2.3 ([24]). *Let $0 \leq m \leq M$ and $0 \leq n \leq N$. Then*

$$\begin{aligned}\sum_{j=m}^{M} \sum_{k=n}^{N} a_{j,k} b_{j,k} = & \sum_{j=m}^{M} \sum_{k=n}^{N} \Delta_{1,1} a_{j,k} B_{j,k} + \sum_{j=m}^{M} \Delta_{1,0} a_{j,N+1} B_{j,N} \\ & - \sum_{j=m}^{M} \Delta_{1,0} a_{j,n} B_{j,n-1} + \sum_{k=n}^{N} \Delta_{0,1} a_{M+1,k} B_{M,k} \\ & - \sum_{k=n}^{N} \Delta_{0,1} a_{m,k} B_{m-1,k} + a_{M+1,N+1} B_{M,N} \\ & - a_{M+1,n} B_{M,n-1} - a_{m,N+1} B_{m-1,N} + a_{m,n} B_{m-1,n-1},\end{aligned}$$

where

$$B_{j,k} := \sum_{r=0}^{j} \sum_{s=0}^{k} b_{r,s},$$

and we agree with conventions $B_{-1,n} := 0$, $B_{m,-1} := 0$, $B_{-1,-1} := 0$.

Throughout this paper, we denote by $D_m(u) = \sum_{i=0}^{m} \cos iu$ and $\widetilde{D}_n(v) = \sum_{j=1}^{n} \sin jv$ the well-known Dirichlet and conjugate Dirichlet's kernels, respectively, where we have agreed to take $\cos 0u := \frac{1}{2}$.

Using Lemma 2.3 we can prove the following.

Lemma 2.4. *Let $(\lambda_{m,n}) \in RBVS_{+,2}^{r_1,r_2;\delta_1,\delta_2}$. Then for $x \in \left(\frac{\pi}{m+1}, \frac{\pi}{m}\right), y \in \left(\frac{\pi}{n+1}, \frac{\pi}{n}\right)$:*

(a)

$$|f^{ss}(x,y)| \leq \frac{C}{m^{\delta_1} n^{\delta_2}} \sum_{i=1}^{m} \sum_{j=1}^{n} ij\lambda_{ij},$$

(b)

$$|f^{cc}(x,y)| \leq C \sum_{i=1}^{m} \sum_{j=1}^{n} i^{1-\delta_1} j^{1-\delta_2} \lambda_{ij},$$

(c)

$$|f^{sc}(x,y)| \leq \frac{C}{m^{\delta_1}} \sum_{i=1}^{m} \sum_{j=1}^{n} ij^{1-\delta_2} \lambda_{ij},$$

(d)

$$|f^{cs}(x,y)| \le \frac{C}{n^{\delta_2}} \sum_{i=1}^{m}\sum_{j=1}^{n} i^{1-\delta_1} j\lambda_{ij},$$

where C is a positive constant.

Proof. (a) First of all, we have

$$\begin{aligned} f^{ss}(x,y) &= \sum_{i=1}^{m}\sum_{j=1}^{n}\lambda_{ij}\sin ix\sin jy + \sum_{i=m+1}^{\infty}\sum_{j=1}^{n}\lambda_{ij}\sin ix \sin jy \\ &\quad + \sum_{i=1}^{m}\sum_{j=n+1}^{\infty}\lambda_{ij}\sin ix\sin jy + \sum_{i=m+1}^{\infty}\sum_{j=n+1}^{\infty}\lambda_{ij}\sin ix\sin jy \\ &:= \sum_{v=1}^{4} s_v^{ss}(x,y). \end{aligned} \tag{9}$$

Using the inequality $|\sin z| \le |z|$ for all $z \in \mathbb{R}$, we immediately get

$$|s_1^{ss}(x,y)| \le \frac{\pi^2}{mn}\sum_{i=1}^{m}\sum_{j=1}^{n} ij\lambda_{ij} \le \frac{\pi^2}{m^{\delta_1}n^{\delta_2}}\sum_{i=1}^{m}\sum_{j=1}^{n} ij\lambda_{ij}. \tag{10}$$

Since

$$\lambda_{m,n} \to 0 \quad \text{as} \quad m+n\to\infty,$$

then, using the summation by parts with respect to first sum, we have

$$s_2^{ss}(x,y) = \sum_{i=m+1}^{\infty}\sum_{j=1}^{n}\Delta_{1,0}\lambda_{ij}\widetilde{D}_i(x)\sin jy - \sum_{i=m+1}^{\infty}\sum_{j=1}^{n}\Delta_{1,0}\lambda_{ij}\widetilde{D}_m(x)\sin jy.$$

Whence, the use of $|\sin z| \le |z|$ for all $z \in \mathbb{R}$, $|\widetilde{D}_i(x)| = \mathcal{O}(1/x)$, and $(\lambda_{m,n}) \in RBVS_{+,2}^{r_1,r_2;\delta_1,\delta_2}$, imply

$$\begin{aligned} |s_2^{ss}(x,y)| &\le C\frac{y}{x}\sum_{i=m+1}^{\infty}\sum_{j=1}^{n} j|\Delta_{1,0}\lambda_{ij}| \\ &\le \frac{m}{n}\frac{C}{m^{r_1+1+\delta_1}}\sum_{i=1}^{m}\sum_{j=1}^{n} i^{r_1+1}j\lambda_{ij} \le \frac{C}{m^{\delta_1}n^{\delta_2}}\sum_{i=1}^{m}\sum_{j=1}^{n} ij\lambda_{ij}. \end{aligned} \tag{11}$$

Applying similar arguments we have verified that

$$|s_3^{ss}(x,y)| \le \frac{C}{m^{\delta_1}n^{\delta_2}}\sum_{i=1}^{m}\sum_{j=1}^{n} ij\lambda_{ij}. \tag{12}$$

Now, we apply Lemma 2.3, to obtain

$$\begin{aligned} s_4^{ss}(x,y) &= \sum_{i=m+1}^{\infty}\sum_{j=n+1}^{\infty}\Delta_{1,1}\lambda_{ij}\widetilde{D}_i(x)\widetilde{D}_j(y) - \sum_{i=m+1}^{\infty}\sum_{j=1}^{n}\Delta_{1,0}\lambda_{i,n+1}\widetilde{D}_i(x)\sin jy \\ &\quad - \sum_{i=1}^{m}\sum_{j=n+1}^{\infty}\Delta_{0,1}\lambda_{m+1,j}\sin ix\widetilde{D}_j(y) + \lambda_{m+1,n+1}\sum_{i=1}^{m}\sum_{j=1}^{n}\sin ix\sin jy. \end{aligned}$$

And consequently,

$$
\begin{aligned}
|s_4^{ss}(x,y)| &\le \frac{C}{xy}\sum_{i=m+1}^{\infty}\sum_{j=n+1}^{\infty}|\Delta_{1,1}\lambda_{ij}| \\
&\le \frac{Cmn}{m^{r_1+1+\delta_1}n^{r_2+1+\delta_2}}\sum_{i=1}^{m}\sum_{j=1}^{n} i^{r_1+1}j^{r_2+1}\lambda_{i,j} \le \frac{C}{m^{\delta_1}n^{\delta_2}}\sum_{i=1}^{m}\sum_{j=1}^{n} ij\lambda_{ij}. \qquad (13)
\end{aligned}
$$

Combining (9)-(13) we get the inequality on (a) as we desired.
(b) We have to estimate the following quantities

$$
s_1^{cc}(x,y) := \sum_{i=1}^{m}\sum_{j=1}^{n}\lambda_{ij}\cos ix\cos jy, \quad s_2^{cc}(x,y) := \sum_{i=m+1}^{\infty}\sum_{j=1}^{n}\lambda_{ij}\cos ix\cos jy
$$

$$
s_3^{cc}(x,y) := \sum_{i=1}^{m}\sum_{j=n+1}^{\infty}\lambda_{ij}\cos ix\cos jy, \quad s_4^{cc}(x,y) := \sum_{i=m+1}^{\infty}\sum_{j=n+1}^{\infty}\lambda_{ij}\cos ix\cos jy.
$$

