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A NEW CLASS OF BERNOULLI POLYNOMIALS ATTACHED TO POLYEXPONENTIAL FUNCTIONS AND RELATED IDENTITIES

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ABSTRACT. In [10], Kim and Kim defined polyexponential functions as well as unipoly functions. They have introduced type 2 poly-Bernoulli polynomials, and gave their analytical properties. Motivated by [10], we introduce partially degenerate polyexponential-Bernoulli polynomials of the second kind. We derive some identities for these polynomials including type 2-Euler polynomials and Stirling numbers of the first kind via generating function methods and analytical means. Finally, we represent Gaussian integral representation of polyexponential-Bernoulli polynomials of the second kind.

1. Introduction

Special numbers and polynomials have significant roles in various branches of mathematics, theoretical physics, and engineering. The problems arising in mathematical physics and engineering are framed in terms of differential equations. Most of these equations can only be treated by using various families of special polynomials which provide new means of mathematical analysis. They are widely used in computational models of scientific and engineering problems. Also, they lead to the derivation of different useful identities in a fairly straight forward way and motivate to consider possible extensions of new families of special polynomials.

The motivation for the new classes of polynomials is because of their intrinsic scientific significance and to the way that they may be demonstrated to be natural solutions of a certain set of (partial) differential equations under some conditions which often appear in the treatment of the electromagnetic wave propagation, quantum beam lifetime in storage rings, etc.

Recent observations including the type 2 degenerate Bernoulli and Euler numbers [3], type 2 degenerate Euler and Bernoulli polynomials [6], type 2 degenerate poly-Bernoulli numbers and polynomials arising from degenerate polyexponential function [11], type 2 degenerate Bernoulli polynomials [17], generalized type 2 degenerate Euler numbers [18], type 2 Daehee and Changhee polynomials derived from p-adic integrals on $\mathbb{Z}_p$ [19], type 2 poly-Apostol-Bernoulli polynomials [20], type 2 degenerate poly-Euler Polynomials [21] and type 2 degenerate central Fubini polynomials [23] have been investigated extensively.

Let us begin with the following definitions of some special polynomials.

Let $B_n(x)$ be the Bernoulli polynomials given by

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$$\sum_{n=0}^{\infty} B_n(x)\frac{t^n}{n!} = \frac{t}{e^t-1}e^{xt} \qquad (|t|<2\pi)\,. \tag{1.1}$$

In the case when $x=0$, $B_n =: B_n(0)$ are called the Bernoulli numbers, *cf.* [3, 5, 9, 11, 16].

In [13], type 2-Bernoulli and type 2-Euler polynomials are defined by means of the following generating function, respectively:

$$\sum_{n=0}^{\infty} b_n(x)\frac{t^n}{n!} = \frac{t}{e^t-e^{-t}}e^{xt} \quad (|t|<\pi) \text{ and } \sum_{n=0}^{\infty} E_n^*(x)\frac{t^n}{n!} = \frac{2}{e^t+e^{-t}}e^{xt} \ \left(|t|<\frac{\pi}{2}\right). \tag{1.2}$$

When $x=0$, $b_n(0) := b_n$ and $E_n^*(0) := E_n^*$ are called type 2-Bernoulli and type 2-Euler numbers.

The Stirling numbers of the first kind are defined by

$$\frac{1}{\ell!}(\log(1+t))^{\ell} = \sum_{m=\ell}^{\infty} S_1(m,\ell)\frac{t^m}{m!} \qquad \text{(see [3, 8, 11, 23])}. \tag{1.3}$$

It is well known from the classical analysis that $e^{xt} = \lim_{\lambda\to 0}(1+\lambda t)^{\frac{x}{\lambda}}$. The degenerate exponential function, $e_\lambda^x(t)$, of e^{xt} may be interpreted without the limit case. Namely, it is defined by

$$e_\lambda^x(t) := (1+\lambda t)^{\frac{x}{\lambda}} = \sum_{m=0}^{\infty}(x)_{m,\lambda}\frac{t^m}{m!} \text{ and } e_\lambda^x(t)\,e_\lambda^y(t) = e_\lambda^{x+y}(t)\,, \tag{1.4}$$

with the assumption $e_\lambda^1(t) := e_\lambda(t)$, where $(x)_{m,\lambda}$ is the λ-falling factorial sequence given by

$$(x)_{m,\lambda} = x(x-\lambda)(x-2\lambda)\cdots(x-(m-1)\lambda) \qquad (m\geq 1). \tag{1.5}$$

with the assumption $(x)_{0,\lambda} := 1$, *cf.* [11].

