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ON HARMONIOUS COLORINGS OF LEXICOGRAPHIC PRODUCT OF GRAPHS

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Abstract. Harmonious coloring was first introduced by Harary and Plantholt in 1982. A harmonious coloring is a proper vertex coloring in which every pair of colors appears on at most one pair of adjacent vertices. The harmonious chromatic number $\chi_H(G)$ of a graph G is the minimum number of colors needed for any harmonious coloring of G. In this paper, we obtain the harmonious chromatic number of lexicographic product of two graphs G and H, denoted by $G[H]$. Path and complete graphs are used to obtain extremal properties of graphs and to obtain upper and lower bounds for some graph parameters. Here, we first consider the graph $G[H]$ where G is the complete graph and H is any simple graph such as the path graph, cycle graph, wheel graph, complete graph, star graph, fan graph or complete bipartite graph. Secondly, we consider G as the path graph and H as the complete graph or path graph respectively. Finally, we consider G as the wheel graph and H as the complete graph.

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1. Introduction

All graphs considered in this paper are non trivial, finite, simple and undirected. Let G be a graph with vertex set V and edge set E. A k-coloring of G is a coloring which consists of k different colors and in this case G is said to be k-colorable. The minimum number k for which there is a k-coloring of the graph G is called the chromatic number of G and denoted by $\chi(G)$. If $\chi(G) = k$, then we can say that G is k-chromatic. Chromatographic graph theory is one of the oldest study areas in graph theory.

The first paper on harmonious graph coloring was published in 1982 by Harary and Plantholt. A harmonious coloring is a proper vertex coloring in which every pair of colors appears on at most one pair of adjacent vertices. The *harmonious chromatic number* $\chi_h(G)$ of a graph G is the minimum number of colors needed for any harmonious coloring of G [3, 5, 6, 7, 8, 10, 11].

In 1991, Mc Diarmid and Xinhua [7] gave upper bounds for harmonious colorings. Lu studied the harmonious chromatic number of a complete binary and trinary tree in [10]. Georges studied on the harmonious colorings of collections of graphs in [3]. Vivin et al. considered on the harmonious

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coloring of central graphs in [8]. The concept of central graph has potential applications in communication networks, data compression and clustering.

2. Preliminaries

Lexicographic product was introduced by Hausdorff in 1914. In graph theory, the *lexicographic product* $G[H]$ of graphs G and H is a graph such that the vertex set of $G \cdot H$ is the cartesian product $V(G) \times V(H)$ of two vertex sets and any two vertices (u, v) and (x, y) are adjacent in $G[H]$ if and only if either

- u is adjacent with x in G or
- $u = x$ and v is adjacent with y in H.

The lexicographic product is also called the composition, [4]. In [1], the lexicographic product was applied to the graphs obtained by algebraic structures.

A graph G is *complete* if every pair of distinct vertices of G are adjacent in G. A complete graph with n vertices is denoted by K_n. A trail is called a *path* if all of its vertices are distinct. A closed trail whose internal vertices are distinct is called a *cycle*. A *wheel graph* is a graph formed by connecting a single central vertex to all vertices of a cycle. If it has n vertices where $n \geq 4$, it is denoted by W_n. In this paper, we define the vertex set of wheel graph of order n as $V(W_n) = \{u_i : \ 1 \leq i \leq n\}$ where u_1 is the hub of the cycle and $\{u_2, \ u_3, \ \dots u_n\}$ are the outer vertices on the cycle in cyclic order. A *complete bipartite graph* $K_{m,n}$ is a bipartite graph with bipartition X and Y such that $|X| = m, \ |Y| = n$ and every vertex in X is adjacent to every vertex in Y. A *star graph* is the complete bipartite graph $K_{1,n}$. The *fan graph* $F_{m,n} \cong \overline{K_m} \vee P_n$ is the graph with vertex set $V(F_{m,n}) = V(\overline{K_m}) \cup V(P_n)$ and edge set $E(F_{m,n}) = E(P_n) \cup \{uv | u \in V(\overline{K_m}), v \in V(P_n)\}$. Clearly, $|V(F_{m,n})| = |m + n|$ and $|E(F_{m,n})| = |n - 1 + mn|$. For the details of the above notions and for other fundamental definitions and results, see, e.g. [2, 9].

3. Main Results

In this section, we obtain the harmonious chromatic number of lexicographic product of two graphs denoted by $G[H]$. First, we consider the case $G[H]$ where G is a complete graph and H is one of the path graph, cycle graph, wheel graph, complete graph, star graph, fan graph or complete bipartite graph. Secondly, we consider G as the path graph and H as the complete graph or path graph, respectively. Finally, we consider the case where G is a wheel graph and H is a complete graph.