Using the summation by parts, Lemma 2.3, and inequalities $|\cos z| \le 1$ for all $z \in \mathbb{R}$, $|D_i(x)| = \mathcal{O}(1/x)$, and $(\lambda_{m,n}) \in RBVS_{+,2}^{r_1,r_2;\delta_1,\delta_2}$, we obtain

$$
|s_1^{cc}(x,y)| \le \sum_{i=1}^{m}\sum_{j=1}^{n}\lambda_{ij} \le \sum_{i=1}^{m}\sum_{j=1}^{n} i^{1-\delta_1}j^{1-\delta_2}\lambda_{ij}, \qquad (14)
$$

$$
\begin{aligned}
|s_2^{cc}(x,y)| &= \left|\sum_{i=m+1}^{\infty}\sum_{j=1}^{n}\Delta_{1,0}\lambda_{ij}D_i(x)\cos jy - \sum_{i=m+1}^{\infty}\sum_{j=1}^{n}\Delta_{1,0}\lambda_{ij}D_m(x)\cos jy\right| \\
&\le \frac{C}{x}\sum_{i=m+1}^{\infty}\sum_{j=1}^{n}|\Delta_{1,0}\lambda_{ij}| \\
&\le \frac{Cm}{m^{r_1+1+\delta_1}}\sum_{i=1}^{m}\sum_{j=1}^{n} i^{r_1+1}\lambda_{ij} \le C\sum_{i=1}^{m}\sum_{j=1}^{n} i^{1-\delta_1}j^{1-\delta_2}\lambda_{ij}, \qquad (15)
\end{aligned}
$$

$$
\begin{aligned}
|s_3^{cc}(x,y)| &= \left|\sum_{i=1}^{m}\sum_{j=n+1}^{\infty}\Delta_{0,1}\lambda_{ij}\cos ixD_j(y) - \sum_{i=1}^{m}\sum_{j=n+1}^{\infty}\Delta_{0,1}\lambda_{ij}\cos ixD_n(y)\right| \\
&\le \frac{C}{y}\sum_{i=1}^{m}\sum_{j=n+1}^{\infty}|\Delta_{0,1}\lambda_{ij}| \\
&\le \frac{Cn}{n^{r_2+1+\delta_2}}\sum_{i=1}^{m}\sum_{j=1}^{n} j^{r_2+1}\lambda_{ij} \le C\sum_{i=1}^{m}\sum_{j=1}^{n} i^{1-\delta_1}j^{1-\delta_2}\lambda_{ij}, \qquad (16)
\end{aligned}
$$

and

$$
\begin{aligned}
|s_4^{cc}(x,y)| &= \Bigg| \sum_{i=m+1}^{\infty}\sum_{j=n+1}^{\infty} \Delta_{1,1}\lambda_{ij} D_i(x) D_j(y) - \sum_{i=m+1}^{\infty}\sum_{j=1}^{n} \Delta_{1,0}\lambda_{i,n+1} D_m(x)\cos jy \\
&\quad - \sum_{i=1}^{m}\sum_{j=n+1}^{\infty} \Delta_{0,1}\lambda_{m+1,j}\cos ix D_n(y) + \lambda_{m+1,n+1}\sum_{i=1}^{m}\sum_{j=1}^{n}\cos ix \cos jy \Bigg| \\
&\le \frac{C}{xy}\sum_{i=m+1}^{\infty}\sum_{j=n+1}^{\infty} |\Delta_{1,1}\lambda_{ij}| \\
(17)\ &\le \frac{Cmn}{m^{r_1+1+\delta_1} n^{r_2+1+\delta_2}} \sum_{i=1}^{m}\sum_{j=1}^{n} i^{r_1+1} j^{r_2+1}\lambda_{ij} \le C\sum_{i=1}^{m}\sum_{j=1}^{n} i^{1-\delta_1} j^{1-\delta_2}\lambda_{ij}.
\end{aligned}
$$

To sum up, the desired inequality on (b) follows from (14)-(17).

(c) The idea of the proof stand "between" the proofs of the cases (a)–(b), and notations used below are clarified by their superscripts. Namely, we have

$$
|s_1^{sc}(x,y)| \le x\sum_{i=1}^{m}\sum_{j=1}^{n} i\lambda_{ij} \le \frac{C}{m^{\delta_1}}\sum_{i=1}^{m}\sum_{j=1}^{n} i j^{1-\delta_2}\lambda_{ij}, \tag{18}
$$

$$
\begin{aligned}
|s_2^{sc}(x,y)| &\le \left| \sum_{i=m+1}^{\infty}\sum_{j=1}^{n} \Delta_{1,0}\lambda_{ij} \widetilde{D}_i(x)\cos jy - \sum_{i=m+1}^{\infty}\sum_{j=1}^{n} \Delta_{1,0}\lambda_{ij} \widetilde{D}_m(x)\cos jy \right| \\
&\le \frac{C}{x}\sum_{i=m+1}^{\infty}\sum_{j=1}^{n} |\Delta_{1,0}\lambda_{ij}| \\
(19)\ &\le \frac{Cm}{m^{r_1+1+\delta_1}}\sum_{i=1}^{m}\sum_{j=1}^{n} i^{r_1+1}\lambda_{ij} \le \frac{C}{m^{\delta_1}}\sum_{i=1}^{m}\sum_{j=1}^{n} i j^{1-\delta_2}\lambda_{ij},
\end{aligned}
$$

$$
\begin{aligned}
|s_3^{sc}(x,y)| &= \left| \sum_{i=1}^{m}\sum_{j=n+1}^{\infty} \Delta_{0,1}\lambda_{ij}\sin ix D_j(y) - \sum_{i=1}^{m}\sum_{j=n+1}^{\infty} \Delta_{0,1}\lambda_{ij}\sin ix D_n(y) \right| \\
&\le \frac{Cx}{y}\sum_{i=1}^{m}\sum_{j=n+1}^{\infty} i|\Delta_{1,0}\lambda_{ij}| \\
(20)\ &\le \frac{Cn}{mn^{r_2+1+\delta_2}}\sum_{i=1}^{m}\sum_{j=1}^{n} i j^{r_2+1}\lambda_{ij} \le \frac{C}{m^{\delta_1}}\sum_{i=1}^{m}\sum_{j=1}^{n} i j^{1-\delta_2}\lambda_{ij},
\end{aligned}
$$

and

$$\begin{aligned}|s_4^{sc}(x,y)| &= \Big|\sum_{i=m+1}^{\infty}\sum_{j=n+1}^{\infty}\Delta_{1,1}\lambda_{ij}\widetilde{D}_i(x)D_j(y) - \sum_{i=m+1}^{\infty}\sum_{j=1}^{n}\Delta_{1,0}\lambda_{i,n+1}\widetilde{D}_m(x)\cos jy \\ &\quad -\sum_{i=1}^{m}\sum_{j=n+1}^{\infty}\Delta_{0,1}\lambda_{m+1,j}\sin ixD_n(y) + \lambda_{m+1,n+1}\sum_{i=1}^{m}\sum_{j=1}^{n}\sin ix\cos jy\Big| \\ &\le \frac{C}{xy}\sum_{i=m+1}^{\infty}\sum_{j=n+1}^{\infty}|\Delta_{1,1}\lambda_{ij}| \\ &\le \frac{Cmn}{m^{r_1+1+\delta_1}n^{r_2+1+\delta_2}}\sum_{i=1}^{m}\sum_{j=1}^{n}i^{r_1+1}j^{r_2+1}\lambda_{ij} \\ (21)\quad &\le \frac{C}{m^{\delta_1}n^{\delta_2}}\sum_{i=1}^{m}\sum_{j=1}^{n}ij\lambda_{ij} \le \frac{C}{m^{\delta_1}}\sum_{i=1}^{m}\sum_{j=1}^{n}ij^{1-\delta_2}\lambda_{ij}.\end{aligned}$$

Collecting (18)–(21) gives (c) to be proved.

(d) In a very similar way, as in the proof of relation on (c), we have verified that

$$|f^{cs}(x,y)| \le \frac{C}{n^{\delta_2}}\sum_{i=1}^{m}\sum_{j=1}^{n}i^{1-\delta_1}j\lambda_{ij}$$

holds too.