The pioneering of this idea was Carlitz ([2]) who considered for degenerate Bernoulli and Euler polynomials as follows:

$$\sum_{n=0}^{\infty}\beta_{n,\lambda}(x)\frac{t^n}{n!} = \frac{t}{e_\lambda(t)-1}e_\lambda^x(t) \quad\text{and}\quad \sum_{n=0}^{\infty}E_{n,\lambda}(x)\frac{t^n}{n!} = \frac{2}{e_\lambda(t)+1}e_\lambda^x(t) \quad (\lambda\in\mathbb{R}), \tag{1.6}$$

respectively. At the point $x=0$ in (1.6), $\beta_{n,\lambda} =: \beta_{n,\lambda}(0)$ and $E_{n,\lambda} =: E_{n,\lambda}(0)$ are called, respectively, the degenerate Bernoulli and Euler numbers, see [2].

In [6], Jang and Kim introduced type 2 degenerate Euler polynomials by the following generating function:

$$\sum_{n=0}^{\infty}\mathcal{E}_{n,\lambda}(x)\frac{t^n}{n!} = \frac{2}{e_\lambda(t)+e_\lambda^{-1}(t)}e_\lambda^x(t)\,.$$

Recently, Kim-Kim [10] have introduced polyexponential function,

$$e_k(t) = \sum_{n=1}^{\infty}\frac{t^n}{(n-1)!n^k}, \tag{1.7}$$

as inverse to the polylogarithm function,

$$\mathrm{Li}_k(t) = \sum_{n=1}^{\infty} \frac{t^n}{n^k} \qquad (|t| < 1; k \in \mathbb{Z}), \tag{1.8}$$

to generalize Bernoulli polynomials as given below

$$\sum_{n=0}^{\infty} \beta_n^{(k)}(x) \frac{t^n}{n!} = \frac{e_k(\log(1+t))}{e^t - 1} e^{xt} \qquad (k \in \mathbb{Z}). \tag{1.9}$$

Upon setting $x = 0$ in (1.9), $\beta_n^{(k)}(0) := \beta_n^{(k)}$ are called the polyexponential-Bernoulli numbers. Since $e_1(t) = e^t - 1$, one has $\beta_n^{(1)}(x) := B_n(x)$ that stands for classical Bernoulli polynomials given by (1.1).

By the motivation of the works of Kim-Kim [6, 10, 13, 18], we first define degenerate polyexponential-Bernoulli polynomials of the second kind. We investigate some new properties of these polynomials and derive some new identities and relations between the degenerate type 2-Euler, type 2-Bernoulli polynomials and Stirling numbers of the first kind.

2. Partially Degenerate polyexponential-Bernoulli polynomials of the second kind

In this section, we begin with the following definition.

Definition 1. Let $\mathcal{B}_{m,\lambda}^{(\ell)}(x)$ be partially degenerate polyexponential-Bernoulli polynomials of the second kind given by

$$\sum_{m=0}^{\infty} \mathcal{B}_{m,\lambda}^{(\ell)}(x) \frac{t^m}{m!} = \frac{e_\ell(\log(1+t))}{e_\lambda(t) - e_\lambda^{-1}(t)} e_\lambda^x(t). \tag{2.1}$$

Definition 2. Let $\widetilde{\beta}_m^{(k)}(x \mid \lambda)$ be partially degenerate polyexponential-Bernoulli polynomials given by

$$\sum_{m=0}^{\infty} \widetilde{\beta}_m^{(\ell)}(x \mid \lambda) \frac{t^m}{m!} = \frac{e_\ell(\log(1+t))}{e_\lambda(t) - 1} e_\lambda^x(t).$$

When $x = 0$, $\widetilde{\beta}_m^{(\ell)}(0 \mid \lambda) := \widetilde{\beta}_m^{(\ell)}(\lambda)$ are called partially degenerate polyexponential-Bernoulli numbers.