Theorem 3.1. *Let G be a complete graph of order $m > 2$ and H be a simple graph of order $n > 3$. Then*

$$\chi_h (G [H]) = \begin{cases} mn, & \text{if } H \cong P_n, \ C_n \text{ or } K_n, \\ mn + 2, & \text{if } H \cong K_{1,n}, \\ m(n + p), & \text{if } H \cong F_{n,p} \text{ or } K_{n,p}. \end{cases}$$

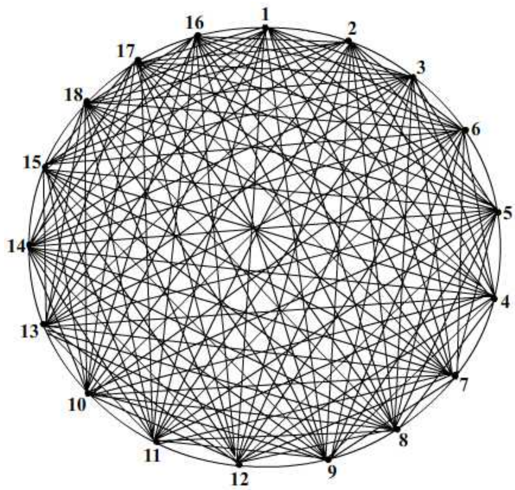

FIGURE 1. Harmonious chromatic number of $K_3[W_6]$ is 18

Proof. First, we define the vertex set of G as $V(G) = \{u_i : 1 \leq i \leq m\}$.

Case 1. Let H be isomorphic to P_n, C_n or K_n. Let $V(H) = \{v_j : 1 \leq j \leq n\}$. By the definition of lexicographic product, let

$$V(G[H]) = \bigcup_{i=1}^{m} \{s_{i,j} : 1 \leq j \leq n\}$$

where $s_{i,j}$ are the vertices (u_i, v_j) where $1 \leq i \leq m$ and $1 \leq j \leq n$. Define a mapping $\sigma : V(G[H]) \to N$ as follows:

$$\begin{aligned} \sigma(s_{i,j}) &= i, \text{ for } 1 \leq i \leq m, \quad j = 1, \\ \sigma(s_{i,j}) &= (j-1)m + i, \text{ for } 1 \leq i \leq m, \quad 2 \leq j \leq n. \end{aligned}$$

Suppose, on the contrary, that $\chi_h(G[H]) \geq mn$. But $|V(G[H])| = mn$ which is a contradiction with the definition of the lexicographic product, therefore only the case $\chi_h(G[H]) \leq mn$ may be possible. If $\chi_h(G[H]) \leq mn$, then it contradicts with the definition of the harmonious coloring that exactly one pair of different colors should exist. Therefore $\chi_h(G[H]) = mn$.

Case 2. Let H be the star graph of order $n+1$. Let $V(H) = \{v_1\} \cup \{v_j : 2 \leq j \leq n+1\}$. By the definition of lexicographic product, we can let $V(G[H]) = \bigcup_{i=1}^{m} \{s_{i,j} : 1 \leq j \leq n+1\}$ where $s_{i,j}$ are the vertices (u_i, v_j) such that $1 \leq i \leq m$ and $1 \leq j \leq n+1$. Define a mapping $\sigma : V(G[H]) \to \mathbb{N}$ as follows:

$$\begin{aligned} \sigma(s_{i,j}) &= i, \text{ for } 1 \leq i \leq m, \quad j = 1; \\ \sigma(s_{i,j}) &= (j-1)m + i, \text{ for } 1 \leq i \leq m, \quad 2 \leq j \leq n+1. \end{aligned}$$

Suppose, on the contrary, that $\chi_h(G[H]) \geq mn + 2$. But the $|V(G[H])|$ is $mn + 2$, so only the case $\chi_h(G[H]) = mn + 2$ is possible. If $\chi_h(G[H]) \leq mn + 2$, then it contradicts with the fact that in a harmonious coloring, exactly one pair of colors should exist. So $\chi_h(G[H]) = mn + 2$ is the only possible case.