The proof is completed. □

Next lemma gives conditions under which the sequence $(\lambda_{m,n}/mn) \in RBVS_{+,2}^{r_1,r_2;\delta_1,\delta_2}$.

Lemma 2.5. *If a sequence $\{\lambda_{m,n}\}$ belongs to $RBVS_{+,2}^{r_1,r_2;\delta_1,\delta_2}$ and satisfies the condition*

$$(22)\quad \sum_{k=m}^{\infty}\sum_{\ell=n}^{\infty}\left|\frac{\lambda_{k,\ell}}{k\ell} - \frac{\lambda_{k+1,\ell}}{(k+1)\ell}\right| \le \frac{C}{m^{r_1+1+\delta_1}n^{r_2+1+\delta_2}}\sum_{i=1}^{m}\sum_{j=1}^{n}i^{r_1+1}j^{r_2+1}\frac{\lambda_{i,j}}{ij}$$

for any $m,n\in\mathbb{N}$, then $\{\lambda_{m,n}/(mn)\} \in RBVS_{+,2}^{r_1,r_2;\delta_1,\delta_2}$.

Proof. We have

$$\begin{aligned}\sum_{k=m}^{\infty}\sum_{\ell=n}^{\infty}\left|\Delta_{1,1}\left(\frac{\lambda_{k,\ell}}{k\ell}\right)\right| &= \sum_{k=m}^{\infty}\sum_{\ell=n}^{\infty}\Bigg|\left(\frac{\lambda_{k,\ell}}{k\ell} - \frac{\lambda_{k+1,\ell}}{(k+1)\ell}\right) \\ &\quad -\left(\frac{\lambda_{k,\ell+1}}{k(\ell+1)} - \frac{\lambda_{k+1,\ell+1}}{(k+1)(\ell+1)}\right)\Bigg| \\ &\le \sum_{k=m}^{\infty}\sum_{\ell=n}^{\infty}\left|\frac{\lambda_{k,\ell}}{k\ell} - \frac{\lambda_{k+1,\ell}}{(k+1)\ell}\right| \\ &\quad +\sum_{k=m}^{\infty}\sum_{\ell=n}^{\infty}\left|\frac{\lambda_{k,\ell+1}}{k(\ell+1)} - \frac{\lambda_{k+1,\ell+1}}{(k+1)(\ell+1)}\right| \\ &\le 2\sum_{k=m}^{\infty}\sum_{\ell=n}^{\infty}\left|\frac{\lambda_{k,\ell}}{k\ell} - \frac{\lambda_{k+1,\ell}}{(k+1)\ell}\right|\end{aligned}$$

$$\leq \frac{C}{m^{r_1+1+\delta_1} n^{r_2+1+\delta_2}} \sum_{i=1}^{m} \sum_{j=1}^{n} i^{r_1+1} j^{r_2+1} \frac{\lambda_{i,j}}{ij},$$

for all $m, n \in \mathbb{N}$, and (in the same way as in [10])

$$\begin{aligned}
\sum_{k=m}^{\infty} \left| \Delta_{1,0} \left(\frac{\lambda_{k,\ell}}{k\ell} \right) \right| &= \frac{1}{\ell} \sum_{k=m}^{\infty} \left| \frac{\lambda_{k,\ell}}{k(k+1)} + \frac{1}{k+1} \Delta_{1,0} \lambda_{k,\ell} \right| \\
&\leq \frac{1}{\ell} \left(\sum_{k=m}^{\infty} \frac{\lambda_{k,\ell}}{k(k+1)} + \frac{1}{m+1} \sum_{k=m}^{\infty} |\Delta_{1,0} \lambda_{k,\ell}| \right) \\
&\leq \frac{1}{\ell} \left(\sum_{k=m}^{\infty} \frac{1}{k(k+1)} \sum_{i=k}^{\infty} |\lambda_{i,\ell} - \lambda_{i+1,\ell}| + \frac{C}{m^{r_1+2+\delta_1}} \sum_{i=1}^{m} i^{r_1+1} \lambda_{i,\ell} \right) \\
&\leq \frac{1}{\ell} \left(\sum_{i=m}^{\infty} |\lambda_{i,\ell} - \lambda_{i+1,\ell}| \sum_{k=i}^{\infty} \frac{1}{k^2} + \frac{C}{m^{r_1+1+\delta_1}} \sum_{i=1}^{m} i^{r_1+1} \frac{\lambda_{i,\ell}}{i} \right) \\
&\leq \frac{1}{\ell} \left(\frac{C}{m} \sum_{i=m}^{\infty} |\lambda_{i,\ell} - \lambda_{i+1,\ell}| + \frac{C}{m^{r_1+1+\delta_1}} \sum_{i=1}^{m} i^{r_1+1} \frac{\lambda_{i,\ell}}{i} \right) \\
&\leq \frac{C}{m^{r_1+1+\delta_1}} \sum_{i=1}^{m} i^{r_1+1} \frac{\lambda_{i,\ell}}{i\ell},
\end{aligned}$$

for each fixed ℓ and all $m \in \mathbb{N}$.

Similarly we have verified the inequality

$$\sum_{\ell=n}^{\infty} \left| \Delta_{0,1} \left(\frac{\lambda_{k,\ell}}{k\ell} \right) \right| \leq \frac{C}{n^{r_2+1+\delta_2}} \sum_{j=1}^{n} j^{r_2+1} \frac{\lambda_{k,j}}{kj},$$

for each fixed k and all $n \in \mathbb{N}$, which means that $(\lambda_{m,n}/mn) \in RBVS_{+,2}^{r_1,r_2;\delta_1,\delta_2}$.

The proof is completed. □

Remark 2.1. *Employing condition (22) in Lemma 2.5 is an indispensable condition (at lest to our best skills), otherwise we would come to estimate the double series*

$$\sum_{k=m}^{\infty} \sum_{n=\ell}^{\infty} \frac{m+n+2}{m(m+1)n(n+1)}$$

which is a divergent one.

Now we are going to prove Lemma 2.6 which shows some interest in itself.

Lemma 2.6. *The following statements hold true:*

(i) *For any four integers r_1, r_1', r_2, r_2' such that $0 \leq r_1' \leq r_1$, $0 \leq r_2' \leq r_2$, $0 < \delta_1 \leq 1$, and $0 < \delta_2 \leq 1$, the embedding relation*

$$RBVS_{+,2}^{r_1,r_2;\delta_1,\delta_2} \subseteq RBVS_{+,2}^{r_1',r_2';\delta_1,\delta_2}$$

holds true.

(ii) *For any two integers $r_1, r_2 \geq 0$, $0 < \delta_1 \leq \delta_1' \leq 1$ and $0 < \delta_2 \leq \delta_2' \leq 1$, the embedding relation*

$$RBVS_{+,2}^{r_1,r_2;\delta_1',\delta_2'} \subseteq RBVS_{+,2}^{r_1,r_2;\delta_1,\delta_2}$$

holds true.

Proof. (i) The conclusion follows because of

$$\begin{aligned}\frac{1}{m^{r_1+1+\delta_1}}\sum_{i=1}^{m} i^{r_1+1}\lambda_{i,j} &\le \frac{m^{r_1-r_1'}}{m^{r_1+1+\delta_1}}\sum_{i=1}^{m} i^{r_1'+1}\lambda_{i,j}\\ &= \frac{1}{m^{r_1'+1+\delta_1}}\sum_{i=1}^{m} i^{r_1'+1}\lambda_{i,j},\end{aligned}$$

$$\begin{aligned}\frac{1}{n^{r_2+1+\delta_2}}\sum_{j=1}^{n} j^{r_2+1}\lambda_{i,j} &\le \frac{n^{r_2-r_2'}}{n^{r_2+1+\delta_2}}\sum_{j=1}^{n} j^{r_2'+1}\lambda_{i,j}\\ &= \frac{1}{n^{r_2'+1+\delta_2}}\sum_{j=1}^{n} j^{r_2'+1}\lambda_{i,j},\end{aligned}$$

and

$$\begin{aligned}&\frac{1}{m^{r_1+1+\delta_1}n^{r_2+1+\delta_2}}\sum_{i=1}^{m}\sum_{j=1}^{n} i^{r_1+1}j^{r_2+1}\lambda_{i,j}\\ &\qquad\le \frac{m^{r_1-r_1'}n^{r_2-r_2'}}{m^{r_1+1+\delta_1}n^{r_2+1+\delta_2}}\sum_{i=1}^{m}\sum_{j=1}^{n} i^{r_1'+1}j^{r_2'+1}\lambda_{i,j}\\ &\qquad= \frac{1}{m^{r_1'+1+\delta_1}n^{r_2'+1+\delta_2}}\sum_{i=1}^{m}\sum_{j=1}^{n} i^{r_1'+1}j^{r_2'+1}\lambda_{i,j}.\end{aligned}$$

(ii) The proof can be done in the same way. Therefore we have omitted its details. □

In the sequel, we shall use the componentwise partial order on $\mathbb{N}\times\mathbb{N}$ given by

$$(m_1,n_1)\le(m_2,n_2) \iff m_1\le m_2 \text{ and } n_1\le n_2$$

for $(m_1,n_1),(m_2,n_2)\in\mathbb{N}\times\mathbb{N}$.

