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EVERY (NOT NECESSARILY CONTINUOUS) FINITE-DIMENSIONAL IRREDUCIBLE LOCALLY BOUNDED REPRESENTATION OF A CONNECTED LIE GROUP IS DETERMINED BY THE CHARACTER OF THE REPRESENTATION

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Abstract. It is proved that every (not necessarily continuous) finite-dimensional irreducible locally bounded representation of a connected Lie group is uniquely determined by the character of the representation.

§ 1. Introduction

An explicit form of continuous irreducible finite-dimensional representations of connected Lie groups in terms of representations of the radical and a Levi subgroup is well known (see, e.g., [1], Ch. 8, § 7).

An explicit form of not necessarily continuous irreducible finite-dimensional representations of connected Lie groups in terms of representations of the radical and a Levi subgroup was obtained in [2].

In the present note, we use the result of [2] to prove that every (not necessarily continuous) finite-dimensional irreducible locally bounded representation of a connected Lie group is uniquely determined by the character of the representation.

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§ 2. Preliminaries

Recall that a (not necessarily continuous) finite-dimensional representation is said to be *locally bounded* if there is a neighborhood of the identity element whose image under the representation is bounded and *finally precontinuous* if there is a normal subgroup whose image under the representation is contained in a ball centered at the identity operator (in the space of the operators in the representation space) whose radius is less than one and the quotient group by this normal subgroup is a Lie group.

Recall now one of the main results of [2].

Theorem 1. *Let G be a connected Lie group, let R be the radical of G, let S be a Levi subgroup of G, and let π be a (not necessarily continuous) irreducible locally bounded finite-dimensional representation of G in a space E. Then there is a (not necessarily continuous) character θ of R satisfying the condition*

$$\theta(grg^{-1}) = \theta(r) \quad \textit{for every} \quad g \in G,\ r \in R, \tag{1}$$

and an (automatically continuous) irreducible representation ρ of S on E such that

$$\pi(g) = \pi(lr) = \theta(r)\rho(s) \quad \text{for every} \quad g = sr,\ s \in S,\ r \in R. \tag{2}$$

Remark. Under the assumptions of Theorem 1, the character χ_π of the representation π has obviously the form

$$\chi_\pi(g) = \chi_\pi(sr) = \theta(r)\chi_\rho(s) \quad \text{for every} \quad g = sr,\ s \in S,\ r \in R, \tag{3}$$

where χ_ρ stands for the character of the representation ρ of S.

We also recall (see, e.g., [3, 4]) that every locally bounded (automatically continuous [5]) irreducible representation of a connected semisimple Lie group is completely determined by the character of the representation, i.e., two irreducible representations of the group are isomorphic if and only if their characters coincide (see, e.g., [3, 4, 5]).

§ 3. Main theorem

Theorem 2. *Every (not necessarily continuous) locally bounded finite-dimensional irreducible representation of a connected Lie group is completely* *determined by the character of the representation, i.e., two locally bounded finite-dimensional irreducible representations of the group are equivalent if and only if their characters coincide.*

Proof. Let G be a connected Lie group, let π be a (not necessarily continuous) locally bounded finite-dimensional representation of G in a vector space E, let R be the radical of G, and let S be a Levi subgroup of G. By Theorem 1, there is a (not necessarily continuous) character θ of R satisfying the condition (1) and an (automatically continuous by [5]) irreducible representation ρ of the semisimple Lie group S on E such that (2) holds. According to the above remark, the character χ_π of the representation π has the form (3), where χ_π stands for the character of the representation ρ of S. Therefore, if π_1 is another (not necessarily continuous) locally bounded finite-dimensional representation of G in a vector space E_1 and the character χ_{π_1} of π_1 coincides with χ_π, where, for some central (not necessarily continuous) character θ_1 of R and for some (automatically continuous) finite-dimensional representation ρ_1 of the Levi subgroup S of G, we have

$$(3) \qquad \chi_{\pi_1}(g) = \chi_{\pi_1}(sr) = \theta_1(r)\chi_{\rho_1}(s) \quad \text{for every} \quad g = sr,\ s \in S,\ r \in R,$$

(χ_{ρ_1} stands for the character of the corresponding representation ρ_1 of S), then the restrictions of χ_π and χ_{π_1} to R coincide. However, these restrictions are equal to $(\dim \pi)\theta$ and $(\dim \pi_1)\theta_1$, respectively. Therefore,

$$\dim \pi = \dim \pi_1$$

(the values of the characters at the identity element of G) and

$$\theta = \theta_1.$$

This implies that

$$\chi_\rho = \chi_{\rho_1}.$$

According to what was noted above, this shows that the representations ρ and ρ_1 are equivalent. This equivalence extends obviously to an equivalence of π and π_1, as was to be proved.

§ 4. Concluding remarks

Using a standard consideration (see, e.g., [6]), one can use the above theorem to show the following corollary.

Corollary. *Every* (*not necessarily continuous*) *locally bounded finite-dimensional finally precontinuous irreducible representation of a connected locally compact group is completely determined by the character of the representation, i.e., two locally bounded finite-dimensional irreducible finally precontinuous representations of the group are equivalent if and only if their characters coincide.*

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References

1. A. O. Barut and R. Rączka, *Theory of Group Representations and Applications*, 2nd ed. (World Scientific Publishing Co., Singapore, 1986).
2. A. I. Shtern, *Not necessarily continuous locally bounded finite-dimensional irreducible representations of connected Lie groups*, Adv. Stud. Contemp. Math. (Kyungshang) **29** (2019), no. 1, 1-5.
3. V. S. Varadarajan, *Lie Groups, Lie Algebras, and Their Representations*, Prentice-Hall Inc., Englewood Cliffs, NJ, 1974.
4. M. A. Naimark, A. I. Štern [Shtern], *Theory of Group Representations*, Springer-Verlag, New York–Heidelberg–Berlin, 1982.
5. A. I. Shtern, *A Version of van der Waerden's Theorem and a Proof of Mishchenko's Conjecture on Homomorphisms of Locally Compact Groups*, Izv. Math. **72** (2008), no. 1, 169–205.
6. A. I. Shtern, *Not Necessarily Continuous Locally Bounded Finally Precontinuous Finite-Dimensional Irreducible Representations of Connected Locally Compact Groups*, Adv. Stud. Contemp. Math. (Kyungshang) **31** (2021), no. 1, 1–5.

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