Proceedings of the Jangjeon Mathematical Society
24 (2021), No. 1, pp. 149 - 152

www.jangjeon.or.kr
http://dx.doi.org/10.17777/pjms2021.24.1.149

PROPERTIES OF PURE PSEUDOREPRESENTATIONS OF COMMUTATIVE GROUPS

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Abstract. We prove that a bounded pure pseudorepresentation of the group of integers $\mathbb{Z}$ with an arbitrarily small defect need not be an ordinary representation of the group. We also prove that a bounded pure pseudorepresentation with a sufficiently small defect of a commutative group on a dual Banach space is an ordinary representation if and only if the image of the pseudorepresentation forms a commutative family of operators.

§ 1. Introduction

Recall that a mapping π of a given group G into the family of invertible operators on a Banach space E is said to be a *quasirepresentation* of G on E if $\pi(e_G) = 1_E$, where e_G stands for the identity element of G and 1_E for the identity operator on the space E, and, in the corresponding operator norm, we have

$$\|\pi(g_1g_2) - \pi(g_1)\pi(g_2)\| \leq \varepsilon, \qquad g_1, g_2 \in G,$$

for some ε, which is usually assumed to be sufficiently small and (its greatest lower bound for π) is called a (the) *defect* of π, and a quasirepresentation π of G is said to be a *pseudorepresentation* of G if $\pi(g^n)$ is conjugate to $\pi(g)^n$, $n \in \mathbb{Z}$, with the help of an operator sufficiently close to the identity operator. In this paper, a pseudorepresentation is said to be *pure* if $\pi(g^n) = \pi(g)^n$ for all $g \in G$ and all $n \in \mathbb{N}$ (the set of positive integers). For generalities concerning pseudorepresentations and quasirepresentations of groups, see [1–4].

2010 *Mathematics Subject Classification.* Primary 22A99 Secondary 22A25, 22A10.

Key words and phrases. pseudorepresentation, defect, pure pseudorepresentation, representation, commutative group.

§ 2. Preliminaries

We claim that a bounded pure pseudorepresentation of a commutative group with an arbitrarily small defect need not be an ordinary representation of the group.

Theorem 1. *A bounded pure pseudorepresentation in a nonone-dimensional Hilbert space of the group $\mathbb{Z}$ of integers with an arbitrarily small defect need not be an ordinary representation of the group.*

Proof. Let π be an arbitrary unitary representation of $\mathbb{Z}$ in a Hilbert space H whose dimension exceeds one such that some representation operators are not multiples of the identity operator (for example, the matrix of the operator $\pi(n)$ in some orthonormal basis in H is diagonal with at least two distinct characters on the diagonal). Then the family of continuous bounded operators commuting with all representation operators differs from the full matrix algebra in H, and hence, for every $\delta \in (0,1)$, there is a continuous bounded operator of the form $A = 1_H + T$, where $\|T\| < \delta$ and T does not commute with the operators $\pi(n)$, $n \neq 0$. Consider the mapping

$$\rho(n) = \begin{cases} \pi(n), & n \geq 0, \\ A\pi(n)A^{-1}, & n < 0. \end{cases} \tag{1}$$

It is immediate that ρ is a pure pseudorepresentation which is not an ordinary representation.

§ 3. Main results

Lemma. *Let E be a dual Banach space, let T be a continuous linear operator on E, and let $\|T^n - 1_E\| \leq q < 1$ for all $n \in \mathbb{N}$. Then $T = 1$.*

Proof. For every $x \in E$ and every $f \in E_*$ (E_* is a predual space of E), the sequence $(T^n - 1_E)x(f)$ is bounded; if I is an invariant mean (Banach limit) on $\mathbb{N}$, then $I_{\mathbb{N}}((T^n - 1_E)x(f))$, where $x \in E$ and $f \in E_*$, defines a continuous linear functional on E_*, which defines uniquely a vector of E and hence defines a continuous linear operator A on E with the norm not exceeding q. Then $I_{\mathbb{N}}((T^n)x(f)) = (1_E + A)x(f)$. Replacing x by Tx, we see that $(1_E + A)T = (1_E + A)$, where $(1_E + A)$ is continuously invertible. Thus, $T = 1_E$.

Theorem 2. *A pure pseudorepresentation π of a commutative group G on a dual Banach space E such that $\|\pi(g)^{-1}\| \le C$ for all $g \in G$ with a sufficiently small defect ε such that $\varepsilon C < 1$ is an ordinary representation of G if and only if the image $\pi(G)$ of the pseudorepresentation forms a commutative family of operators on E.*

Proof. It is immediate that the family $\pi(g)$ is a commutative operator family in the space $\mathcal{L}(E)$ of continuous linear operators on E if π is an ordinary representation.

Let now π be a pure pseudorepresentation of a commutative group G on the Banach space H with the commutative set $\pi(G)$ and let $\|\pi(g)^{-1}\| \le C$ for all $g \in G$ (recall that $\|\pi(g^{-1})\pi(g) - 1_E\| \le \varepsilon$). Let $a, b \in G$. By assumption, $\pi(a^n) = \pi(a)^n$ and $\pi(b^n) = \pi(b)^n$ for every $n \in \mathbb{N}$. Then (since G is commutative)

$$\pi((ab)^n) = \pi(ab)^n = \pi(a^n b^n), \qquad n \in \mathbb{N},$$

where

$$\|\pi(a^n b^n) - \pi(a^n)\pi(b^n)\| = \|\pi(ab)^n - \pi(a)^n \pi(b)^n\| < \varepsilon.$$

This implies that

$$\|1_E - \pi(a)^n \pi(b)^n \pi(ab)^{-n}\| < \varepsilon C$$

and

$$\|1_E - T^n\| < \varepsilon C \quad \text{for} \quad T = \pi(a)\pi(b)\pi(ab)$$

by the assumed commutativity of $\pi(G)$. Since $\varepsilon C < 1$ by assumption, it follows from the lemma that $T = 1_E$, or $\pi(ab) = \pi(a)\pi(b)$ for all $a, b \in G$, as was to be proved.

§ 4. Comments

Corollary. *A bounded pure one-dimensional pseudorepresentation of a commutative group with a sufficiently small defect is an ordinary representation.*

Proof. This follows immediately from Theorem 2, since the family of linear operators on a one-dimensional space is automatically commutative.

Acknowledgments

I thank Professor Taekyun Kim for the invitation to publish this paper in the Proceedings of the Jangjeon Mathematical Society.

Funding

The research was partially supported by the Moscow Center for Fundamental and Applied Mathematics.

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