For the sake of completeness we recall that: A double sequence $\{w_{m,n}\}$ is called a Cauchy double sequence if for every $\varepsilon>0$, there is $(m_0,n_0)\in\mathbb{N}\times\mathbb{N}$ such that

$$|w_{m,n}-w_{p,q}|<\varepsilon \quad \text{for all} \quad (m,n),\,(p,q)\ge(m_0,n_0).$$

Now, we pass to next section which contains main results of the paper.

3. Main Results

At first we prove the following.

Theorem 3.1. *If $(\lambda_{k,\ell})\in RBVS_{+,2}^{r_1,r_2;\delta_1,\delta_2}$, then the series (1)–(4) converge in the sense of Pringsheim on $\mathbb{I}_{r,s}:=\left[\frac{\pi}{r+1},\frac{\pi}{r}\right]\times\left[\frac{\pi}{s+1},\frac{\pi}{s}\right]$, $(r,s=1,2,\dots)$, and*

$$f^{ss}(x,y)=\sum_{k=1}^{\infty}\sum_{\ell=1}^{\infty}\Delta_{1,1}\lambda_{k,\ell}\widetilde{B}_k(x)\widetilde{B}_\ell(y),\quad f^{cc}(x,y)=\sum_{k=1}^{\infty}\sum_{\ell=1}^{\infty}\Delta_{1,1}\lambda_{k,\ell}B_k(x)B_\ell(y),$$

$$f^{sc}(x,y)=\sum_{k=1}^{\infty}\sum_{\ell=1}^{\infty}\Delta_{1,1}\lambda_{k,\ell}\widetilde{B}_k(x)B_\ell(y),\quad f^{cs}(x,y)=\sum_{k=1}^{\infty}\sum_{\ell=1}^{\infty}\Delta_{1,1}\lambda_{k,\ell}B_k(x)\widetilde{B}_\ell(y),$$

where $\widetilde{B}_r(u) := \sum_{p=1}^{r} \sin pu$ *and* $B_s(v) := \sum_{q=1}^{s} \cos qv$.

Proof. Let $(m,n) \geq (r,s)$, that is, $m > r$, $n > s$, and $(x,y) \in \mathbb{I}_{r,s}$, $(r,s = 1,2,\dots)$. Then for the series (1) we can write

$$\begin{aligned} s_{m,n}(x,y) - s_{r,s}(x,y) &= \sum_{k=1}^{r}\sum_{\ell=s+1}^{n}(\cdot) + \sum_{k=r+1}^{m}\sum_{\ell=1}^{s}(\cdot) + \sum_{k=r+1}^{m}\sum_{\ell=s+1}^{n}(\cdot) \\ &:= t_{r,n}(x,y) + t_{m,s}(x,y) + t_{m,n}(x,y). \end{aligned} \tag{23}$$

Using Lemma 11 we have

$$\begin{aligned} t_{m,n}(x,y) &= \sum_{k=r+1}^{m}\sum_{\ell=s+1}^{n} \lambda_{k,\ell} \sin kx \sin \ell y \\ &= \sum_{k=r+1}^{m}\sum_{\ell=s+1}^{n} \Delta_{1,1}\lambda_{k,\ell} B_{k,\ell}(x,y) + \sum_{k=r+1}^{m} \Delta_{1,0}\lambda_{k,n+1} B_{k,n}(x,y) \\ &\quad - \sum_{k=r+1}^{m} \Delta_{1,0}\lambda_{k,s+1} B_{k,s}(x,y) + \sum_{\ell=s+1}^{n} \Delta_{0,1}\lambda_{m+1,\ell} B_{m,\ell}(x,y) \\ &\quad - \sum_{\ell=s+1}^{n} \Delta_{0,1}\lambda_{r+1,\ell} B_{r,\ell}(x,y) + \lambda_{m,n} B_{m+1,n+1}(x,y) \\ &\quad -\lambda_{m+1,s+1} B_{m,s}(x,y) - \lambda_{r+1,n+1} B_{r,n}(x,y) + \lambda_{r+1,s+1} B_{r,s}(x,y), \end{aligned}$$

where

$$B_{r,s}(x,y) = \sum_{p=1}^{r}\sum_{q=1}^{s} \sin px \sin qy.$$

Since, $|B_{r,s}(x,y)| = \mathcal{O}((xy)^{-1})$ for $(x,y) \in \mathbb{I}_{r,s}$, and $(\lambda_{k,\ell}) \in RBVS_{+,2}^{r_1,r_2;\delta_1,\delta_2}$, then for any real number $\varepsilon > 0$, as small as we wish, we have

$$\begin{aligned} |t_{m,n}(x,y)| &= \mathcal{O}((xy)^{-1}) \Bigg[\sum_{k=r+1}^{m}\sum_{\ell=s+1}^{n} |\Delta_{1,1}\lambda_{k,\ell}| + \sum_{k=r+1}^{m} |\Delta_{1,0}\lambda_{k,n+1}| \\ &\quad + \sum_{k=r+1}^{m} |\Delta_{1,0}\lambda_{k,s+1}| + \sum_{\ell=s+1}^{n} |\Delta_{0,1}\lambda_{m+1,\ell}| + \sum_{\ell=s+1}^{n} |\Delta_{0,1}\lambda_{r+1,\ell}| \\ &\quad +\lambda_{m,n} + \lambda_{m+1,s+1} + \lambda_{r+1,n+1} + \lambda_{r+1,s+1} \Bigg] \\ &= \mathcal{O}((xy)^{-1}) \Bigg[\frac{1}{r^{r_1+1+\delta_1} s^{r_2+1+\delta_2}} \sum_{i=1}^{r}\sum_{j=1}^{s} i^{r_1+1} j^{r_2+1} \lambda_{i,j} \\ &\quad + \frac{1}{r^{r_1+1+\delta_1}} \sum_{i=1}^{r} i^{r_1+1}\lambda_{i,n+1} + \frac{1}{r^{r_1+1+\delta_1}} \sum_{i=1}^{r} i^{r_1+1}\lambda_{i,s+1} \\ &\quad + \frac{1}{s^{r_2+1+\delta_2}} \sum_{j=1}^{s} j^{r_2+1}\lambda_{m+1,j} + \frac{1}{s^{r_2+1+\delta_2}} \sum_{j=1}^{s} j^{r_2+1}\lambda_{r+1,j} \end{aligned}$$

$$+\frac{4}{r^{r_1+1+\delta_1}s^{r_2+1+\delta_2}}\sum_{i=1}^{r}\sum_{j=1}^{s}i^{r_1+1}j^{r_2+1}\lambda_{i,j}\Bigg]$$

$$=\mathcal{O}((xy)^{-1})\frac{9}{r^{r_1+1+\delta_1}s^{r_2+1+\delta_2}}\sum_{i=1}^{r}\sum_{j=1}^{s}i^{r_1+1}j^{r_2+1}\lambda_{i,j}$$

$$=\mathcal{O}((xy)^{-1})r^{1-\delta_1}s^{1-\delta_2}\frac{\varepsilon}{3}=\mathcal{O}((xy)^{-2})\frac{\varepsilon}{3}, \tag{24}$$

for all $r>r_0(\varepsilon)\in\mathbb{N}$, $s>s_0(\varepsilon)\in\mathbb{N}$, and $\mathbb{I}_{r,s}$, $(r,s=1,2,\dots)$.