Remark 2.1. Letting λ tend to 0 in (2.1) yields

$$\sum_{m=0}^{\infty} \mathcal{B}_m^{(\ell)}(x) \frac{t^m}{m!} = \frac{e_\ell(\log(1+t))}{e^t - e^{-t}} e^{xt}, \tag{2.2}$$

where $\mathcal{B}_m^{(\ell)}(x)$ are polyexponential-Bernoulli polynomials of the second kind.

Remark 2.2. Taking $\ell = 1$ in (2.1) yields

$$\sum_{m=0}^{\infty} b_{m,\lambda}(x) \frac{t^m}{m!} = \frac{t}{e_\lambda(t) - e_\lambda^{-1}(t)} e_\lambda^x(t),$$

with the assumption $B_{m,\lambda}^{(1)}(x) := b_{m,\lambda}(x)$, which is called degenerate Bernoulli polynomials of the second kind.

Remark 2.3. When $\ell = 1$ and $\lambda \to 0$ in Eq. (2.1), we have

$$\lim_{\lambda\to 0}\sum_{m=0}^{\infty}\mathcal{B}_{m,\lambda}^{(1)}(x)\frac{t^m}{m!}=\sum_{m=0}^{\infty}b_m(x)\frac{t^m}{m!}=\frac{t}{e^t-e^{-t}}e^{xt}, \tag{2.3}$$

where $b_m(x)$ are called type 2-Bernoulli polynomials.

By (1.7), it is easy to see that

$$\begin{aligned}\frac{d}{dt}e_\ell(\log(1+t)) &= \frac{d}{dt}\sum_{n=1}^{\infty}\frac{(\log(1+t))^n}{(n-1)!n^\ell}\\ &= \sum_{n=1}^{\infty}\frac{n(\log(1+t))^{n-1}\frac{1}{1+t}}{(n-1)!n^\ell}\\ &= \frac{1}{(1+t)\log(1+t)}e_{l-1}(\log(1+t)).\end{aligned}$$

So we have

$$\begin{aligned}\sum_{m=0}^{\infty}\mathcal{B}_{m,\lambda}^{(\ell)}(x)\frac{t^m}{m!} &= \frac{e_\lambda^x(t)}{e_\lambda(t)-e_\lambda^{-1}(t)}\int_0^t\frac{1}{(1+t_1)\log(1+t_1)}\int_0^{t_1}\frac{1}{(1+t_2)\log(1+t_2)}\times\\ &\cdots\int_0^{t_{\ell-2}}\frac{1}{(1+t_{\ell-1})\log(1+t_{\ell-1})}e_1(\log(1+t_{\ell-1}))\prod_{i=1}^{\ell-1}dt_i\\ &= \frac{e_\lambda^x(t)}{e_\lambda(t)-e_\lambda^{-1}(t)}\int_0^t\frac{1}{(1+t_1)\log(1+t_1)}\int_0^{t_1}\frac{1}{(1+t_2)\log(1+t_2)}\times\\ &\cdots\int_0^{t_{\ell-2}}\frac{t_{\ell-1}}{(1+t_{\ell-1})\log(1+t_{\ell-1})}\prod_{i=1}^{\ell-1}dt_i.\end{aligned}$$