Case 3. Let H be isomorphic to $K_{n,p}$ or $F_{n,p}$. Let $V(H) = \{v_j : 1 \le j \le n \text{ or } n+1 \le j \le p)\}$. By the definition of lexicographic product, let

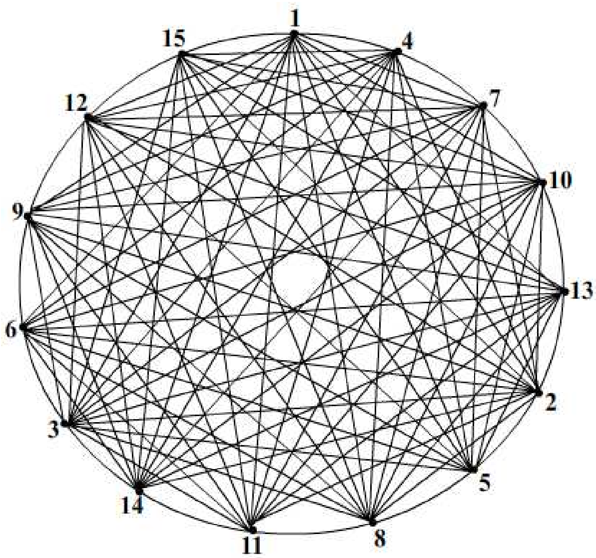

FIGURE 2. Harmonious chromatic number of $K_3[F_{1,4}]$ is 15

$V(G[H]) = \bigcup_{i=1}^{m} \{s_{i,j} : 1 \le j \le n \text{ or } n+1 \le j \le p\}$ where $s_{i,j}$ are the vertices (u_i, v_j) with $1 \le i \le m$ and $1 \le j \le n$ or $n+1 \le j \le p$. Define a mapping $\sigma : V(G[H]) \to \mathbb{N}$ as follows:

$$\begin{aligned} \sigma(s_{i,j}) &= i; && \text{for } 1 \le i \le m, \quad j = 1, \\ \sigma(s_{i,j}) &= (j-1)n + i, && \text{for } 2 \le j \le n + p. \end{aligned}$$

Suppose, on the contrary, that $\chi_h(G[H]) \ge m(n+p)$. As $|V(G[H])| = m(n+p)$, only the case $\chi_h(G[H]) \le m(n+p)$ is possible. Also if $\chi_h(G[H]) \le m(n+p)$, then it contradicts with the definition of harmonious coloring saying that exactly one pair of colors should exist. So $\chi_h(G[H]) = m(n+p)$. □

Theorem 3.2. *Let G be a path graph of order* 2 *or* 3 *and H be a simple graph of order n. Then if $H \cong K_n$ or P_n, then*

$$\chi_h(G[H]) = mn.$$

Proof. First, we take the vertex set of G of order m by $V(G) = \{u_i : 1 \le i \le m\}$ where $m = 2$ or 3, and the vertex set of H of order n by $V(H) = \{v_j : 1 \le j \le n\}$. By the definition of lexicographic product, we let $V(G[H]) = \bigcup_{i=1}^{m} \{s_{i,j} : 1 \le j \le n\}$ where $s_{i,j}$ are the vertices (u_i, v_j) with

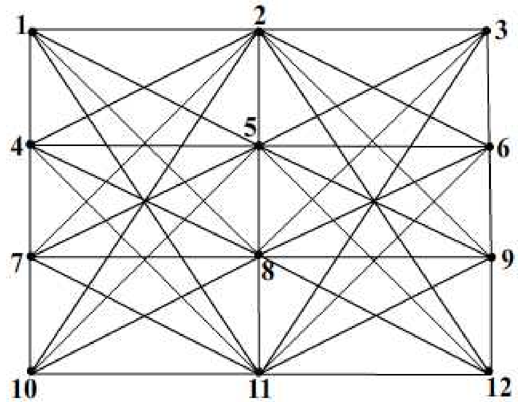

FIGURE 3. Harmonious chromatic number of $P_3[P_4]$ is 12

$1 \leq i \leq m,\ 1 \leq j \leq n$. Define a mapping $\sigma:\ V(G[H]) \to \mathbb{N}$ as follows:

$$\begin{array}{lll} \sigma(s_{i,j}) & = i, & \text{for } 1 \leq i \leq m, \quad j = 1; \\ \sigma(s_{i,j}) & = (j-1)m + i, & \text{for } 1 \leq i \leq m, \quad 2 \leq j \leq n. \end{array}$$

Sİmilarly to Theorem 3.1, we complete the proof. □

Theorem 3.3. *Let G be a wheel graph of order m and H be a simple graph of order n. Then if $H \cong K_n$, then*

$$\chi_h(G[H]) = mn.$$

Proof. This proof follows by Theorems 3.1 and 3.2. □

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