Similarly, we have obtained that

$$|t_{r,n}(x,y)|=\mathcal{O}((xy)^{-2})\frac{\varepsilon}{3}, \tag{25}$$

and

$$|t_{m,s}(x,y)|=\mathcal{O}((xy)^{-2})\frac{\varepsilon}{3}, \tag{26}$$

for all $r>r_0(\varepsilon)\in\mathbb{N}$, $s>s_0(\varepsilon)\in\mathbb{N}$, and $(x,y)\in\mathbb{I}_{r,s}$, $(r,s=1,2,\dots)$.

Whence, using (23)–(26) we conclude that

$$|s_{m,n}(x,y)-s_{r,s}(x,y)|<\varepsilon$$

for all $r>r_0(\varepsilon)\in\mathbb{N}$, $s>s_0(\varepsilon)\in\mathbb{N}$, and $(x,y)\in\mathbb{I}_{r,s}$, $(r,s=1,2,\dots)$.

Now, the existence of $\lim_{m+n\to\infty}s_{m,n}(x,y)$ and the use of Lemma 11 imply

$$\begin{aligned}
f^{ss}(x,y) &= \lim_{m+n\to\infty}s_{m,n}(x,y)=\lim_{m+n\to\infty}\sum_{i=1}^{m}\sum_{j=1}^{n}a_{ij}\sin ix\sin jy\\
&= \lim_{m+n\to\infty}\Bigg[\sum_{k=1}^{m}\sum_{\ell=1}^{n}\Delta_{1,1}\lambda_{k,\ell}B_{k,\ell}(x,y)+\sum_{k=1}^{m}\Delta_{1,0}\lambda_{k,n+1}B_{k,n}(x,y)\\
&\quad -\sum_{k=1}^{m}\Delta_{1,0}\lambda_{k,1}B_{k,0}(x,y)+\sum_{\ell=1}^{n}\Delta_{0,1}\lambda_{m+1,\ell}B_{m,\ell}(x,y)\\
&\quad -\sum_{\ell=1}^{n}\Delta_{0,1}\lambda_{1,\ell}B_{0,\ell}(x,y)+\lambda_{m+1,n+1}B_{m,n}(x,y)\\
&\quad -\lambda_{m+1,1}B_{m,0}(x,y)-\lambda_{1,n+1}B_{0,n}(x,y)+\lambda_{1,1}B_{0,0}(x,y)\Bigg]\\
&= \sum_{k=1}^{\infty}\sum_{\ell=1}^{\infty}\Delta_{1,1}\lambda_{k,\ell}B_{k,\ell}(x,y).
\end{aligned}$$

The proofs, pertaining to given statements for the series (2)–(4), can be obtained in the same lines. Therefore we have omitted their details.

The proof is completed. □

Theorem 3.2. *Suppose that* $(\lambda_{k,\ell})\in RBVS^{r_1,r_2;\delta_1,\delta_2}_{+,2}$, $1\le p<\infty$, *and* $f^{ss}(x,y)$ *the sum-function of the series (1). If the sequence* $(\gamma_{k,\ell})$ *satisfies the condition that there exist some* $\varepsilon_9,\varepsilon_{10}>0$ *such that the sequence*

$\{\gamma_{k,\ell}k^{-1-\delta_1 p+\varepsilon_9}\}$ *is almost decreasing for each* ℓ, *and the sequence* $\{\gamma_{k,\ell}\ell^{-1-\delta_2 p+\varepsilon_{10}}\}$ *is almost decreasing for each* k, *then the condition*

$$\sum_{k=1}^{\infty}\sum_{\ell=1}^{\infty}\gamma_{k,\ell}k^{p(2-\delta_1)-2}\ell^{p(2-\delta_2)-2}\lambda_{k,\ell}^{p}<+\infty$$

is sufficient for the validity of the condition

$$\gamma(x,y)|f^{ss}(x,y)|^p\in L(0,\pi)^2.$$

Proof. Using Lemma 2.4 part (a), we have

$$\begin{aligned}
I^{ss} &:= \int_0^{\pi}\int_0^{\pi}\gamma(x,y)|f^{ss}(x,y)|^p dxdy\\
&= \sum_{k=1}^{\infty}\sum_{\ell=1}^{\infty}\int_{\frac{\pi}{k+1}}^{\frac{\pi}{k}}\int_{\frac{\pi}{\ell+1}}^{\frac{\pi}{\ell}}\gamma(x,y)|f^{ss}(x,y)|^p dxdy\\
&\leq C\sum_{k=1}^{\infty}\sum_{\ell=1}^{\infty}\frac{\gamma_{k,\ell}}{k^{p\delta_1+2}\ell^{p\delta_2+2}}\left(\sum_{i=1}^{k}\sum_{j=1}^{\ell}ij\lambda_{ij}\right)^p. \qquad (27)
\end{aligned}$$

Applying Lemma 2.1 in (27) we get

$$\begin{aligned}
I^{ss} &\leq C\sum_{k=1}^{\infty}\frac{1}{k^{p\delta_1+2}}\left(\sum_{\ell=1}^{\infty}\frac{\gamma_{k,\ell}}{\ell^{p\delta_2+2}}\left(\sum_{j=1}^{\ell}\left(\sum_{i=1}^{k}ij\lambda_{ij}\right)\right)^p\right)\\
&\leq C\sum_{k=1}^{\infty}\frac{1}{k^{p\delta_1+2}}\sum_{\ell=1}^{\infty}\left(\frac{\gamma_{k,\ell}}{\ell^{p\delta_2+2}}\right)^{1-p}\left(\sum_{i=1}^{k}i\ell\lambda_{i,\ell}\right)^p\left(\sum_{j=\ell}^{\infty}\frac{\gamma_{k,j}}{j^{p\delta_2+2}}\right)^p.
\end{aligned}$$

Since the sequence $\{\gamma_{k,\ell}\ell^{-1-\delta_2 p+\varepsilon_{10}}\}$ is almost decreasing for each k, then

$$\left(\sum_{j=\ell}^{\infty}\frac{\gamma_{k,j}}{j^{p\delta_2+2}}\right)^p=\left(\sum_{j=\ell}^{\infty}\frac{\gamma_{k,j}j^{-p\delta_2-1+\varepsilon_{10}}}{j^{1+\varepsilon_{10}}}\right)^p\leq C\left(\gamma_{k,\ell}\ell^{-p\delta_2-1}\right)^p,$$

and hence

$$\begin{aligned}
I^{ss} &\leq C\sum_{k=1}^{\infty}\frac{1}{k^{p\delta_1+2}}\sum_{\ell=1}^{\infty}\left(\frac{\gamma_{k,\ell}}{\ell^{p\delta_2+2}}\right)^{1-p}\left(\sum_{i=1}^{k}i\ell\lambda_{i,\ell}\right)^p\left(\gamma_{k,\ell}\ell^{-p\delta_2-1}\right)^p\\
&= C\sum_{k=1}^{\infty}\frac{1}{k^{p\delta_1+2}}\sum_{\ell=1}^{\infty}\gamma_{k,\ell}\ell^{p(2-\delta_2)-2}\left(\sum_{i=1}^{k}i\lambda_{i,\ell}\right)^p. \qquad (28)
\end{aligned}$$

Applying Lemma 2.1 once again, but now in (28), we have that

$$\begin{aligned}
I^{ss} &\leq C\sum_{k=1}^{\infty}\sum_{\ell=1}^{\infty}\ell^{p(2-\delta_2)-2}\left(\frac{\gamma_{k,\ell}}{k^{p\delta_1+2}}\right)^{1-p}(k\lambda_{k,\ell})^p\left(\sum_{i=k}^{\infty}\frac{\gamma_{i,\ell}}{i^{p\delta_1+2}}\right)^p\\
&\leq C\sum_{k=1}^{\infty}\sum_{\ell=1}^{\infty}\ell^{p(2-\delta_2)-2}\left(\frac{\gamma_{k,\ell}}{k^{p\delta_1+2}}\right)^{1-p}(k\lambda_{k,\ell})^p\left(k^{-p\delta_1-1}\gamma_{k,\ell}\right)^p\\
&= C\sum_{k=1}^{\infty}\sum_{\ell=1}^{\infty}\gamma_{k,\ell}k^{p(2-\delta_1)-2}\ell^{p(2-\delta_2)-2}\lambda_{k,\ell}^{p}<+\infty,
\end{aligned}$$

because of the assumption that the sequence $\{\gamma_{k,\ell}k^{-1-\delta_1 p+\varepsilon_9}\}$ is almost decreasing for each ℓ, and the proof is done. $\square$