Here, in particular $\ell = 2$, we have

$$\begin{aligned}\sum_{m=0}^{\infty}\mathcal{B}_{m,\lambda}^{(2)}(x)\frac{t^m}{m!} &= \frac{e_\lambda^x(t)}{e_\lambda(t)-e_\lambda^{-1}(t)}\int_0^t\frac{t_1}{(1+t_1)\log(1+t_1)}dt_1\\ &= \left(\sum_{m=0}^{\infty}b_{m,\lambda}(x)\frac{t^m}{m!}\right)\left(\sum_{n=0}^{\infty}\frac{B_n^{(n)}}{n+1}\frac{t^n}{n!}\right)\\ &= \sum_{m=0}^{\infty}\left(\sum_{\ell=0}^{m}\binom{m}{\ell}b_{m-\ell,\lambda}(x)\frac{B_\ell^{(\ell)}}{\ell+1}\right)\frac{t^m}{m!},\end{aligned}$$

where $B_\ell^{(\alpha)}$ are called Bernoulli numbers of order α given by

$$\sum_{\ell=0}^{\infty}B_\ell^{(\alpha)}\frac{t^\ell}{\ell!}=\left(\frac{t}{e^t-1}\right)^\alpha, \text{ see [15].}$$

Thus we state the following theorem.

Theorem 2.4. *The following identitiy holds true:*

$$\mathcal{B}_{m,\lambda}^{(2)}(x)=\sum_{\ell=0}^{m}\binom{m}{\ell}b_{m-\ell,\lambda}(x)\frac{B_\ell^{(\ell)}}{\ell+1}.$$

Theorem 2.5. *Let n be a nonnegative number. Then, the following equality holds*

$$\mathcal{B}_{n,\lambda}^{(\ell)}(x)=\sum_{m=0}^{n}\binom{n}{m}\mathcal{B}_{n,\lambda}^{(\ell)}\ (x)_{n-m,\lambda}\,. \tag{2.4}$$

Proof. It is proved by using (1.4) and (2.1) that

$$\sum_{n=0}^{\infty}\mathcal{B}_{n,\lambda}^{(\ell)}(x)\frac{t^n}{n!} = \frac{e_\ell(\log(1+t))}{e_\lambda(t)-e_\lambda^{-1}(t)}e_\lambda^x(t) \tag{2.5}$$

$$= \left(\sum_{n=0}^{\infty}\mathcal{B}_{n,\lambda}^{(\ell)}\frac{t^n}{n!}\right)\left(\sum_{n=0}^{\infty}(x)_{n,\lambda}\frac{t^n}{n!}\right)$$

$$= \sum_{n=0}^{\infty}\left(\sum_{m=0}^{n}\binom{n}{m}\mathcal{B}_{m,\lambda}^{(\ell)}\ (x)_{n-m,\lambda}\right)\frac{t^n}{n!} \tag{2.6}$$

Therefore, by equating the left-hand side of Eq. (2.5) and the right-hand side of Eq. (2.6) of the coefficients $\frac{t^n}{n!}$, we arrive at the desired result. □

Theorem 2.6. *Let n be nonnegative number. Then the identity*

$$\mathcal{B}_{m,\lambda}^{(k)}(x)=\frac{1}{2}\sum_{\ell=0}^{m}\binom{m}{\ell}\widetilde{\beta}_{\ell}^{(k)}(\lambda)\frac{\mathcal{E}_{m-\ell,2\lambda}(2x+1)}{2^{m-\ell}}$$

holds true. In particular, we have

$$\mathcal{B}_{m,\lambda}^{(k)}=\frac{1}{2}\sum_{\ell=0}^{m}\binom{m}{\ell}\widetilde{\beta}_{\ell}^{(k)}(\lambda)\frac{\mathcal{E}_{m-\ell,2\lambda}(1)}{2^{m-\ell}}.$$

Proof. Recall from (2.1) that

$$\sum_{n=0}^{\infty}\mathcal{B}_{n,\lambda}^{(k)}(x)\frac{t^n}{n!}=\frac{e_k(\log(1+t))}{e_\lambda(t)-e_\lambda^{-1}(t)}e_\lambda^x(t)\,. \tag{2.7}$$