Theorem 3.3. *Suppose that* $(\lambda_{k,\ell}) \in RBVS_{+,2}^{r_1,r_2;\delta_1,\delta_2}$, $1 \le p < \infty$, *and* $f^{ss}(x,y)$ *the sum-function of the series (1). If the sequence* $(\gamma_{k,\ell})$ *satisfies the condition that there exist some* $\varepsilon_{11}, \varepsilon_{12} > 0$ *such that the sequence* $\{\gamma_{k,\ell}k^{p-1-\varepsilon_{11}}\}$ *is almost increasing for each* ℓ, *and the sequence* $\{\gamma_{k,\ell}\ell^{p-1-\varepsilon_{12}}\}$ *is almost increasing for each* k, *then the condition*

$$HL := \sum_{k=1}^{\infty}\sum_{\ell=1}^{\infty}\gamma_{k,\ell}k^{p\delta_1-2}\ell^{p\delta_2-2}\lambda_{k,\ell}^p < +\infty$$

is necessary for the validity of the condition

$$\gamma(x,y)|f^{ss}(x,y)|^p \in L(0,\pi)^2.$$

Proof. Let p, q be two real numbers such that $1 < p < +\infty$ and $p + q = pq$. Applying Hölder's inequality, we get

$$\begin{aligned} I &:= \int_0^\pi\int_0^\pi |g(x,y)|dxdy \\ &\le \left(\int_0^\pi\int_0^\pi \gamma(x,y)|g(x,y)|^p dxdy\right)^{1/p}\left(\int_0^\pi\int_0^\pi (\gamma(x,y))^{-q/p}dxdy\right)^{1/q}. \end{aligned}$$

Because of our assumption and

$$\begin{aligned} \int_0^\pi\int_0^\pi (\gamma(x,y))^{-q/p}dxdy &= \sum_{r_1=1}^{\infty}\sum_{r_2=1}^{\infty}\int_{\frac{\pi}{r_1+1}}^{\frac{\pi}{r_1}}\int_{\frac{\pi}{r_2+1}}^{\frac{\pi}{r_2}} (\gamma^{-1}(x,y))^{1/(p-1)}dxdy \\ &\le C\sum_{r_1=1}^{\infty}\sum_{r_2=1}^{\infty}(\gamma_{r_1,r_2})^{1/(1-p)}\int_{\frac{\pi}{r_1+1}}^{\frac{\pi}{r_1}}\int_{\frac{\pi}{r_2+1}}^{\frac{\pi}{r_2}} dxdy \\ &\le C(\gamma_{1,1})^{1/(1-p)}\sum_{r_1=1}^{\infty}\sum_{r_2=1}^{\infty}\frac{\pi^2}{(r_1r_2)^2} < +\infty, \end{aligned}$$

we obtain that $I \le +\infty$, which in fact shows that $g\,(x,y) \in L(0,\pi)^2$.

Now let $p = 1$. Then, since $\{\gamma_{r_1,r_2}\}$ is almost increasing, we have

$$\begin{aligned} \int_0^\pi\int_0^\pi |g(x,y)|dxdy &\le \sum_{r_1=1}^{\infty}\sum_{r_2=1}^{\infty}\frac{1}{C\gamma_{r_1,r_2}}\int_{\frac{\pi}{r_1+1}}^{\frac{\pi}{r_1}}\int_{\frac{\pi}{r_2+1}}^{\frac{\pi}{r_2}} \gamma(x,y)|g(x,y)|dxdy \\ &\le \frac{1}{C\gamma_{1,1}}\int_0^\pi\int_0^\pi \gamma(x,y)|g(x,y)|dxdy < +\infty. \end{aligned}$$

Thus, for all $p \in [1,+\infty)$ we verified that $g\,(x,y) \in L(0,\pi)^2$. Using this fact we can integrate the function $g\,(x,y)$, so that we get

$$\begin{aligned} F^{ss}(x,y) &:= \int_0^x\int_0^y g(u,v)dudv \\ &= \sum_{k=1}^{\infty}\sum_{\ell=1}^{\infty}\lambda_{k,\ell}\int_0^x\int_0^y \sin ku\sin\ell v dudv = 4\sum_{k=1}^{\infty}\frac{\lambda_{k,\ell}}{k\ell}\sin^2\frac{kx}{2}\sin^2\frac{\ell y}{2}. \end{aligned}$$

If we denote

$$d_{r_1,r_2} := \int_{\frac{\pi}{r_1+1}}^{\frac{\pi}{r_1}} \int_{\frac{\pi}{r_2+1}}^{\frac{\pi}{r_2}} |g(x,y)|dxdy, \;\; r_1, r_2 \in \mathbb{N},$$

then we have

$$\begin{aligned} F^{ss}\left(\frac{\pi}{m}, \frac{\pi}{n}\right) &\geq C\sum_{k=1}^{m}\sum_{\ell=1}^{n}\frac{\lambda_{k,\ell}}{k\ell}\left(\frac{k\ell}{mn}\right)^2 = \frac{C}{(mn)^2}\sum_{k=1}^{m}\sum_{\ell=1}^{m} k\ell\lambda_{k,\ell} \\ &\geq \frac{C}{m^{r_1+2}n^{r_2+2}}\sum_{k=1}^{m}\sum_{\ell=1}^{n} k^{r_1+1}\ell^{r_2+1}\lambda_{k,\ell} \\ &= \frac{Cm^{\delta_1-1}n^{\delta_2-1}}{m^{r_1+\delta_1+1}n^{r_2+\delta_2+1}}\sum_{k=1}^{m}\sum_{\ell=1}^{n} k^{r_1+1}\ell^{r_2+1}\lambda_{k,\ell} \\ &\geq Cm^{\delta_1-1}n^{\delta_2-1}\sum_{k=m}^{\infty}\sum_{\ell=n}^{\infty}|\Delta_{1,1}\lambda_{k,\ell}| \geq Cm^{\delta_1-1}n^{\delta_2-1}\lambda_{m,n} \end{aligned}$$

or

$$\lambda_{m,n} \leq Cm^{1-\delta_1}n^{1-\delta_2}F^{ss}\left(\frac{\pi}{m}, \frac{\pi}{n}\right),$$

taking into account that $\{\lambda_{k,\ell}\} \in RBVS_{+,2}^{r_1,r_2;\delta_1,\delta_2}$.

Whence,

$$HL = \sum_{k=1}^{\infty}\sum_{\ell=1}^{\infty}\gamma_{k,\ell}k^{p\delta_1-2}\ell^{p\delta_2-2}\lambda_{k,\ell}^p \leq C\sum_{k=1}^{\infty}\sum_{\ell=1}^{\infty}\gamma_{k,\ell}(k\ell)^{p-2}\left(\sum_{i=k}^{\infty}\sum_{j=\ell}^{\infty}d_{i,j}\right)^p.$$

Then, applying Lemma 2.2 and the assumption that the sequence $\{\gamma_{i,j}i^{p-1-\varepsilon_{11}}\}$ is almost increasing, we get

$$\begin{aligned} HL &\leq \sum_{k=1}^{\infty}\sum_{\ell=1}^{\infty}\gamma_{k,\ell}(k\ell)^{p-2}\left(\sum_{i=k}^{\infty}\sum_{j=\ell}^{\infty}d_{i,j}\right)^p \\ &\leq C\sum_{k=1}^{\infty}\sum_{\ell=1}^{\infty}(\gamma_{k,\ell}k^{p-2})^{1-p}\ell^{p-2}\left(\sum_{j=\ell}^{\infty}d_{k,j}\right)^p\left(\sum_{i=1}^{k}\gamma_{i,\ell}i^{p-2}\right)^p \\ &= C\sum_{k=1}^{\infty}\sum_{\ell=1}^{\infty}(\gamma_{k,\ell}k^{p-2})^{1-p}\ell^{p-2}\left(\sum_{j=\ell}^{\infty}d_{k,j}\right)^p\left(\sum_{i=1}^{k}\frac{\gamma_{i,\ell}i^{p-1-\varepsilon_{11}}}{i^{1-\varepsilon_{11}}}\right)^p \\ &\leq C\sum_{k=1}^{\infty}\sum_{\ell=1}^{\infty}(\gamma_{k,\ell}k^{p-2})^{1-p}\ell^{p-2}\left(\sum_{j=\ell}^{\infty}d_{k,j}\right)^p\left(\gamma_{k,\ell}k^{p-1}\right)^p \\ &= C\sum_{k=1}^{\infty}\sum_{\ell=1}^{\infty}\gamma_{k,\ell}k^{2(p-1)}\ell^{p-2}\left(\sum_{j=\ell}^{\infty}d_{k,j}\right)^p. \end{aligned}$$

Once again, applying Lemma 2.2 and the assumption that the sequence $\{\gamma_{i,j}j^{p-1-\varepsilon_{12}}\}$ is almost increasing, we obtain

$$\begin{aligned} HL &\le C\sum_{k=1}^{\infty}\sum_{\ell=1}^{\infty} k^{2(p-1)}(\gamma_{k,\ell}\ell^{p-2})^{1-p}d_{k,\ell}^p\left(\sum_{j=1}^{\ell}\gamma_{k,j}j^{p-2}\right)^p \\ &= C\sum_{k=1}^{\infty}\sum_{\ell=1}^{\infty} k^{2(p-1)}(\gamma_{k,\ell}\ell^{p-2})^{1-p}d_{k,\ell}^p\left(\sum_{j=1}^{\ell}\frac{\gamma_{k,j}j^{p-1-\varepsilon_{12}}}{j^{1-\varepsilon_{12}}}\right)^p \\ &\le C\sum_{k=1}^{\infty}\sum_{\ell=1}^{\infty} k^{2(p-1)}(\gamma_{k,\ell}\ell^{p-2})^{1-p}d_{k,\ell}^p\left(\gamma_{k,\ell}\ell^{p-1}\right)^p \\ &= C\sum_{k=1}^{\infty}\sum_{\ell=1}^{\infty}\gamma_{k,\ell}(k\ell)^{2(p-1)}d_{k,\ell}^p. \end{aligned}$$

Further, let $1 < p < +\infty$ and $q = \frac{p}{p-1}$. Then applying Hölder's inequality, we get

$$d_{k,\ell}^p = \left(\int_{\frac{\pi}{k+1}}^{\frac{\pi}{k}}\int_{\frac{\pi}{\ell+1}}^{\frac{\pi}{\ell}}|g(x,y)|dxdy\right)^p \le C(k\ell)^{2(1-p)}\int_{\frac{\pi}{k+1}}^{\frac{\pi}{k}}\int_{\frac{\pi}{\ell+1}}^{\frac{\pi}{\ell}}|g(x,y)|^p dxdy.$$

Thus,

$$\begin{aligned} HL &\le C\sum_{k=1}^{\infty}\sum_{\ell=1}^{\infty}\gamma_{k,\ell}\int_{\frac{\pi}{k+1}}^{\frac{\pi}{k}}\int_{\frac{\pi}{\ell+1}}^{\frac{\pi}{\ell}}|g(x,y)|^p dxdy \\ &\le C\sum_{k=1}^{\infty}\sum_{\ell=1}^{\infty}\int_{\frac{\pi}{k+1}}^{\frac{\pi}{k}}\int_{\frac{\pi}{\ell+1}}^{\frac{\pi}{\ell}}\gamma(x,y)|g(x,y)|^p dxdy \\ &= C\int_0^{\pi}\int_0^{\pi}\gamma(x,y)|g(x,y)|^p dxdy < +\infty. \end{aligned}$$

For the case $p = 1$, we have

$$\begin{aligned} HL &\le C\sum_{k=1}^{\infty}\sum_{\ell=1}^{\infty}\gamma_{k,\ell}d_{k,\ell} \le C\sum_{k=1}^{\infty}\sum_{\ell=1}^{\infty}\gamma_{k,\ell}\int_{\frac{\pi}{k+1}}^{\frac{\pi}{k}}\int_{\frac{\pi}{\ell+1}}^{\frac{\pi}{\ell}}|g(x,y)|dxdy \\ &\le C\sum_{k=1}^{\infty}\sum_{\ell=1}^{\infty}\int_{\frac{\pi}{k+1}}^{\frac{\pi}{k}}\int_{\frac{\pi}{\ell+1}}^{\frac{\pi}{\ell}}\gamma(x,y)|g(x,y)|dxdy \le C\int_0^{\pi}\int_0^{\pi}\gamma(x,y)|g(x,y)|dxdy. \end{aligned}$$

as well.

The proof is completed. □

Theorem 3.4. *Suppose that* $(\lambda_{k,\ell}) \in RBVS_{+,2}^{r_1,r_2;\delta_1,\delta_2}$, $1 \le p < \infty$, *and* $f^{cc}(x,y)$ *the sum-function of the series (2). If the sequence* $(\gamma_{k,\ell})$ *satisfies the condition that there exist some* $\varepsilon_{13}, \varepsilon_{14} > 0$ *such that the sequence* $\{\gamma_{k,\ell}k^{-1+\varepsilon_{13}}\}$ *is almost decreasing for each* ℓ, *and the sequence* $\{\gamma_{k,\ell}\ell^{-1+\varepsilon_{14}}\}$ *is almost decreasing for each* k, *then it follows from condition*

$$\sum_{k=1}^{\infty}\sum_{\ell=1}^{\infty}\gamma_{k,\ell}k^{p(2-\delta_1)-2}\ell^{p(2-\delta_2)-2}\lambda_{k,\ell}^p < +\infty \tag{29}$$

for the coefficients of the series (2) that

$$\gamma(x,y)|f^{cc}(x,y)|^p \in L(0,\pi)^2. \tag{30}$$

Proof. Lemma 12, part (b), implies

$$\begin{aligned} I^{cc} &:= \int_0^\pi \int_0^\pi \gamma(x,y)|f^{cc}(x,y)|^p dxdy \\ &= \sum_{k=1}^\infty \sum_{\ell=1}^\infty \int_{\frac{\pi}{k+1}}^{\frac{\pi}{k}} \int_{\frac{\pi}{\ell+1}}^{\frac{\pi}{\ell}} \gamma(x,y)|f^{cc}(x,y)|^p dxdy \\ &\le C\sum_{k=1}^\infty \sum_{\ell=1}^\infty \frac{\gamma_{k,\ell}}{(k\ell)^2}\left(\sum_{i=1}^k \sum_{j=1}^\ell i^{1-\delta_1} j^{1-\delta_2}\lambda_{ij}\right)^p. \end{aligned} \tag{31}$$

The use of the Lemma 2.1 in (31) gives

$$I^{cc} \le C\sum_{k=1}^\infty \frac{1}{k^2}\sum_{\ell=1}^\infty \left(\frac{\gamma_{k,\ell}}{\ell^2}\right)^{1-p}\left(\sum_{i=1}^k i^{1-\delta_1}\ell^{1-\delta_2}\lambda_{i,\ell}\right)^p \left(\sum_{j=\ell}^\infty \frac{\gamma_{k,j}}{j^2}\right)^p.$$

By assumption the sequence $\{\gamma_{k,\ell}\ell^{-1+\varepsilon_{14}}\}$ is almost decreasing for each k, therefore

$$\left(\sum_{j=\ell}^\infty \frac{\gamma_{k,j}j^{-1+\varepsilon_{10}}}{j^{1+\varepsilon_{10}}}\right)^p \le C\left(\gamma_{k,\ell}\ell^{-1}\right)^p,$$

and whence

$$I^{cc} \le C\sum_{k=1}^\infty \frac{1}{k^2}\sum_{\ell=1}^\infty \gamma_{k,\ell}\ell^{p(2-\delta_2)-2}\left(\sum_{i=1}^k i^{1-\delta_1}\lambda_{i,\ell}\right)^p. \tag{32}$$

The use of the Lemma 2.1 once again, in (32), we find

$$\begin{aligned} I^{cc} &\le C\sum_{k=1}^\infty \sum_{\ell=1}^\infty \ell^{p(2-\delta_2)-2}\left(\frac{\gamma_{k,\ell}}{k^2}\right)^{1-p}\left(k^{1-\delta_1}\lambda_{k,\ell}\right)^p\left(k^{-1}\gamma_{k,\ell}\right)^p \\ &= C\sum_{k=1}^\infty \sum_{\ell=1}^\infty \gamma_{k,\ell}k^{p(2-\delta_1)-2}\ell^{p(2-\delta_2)-2}\lambda_{k,\ell}^p < +\infty, \end{aligned}$$

since the sequence $\{\gamma_{k,\ell}k^{-1+\varepsilon_{13}}\}$ is almost decreasing for each ℓ.