By the simple calculation, it becomes

$$\frac{e_k(\log(1+t))}{e_\lambda(t)-e_\lambda^{-1}(t)}e_\lambda^x(t) = \frac{e_k(\log(1+t))}{(e_\lambda(t)+1)(e_\lambda(t)-1)}e_\lambda^{x+1}(t)$$

$$= \frac{1}{2}\frac{e_k(\log(1+t))}{e_\lambda(t)-1}\frac{2}{e_\lambda(t)+1}e_\lambda^{x+1}(t)$$

$$= \frac{1}{2}\left(\sum_{n=0}^{\infty}\widetilde{\beta}_{n}^{(k)}(\lambda)\frac{t^n}{n!}\right)\left(\sum_{n=0}^{\infty}\frac{\mathcal{E}_{n,2\lambda}(2x+1)}{2^n}\frac{t^n}{n!}\right)$$

$$= \frac{1}{2}\sum_{n=0}^{\infty}\left(\sum_{\ell=0}^{n}\binom{n}{\ell}\widetilde{\beta}_{\ell}^{(k)}(\lambda)\frac{\mathcal{E}_{n-\ell,2\lambda}(2x+1)}{2^{n-\ell}}\right)\frac{t^n}{n!}. \tag{2.8}$$

By comparing the coefficients of the same powers in t of (2.7) and (2.8), we complete the proof of Theorem. □

The following identity is well known from [12] that

$$\frac{1}{t}e_k\left(\log\left(1+t\right)\right)=\sum_{n=0}^{\infty}\left\{\sum_{\ell=0}^{n}\frac{1}{\left(\ell+1\right)^{k-1}}\frac{S_1\left(n+1,\ell+1\right)}{n+1}\right\}\frac{t^n}{n!}. \tag{2.9}$$

We now state the following Theorem which is the sums of the products of degenerate polyexponential-Bernoulli polynomials of the second kind and Stirling numbers of the first kind.

Theorem 2.7. *Let $n\in\mathbb{N}_0$. Then the sums for the products of $b_{m,\lambda}(x)$ and $S_1(n,m)$ holds*

$$\mathcal{B}_{n,\lambda}^{(k)}\left(x\right)=\sum_{m=0}^{n}\binom{n}{m}\sum_{\ell=0}^{m}\frac{1}{\left(\ell+1\right)^{k-1}}\frac{S_1\left(m+1,\ell+1\right)}{m+1}b_{n-m,\lambda}\left(x\right).$$

Proof. By making use of (2.1), we see that

$$\begin{aligned}
\sum_{n=0}^{\infty}\mathcal{B}_{n,\lambda}^{(k)}\left(x\right)\frac{t^n}{n!} &= \frac{e_k\left(\log\left(1+t\right)\right)}{e_\lambda\left(\gamma\right)-e_\lambda^{-1}\left(\gamma\right)}e_\lambda^x\left(t\right)\\
&= \frac{1}{t}e_k\left(\log\left(1+t\right)\right)\frac{t}{e_\lambda\left(t\right)-e_\lambda^{-1}\left(t\right)}e_\lambda^x\left(t\right) &\text{(2.10)}\\
&= \left(\sum_{n=0}^{\infty}\left\{\sum_{\ell=0}^{n}\frac{1}{\left(\ell+1\right)^{k-1}}\frac{S_1\left(n+1,\ell+1\right)}{n+1}\right\}\frac{t^n}{n!}\right)\left(\sum_{n=0}^{\infty}b_{n,\lambda}\left(x\right)\frac{t^n}{n!}\right) &\text{(2.11)}\\
&= \sum_{n=0}^{\infty}\left(\sum_{m=0}^{n}\binom{n}{m}\sum_{\ell=0}^{m}\frac{1}{\left(\ell+1\right)^{k-1}}\frac{S_1\left(m+1,\ell+1\right)}{m+1}b_{n-m,\lambda}\left(x\right)\right)\frac{t^n}{n!}. &\text{(2.12)}
\end{aligned}$$

Thus we complete the proof of the Theorem. □

For $\ell=1$ in (2.1), we have the symmetric property of degenerate Bernoulli polynomials of the second kind as follows:

$$b_{m,\lambda}\left(x\right)=\left(-1\right)^m b_{m,-\lambda}\left(-x\right).$$

Thus we note that

$$b_{m,\lambda}\left(1\right)=b_{m,-\lambda}\left(-1\right)\left(-1\right)^m. \tag{2.13}$$

Let us now give the following Theorem.