The proof is completed. □

Theorem 3.5. $(\lambda_{k,\ell}) \in RBVS_{+,2}^{r_1,r_2;\delta_1,\delta_2}$, $1 \le p < \infty$, *and* $f^{cc}(x,y)$ *the sum-function of the series (2). If the sequence* $(\gamma_{k,\ell})$ *satisfies the condition that there exist some* $\varepsilon_{15},\varepsilon_{16} > 0$ *such that the sequence* $\{\gamma_{k,\ell}k^{p-1+\varepsilon_{15}}\}$ *is almost increasing for each* ℓ*, and the sequence* $\{\gamma_{k,\ell}\ell^{p-1+\varepsilon_{16}}\}$ *is almost increasing for each* k*, then the conditions*

$$\sum_{k=1}^\infty \sum_{\ell=1}^\infty \gamma_{k,\ell}k^{p\delta_1-2}\ell^{p\delta_2-2}\lambda_{k,\ell}^p < +\infty$$

and (22) are necessary for the validity of condition

$$\gamma(x,y)|f^{cc}(x,y)|^p \in L(0,\pi)^2.$$

Proof. The conclusion that $f^{cc}(x,y) \in L(0,\pi)^2$ can be derived in the same way as in the proof of Theorem 3.3. This enables us to integrate $f^{cc}(x,y)$ so that to obtain

$$H(x,y) = \int_0^x \int_0^y f(u,v)\,dudv = \sum_{i=1}^{\infty}\sum_{j=1}^{\infty} \frac{\lambda_{i,j}}{ij} \sin ix \sin jy.$$

Now, Lemma 2.5 ensures the applicability of Theorem 3.3 to the function $H(x,y)$, so that

$$\sum_{k=1}^{\infty}\sum_{\ell=1}^{\infty} \gamma_{k,\ell}^{*} k^{p\delta_1-2} \ell^{p\delta_2-2} \lambda_{k,\ell}^{p} \le C \int_0^{\pi}\int_0^{\pi} \gamma^{*}(x,y) \left|H(x,y)\right|^p dxdy,$$

where $\left\{\gamma_{k,\ell}^{*}\right\}$ satisfies the following condition: there exist some $\varepsilon_{17}, \varepsilon_{18} > 0$ such that the sequence $\{\gamma_{k,\ell} k^{p-1-\varepsilon_{17}}\}$ is almost increasing for each ℓ, and the sequence $\{\gamma_{k,\ell} \ell^{p-1-\varepsilon_{18}}\}$ is almost increasing for each k. For $\gamma_{k,\ell}^{*} = \gamma_{k,\ell}(k\ell)^p$, this condition is obviously satisfied. The last inequality implies that

$$\begin{aligned}
HL &:= \sum_{k=1}^{\infty}\sum_{\ell=1}^{\infty} \gamma_{k,\ell} k^{p\delta_1-2} \ell^{p\delta_2-2} \lambda_{k,\ell}^{p} \\
&= \sum_{k=1}^{\infty}\sum_{\ell=1}^{\infty} \gamma_{k,\ell}^{*} k^{p\delta_1-2} \ell^{p\delta_2-2} \left(\frac{\lambda_{k,\ell}}{k\ell}\right)^p \\
&\le C \int_0^{\pi}\int_0^{\pi} \frac{\gamma(x,y)}{(xy)^p} \left|H(x,y)\right|^p dxdy \\
&\le C \sum_{m=1}^{\infty}\sum_{n=1}^{\infty} \int_{\frac{\pi}{m+1}}^{\frac{\pi}{m}} \int_{\frac{\pi}{n+1}}^{\frac{\pi}{n}} \frac{\gamma(x,y)}{(xy)^p} \left(\int_0^x \int_0^y \left|f^{cc}(u,v)\right| dudv\right)^p dxdy \\
&\le C \sum_{m=1}^{\infty}\sum_{n=1}^{\infty} \gamma_{m,n}(mn)^{p-2} \left(\int_0^{\frac{\pi}{m}} \int_0^{\frac{\pi}{n}} \left|f^{cc}(u,v)\right| dudv\right)^p \\
&= C \sum_{m=1}^{\infty}\sum_{n=1}^{\infty} \gamma_{m,n}(mn)^{p-2} \left(\sum_{i=m}^{\infty}\sum_{j=n}^{\infty} \int_{\frac{\pi}{i+1}}^{\frac{\pi}{i}} \int_{\frac{\pi}{j+1}}^{\frac{\pi}{j}} \left|f^{cc}(u,v)\right| dudv\right)^p .
\end{aligned}$$

For the sake of brevity, we denote

$$h_{i,j} = \int_{\frac{\pi}{i+1}}^{\frac{\pi}{i}} \int_{\frac{\pi}{j+1}}^{\frac{\pi}{j}} \left|f^{cc}(u,v)\right| dudv;\ k,\ell \in \mathbb{N}.$$

Thus, using Lemma 2.2 twice, we get

$$HL \le C \sum_{m=1}^{\infty}\sum_{n=1}^{\infty} \gamma_{m,n}(mn)^{2p-2} h_{m,n}^{p}.$$

Let $1 < p < +\infty$ and $q = \frac{p}{p-1}$. Then we apply Hölder's inequality, in order to find

$$h_{m,n}^{p} = \left(\int_{\frac{\pi}{m+1}}^{\frac{\pi}{m}} \int_{\frac{\pi}{n+1}}^{\frac{\pi}{n}} \left|f^{cc}(u,v)\right| dudv\right)^p \le C(mn)^{2(1-p)} \int_{\frac{\pi}{m+1}}^{\frac{\pi}{m}} \int_{\frac{\pi}{n+1}}^{\frac{\pi}{n}} |f^{cc}(x,y)|^p dxdy.$$

Consequently, we obtain

$$\begin{aligned} HL &\le C\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}\gamma_{m,n}\int_{\frac{\pi}{m+1}}^{\frac{\pi}{m}}\int_{\frac{\pi}{n+1}}^{\frac{\pi}{n}}|f^{cc}(x,y)|^p dxdy \\ &\le C\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}\int_{\frac{\pi}{m+1}}^{\frac{\pi}{m}}\int_{\frac{\pi}{n+1}}^{\frac{\pi}{n}}\gamma(x,y)|f^{cc}(x,y)|^p dxdy \\ &\le C\int_0^{\pi}\int_0^{\pi}\gamma(x,y)|f^{cc}(x,y)|^p dxdy. \end{aligned}$$

Finally, for $p=1$, we also have

$$HL \le C\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}\gamma_{m,n}f_{m,n} \le C\int_0^{\pi}\gamma(x,y)|f^{cc}(x,y)|dxdy.$$

The proof is completed. □

Remark 3.1. *Similar results can be obtained pertaining to the functions $f^{sc}(x,y)$ and $f^{cs}(x,y)$ defined by (3) and (4) respectively.*

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University of Prishtina, Faculty of Education, Department of Mathematics and Informatics, Avenue "Mother Theresa " no. 5, Prishtinë 10000, Republic of Kosovo

Email address: `xhevat.krasniqi@uni-pr.edu`

Department of Mathematics, School of Advanced Sciences, Vellore Institute of Technology (VIT) University, Vellore 632 014, Tamil Nadu, India

Email address: `lakshminarayanmishra04@gmail.com`