Theorem 2.8. *The following recurrence relation holds:*

$$\mathcal{B}_{n,\lambda}^{(\ell)}\left(1\right)-\mathcal{B}_{n,\lambda}^{(\ell)}\left(-1\right)=\sum_{k=1}^{n}\frac{1}{k^{\ell-1}}S_1\left(n,k\right). \tag{2.14}$$

Proof. By (2.4), we consider

$$\begin{aligned} e_\ell(\log(1+t)) &= \left(e_\lambda(t)-e_\lambda^{-1}(t)\right)\frac{e_\ell(\log(1+t))}{e_\lambda(t)-e_\lambda^{-1}(t)} \\ &= \left\{\sum_{n=0}^{\infty}\left((1)_{n,\lambda}-(-1)_{n,\lambda}\right)\frac{t^n}{n!}\right\}\left\{\sum_{n=0}^{\infty}\mathcal{B}_{n,\lambda}^{(\ell)}\frac{t^n}{n!}\right\} \\ &= \sum_{n=0}^{\infty}\left\{\sum_{m=0}^{n}\binom{n}{m}\mathcal{B}_{m,\lambda}^{(\ell)}\left((1)_{n-m,\lambda}-(-1)_{n-m,\lambda}\right)\right\}\frac{t^n}{n!} \\ &= \sum_{n=0}^{\infty}\left\{\mathcal{B}_{n,\lambda}^{(\ell)}(1)-\mathcal{B}_{n,\lambda}^{(\ell)}(-1)\right\}\frac{t^n}{n!}. \end{aligned} \tag{2.15}$$

On the other hand, it follows from (2.9) that

$$e_\ell(\log(1+t))=\sum_{n=1}^{\infty}\left\{\sum_{k=1}^{n}\frac{1}{k^{\ell-1}}S_1(n,k)\right\}\frac{t^n}{n!}. \tag{2.16}$$

Comparing the same coefficients of t on the both sides of Eqs. (2.15) and (2.16) completes the proof. □

When $\ell=1$ and $\lambda\to 0$ in (2.14), we have

$$b_n(1)-b_n(-1)=\sum_{k=1}^{n}S_1(n,k).$$

From here, by (2.13), we see that

$$\sum_{k=1}^{2n}S_1(2n,k)=0$$

and

$$b_{2n+1}(1)=\frac{1}{2}\sum_{k=1}^{2n+1}S_1(2n+1,k).$$

3. **Conclusion**

The pioneering of "degenerate" notion was Carlitz in [2]. Kim and his research team have applied Carlitz's idea to many known special functions and polynomials, see [9-20]. This was a good way in order to introduce new generalizations of known special functions and polynomials. In this paper, motivated by their works, we have studied partially degenerate polyexponential-Bernoulli polynomials of the second kind. We have derived their explicit, closed and summation formulae by making use of their generating function, series manipulation and analytical means as has been shown in the paper.

Gaussian integral representation plays an important role in classical problems arising from quantum optics and quantum mechanics. They are studied to calculate the optical mode overlapping and transition rates between quantum eigenstates of the harmonic oscillator, *cf.* [22].

Recall from (2.2) that

$$\sum_{m=0}^{\infty}\mathcal{B}_m^{(\ell)}(x)\frac{t^m}{m!}=\frac{e_\ell(\log(1+t))}{e^t-e^{-t}}e^{xt}, \tag{3.1}$$

where $\mathcal{B}_m^{(\ell)}(x)$ are polyexponential-Bernoulli polynomials of the second kind.

By (3.1), we consider the Gaussian integral representation of polyexponential-Bernoulli polynomials of the second kind as follows:

$$T_n^{(\ell)}(\alpha,\beta,\mu):=T_n^{(\ell)}:=\int_{-\infty}^{\infty}\mathcal{B}_n^{(\ell)}(\alpha x)e^{-\beta x^2+\mu x}dx. \tag{3.2}$$

By (3.2), we have

$$\sum_{n=0}^{\infty}T_n^{(\ell)}\frac{t^n}{n!}=\sum_{n=0}^{\infty}\left(\int_{-\infty}^{\infty}\mathcal{B}_n^{(\ell)}(\alpha x)e^{-\beta x^2+\mu x}dx\right)\frac{t^n}{n!}.$$

It follows from (2.3) that

$$\sum_{n=0}^{\infty}T_n^{(\ell)}\frac{t^n}{n!}=\frac{e_\ell(\log(1+t))}{e^t-e^{-t}}\int_{-\infty}^{\infty}e^{(\alpha t+\mu)x-\beta x^2}dx.$$

Since

$$\int_{-\infty}^{\infty}e^{\beta x-\alpha x^2+\mu}dx=\frac{\sqrt{\pi}}{\sqrt{\alpha}}e^{\frac{\beta^2}{4\alpha}+\mu},$$

which represents Gaussian integral, we find

$$\sum_{n=0}^{\infty}T_n^{(\ell)}\frac{t^n}{n!}=\frac{\sqrt{\pi}}{\sqrt{\beta}}\frac{e_\ell(\log(1+t))}{e^t-e^{-t}}\exp\left(\frac{\alpha^2}{4\beta}t^2+\frac{\mu^2}{4\beta}+\frac{\alpha\mu}{2\beta}t\right)$$

with the assumption $\exp(t):=e^t$. Recall from [22] that the 2-variable Hermite polynomials are defined by means of the following generating function:

$$\sum_{m=0}^{\infty}H_m(x,y)\frac{t^m}{m!}=\exp\left(xt+yt^2\right). \tag{3.3}$$

By (2.3) and (3.3), we derive

$$\begin{aligned}\sum_{n=0}^{\infty}T_n^{(\ell)}\frac{t^n}{n!} &= \frac{\sqrt{\pi}}{\sqrt{\beta}}e^{\frac{\mu^2}{4\beta}}\left(\sum_{n=0}^{\infty}\mathcal{B}_n^{(\ell)}\frac{t^n}{n!}\right)\left(\sum_{n=0}^{\infty}H_n\left(\frac{\alpha\mu}{2\beta},\frac{\alpha^2}{4\beta}\right)\frac{t^n}{n!}\right)\\ &= \frac{\sqrt{\pi}}{\sqrt{\beta}}e^{\frac{\mu^2}{4\beta}}\sum_{n=0}^{\infty}\left(\sum_{k=0}^{n}\binom{n}{k}\mathcal{B}_{n-k}^{(\ell)}H_k\left(\frac{\alpha\mu}{2\beta},\frac{\alpha^2}{4\beta}\right)\right)\frac{t^n}{n!}.\end{aligned}$$

By comparing the coefficients of $\frac{t^n}{n!}$ on the above, we obtain

$$T_n^{(\ell)}=\frac{\sqrt{\pi}}{\sqrt{\beta}}e^{\frac{\mu^2}{4\beta}}\sum_{k=0}^{n}\binom{n}{k}\mathcal{B}_{n-k}^{(\ell)}H_k\left(\frac{\alpha\mu}{2\beta},\frac{\alpha^2}{4\beta}\right). \tag{3.4}$$

Thus, by (3.2) and (3.4), we finalize our paper with the following result:

$$\int_{-\infty}^{\infty}\mathcal{B}_n^{(\ell)}(\alpha x)\;e^{-\beta x^2+\mu x}dx=\frac{\sqrt{\pi}}{\sqrt{\beta}}e^{\frac{\mu^2}{4\beta}}\sum_{k=0}^{n}\binom{n}{k}\mathcal{B}_{n-k}^{(\ell)}H_k\left(\frac{\alpha\mu}{2\beta},\frac{\alpha^2}{4\beta}\right).$$

Seemingly that these types of polynomials will be continued to be studied for a while due to their interesting reflections in the fields of mathematics, statistics and sciences.

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