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LABEL INCIDENCE ENERGY OF PARTIAL EDGE LABELED GRAPH

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ABSTRACT. Let $G = (V, E)$ be a simple graph with vertex set $V = \{v_1, v_2, \ldots, v_n\}$ and edge set $E = \{e_1, e_2, \ldots, e_m\}$. The label incidence matrix $B_l(G)$ of G is the $n \times m$ matrix whose (i, j)-entry is a if 0 labeled edge incident to 0 labeled vertex, b if 1 labeled edge incident to 1 labeled vertex, c if unlabeled edge incident to 0 or 1 labeled vertex and 0 otherwise. The label incidence energy $IE_l(G)$ is the sum of the singular values of $B_l(G)$. In this paper we give lower and upper bounds for $IE_l(G)$ in terms of graph parameters and we study label incidence energy of some families of graph.

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1. INTRODUCTION AND PRELIMINARIES

Let G be a graph with vertex set $V(G)$ and edge set $E(G)$. Consider a set $A = \{0, 1\}$. A mapping $l : V(G) \to A$ is called binary vertex labeling of G and $l(v)$ is called the label of the vertex v under l. For an edge $e = uv$, the induced edge labeling $f^* : E(G) \to \{0, 1\}$ is given by $f^*(e) = |f(u) - f(v)|$. Let $v_f(0)$ and $v_f(1)$ be the number of vertices of graph G having labels 0 and 1 respectively under f, while $e_{f^*}(0)$ and $e_{f^*}(1)$ be the number of edges having labels 0 and 1 respectively under f^*[1, 2].

The energy of a graph [3] was defined by I. Gutman in 1978 as sum of the absolute eigenvalues of the adjacency matrix of a graph, belongs to the most popular graph invariants in chemical graph theory. It originates from the π-electron energy in the Hückel molecular orbital model, but has also gained purely mathematical interest.

Nikiforov [4] extended the concept of graph energy to any matrix by defining it as the sum of singular values of matrix. Motivated by this idea, Jooyandesh et al. [5] introduced the concept of incidence energy $IE(G)$ of a graph G, defining it as the sum of singular values of the incidence matrix $I(G)$ i. e., $\sum_{i=1}^{n} \sqrt{q_i}$, where $\sqrt{q_1}, \sqrt{q_2}, \ldots, \sqrt{q_n}$, are the singular values of $I(G)$. Some basic properties of incidence energy were established in [6, 7, 8, 9, 10, 11].

The energy of binary labeled graph [12] was introduced in 2013. The entries of the label adjacency matrix $A_l(G) = l_{ij}$ are as follows:

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$$l_{ij} = \begin{cases} a, & \text{if } v_iv_j \in E \text{ and } l(v_i) = l(v_j) = 0, \\ b, & \text{if } v_iv_j \in E \text{ and } l(v_i) = l(v_j) = 1, \\ c, & \text{if } v_iv_j \in E \text{ and } l(v_i) = 0,\ l(v_j) = 1 \text{ or vice-versa}, \\ 0, & \text{otherwise.} \end{cases}$$

where a, b, c are distinct nonzero real numbers. The set of eigenvalues $\mu_1, \mu_2, \ldots, \mu_n$ of $A_l(G)$ is the label spectrum of G. The label energy is defined as $E_l(G)=\sum_{i=1}^{n} |\mu_i|$. Since $A_l(G)$ is symmetric matrix with zero trace, these eigenvalues of binary labeled graph are real with sum equal to zero. Thus $\mu_1 \geq \mu_2 \geq \cdots \geq \mu_n$ and $\sum_{i=1}^{n} \mu_i=0$.

Lee, Liu and Tan [13] considered a new labeling problem in graph theory. For each vertex labeling $f : V(G) \to \{0, 1\}$, a partial edge labeling f^* of graph G is defined in the following way.
For each edge uv in G,

$$f^*(uv) = \begin{cases} 0, & \text{if } f(u) = f(v) = 0 \\ 1, & \text{if } f(u) = f(v) = 1 \end{cases}$$

Note that if $f(u) \neq f(v)$, then the edge uv will not be labeled by f^*. Thus f^* is a partial function from $E(G)$ into the set $\{0, 1\}$.

Motivated by the label energy of a graph, now we introduce label incidence matrix. The label incidence matrix $B_l = B_l(G) = [b_{ij}]$ of a partial edge labeled graph G is the matrix of order $m \times n$, where

$$b_{ij} = \begin{cases} a, & \text{if the vertex } v_i \text{ is incident to the edge } e_j \text{ and } l(v_i) = l(e_j) = 0, \\ b, & \text{if the vertex } v_i \text{ is incident to the edge } e_j \text{ and } l(v_i) = l(e_j) = 1, \\ c, & \text{if the vertex } v_i \text{ is incident to the unlabeled edge } e_j \text{ and } l(v_i) = 0 \text{ or } l(v_i) = 1, \\ 0, & \text{otherwise.} \end{cases}$$

where a, b and c are non zero real numbers. Let $\lambda_1, \lambda_2, \ldots, \lambda_n$ be the eigenvalues of $B_l(G)B_l(G)^T$. Then $\sqrt{\lambda_1}, \sqrt{\lambda_2}, \ldots, \sqrt{\lambda_n}$ assumed in non-increasing order are the singular values of the label incidence matrix $B_l(G)$ of a partial edge labeled graph G. The label incidence energy of a partial edge labeled graph is defined as sum of the singular values of $B_l(G)$, i.e., $IE_l(G) = \sum_{i=1}^{n} \sqrt{\lambda_i}$.

Example 1.1. *Consider a partial edge labeled graph G,*

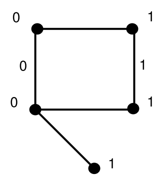

FIGURE 1. Graph G

The label incidence matrix of graph G is given by

$$B_l(G) = \begin{bmatrix} c & a & 0 & 0 & 0 \\ c & 0 & b & 0 & 0 \\ 0 & 0 & b & c & 0 \\ 0 & a & 0 & c & c \\ 0 & 0 & 0 & 0 & c \end{bmatrix}$$

and

$$B_l(G)B_l(G)^T = \begin{bmatrix} a^2+c^2 & c^2 & 0 & a^2 & 0 \\ c^2 & b^2+c^2 & b^2 & 0 & 0 \\ 0 & b^2 & b^2+c^2 & c^2 & 0 \\ a^2 & 0 & c^2 & 2c^2+a^2 & c^2 \\ 0 & 0 & 0 & c^2 & c^2 \end{bmatrix}$$

By taking $a = 1$, $b = 2$ and $c = 3$, the singular values of label incidence matrix will be $\sqrt{\lambda_1} = 0, \sqrt{\lambda_2} = 1.7920, \sqrt{\lambda_3} = 3.3406, \sqrt{\lambda_4} = 4.5513$ and $\sqrt{\lambda_5} = 5.3773$. The label incidence energy of graph G is $IE_l(G) = 15.0692$.

The present paper is organized as follows. In Section 2, we give some basic results. In Section 3, we establish incidence label energy of some classes of graphs.

2. Some basic results

Theorem 2.1. *If $\sqrt{\lambda_1}, \sqrt{\lambda_2}, \ldots, \sqrt{\lambda_n}$ represents singular values of partial edge labeled incidence matrix B_l, then*

(1) $\sum_{i=1}^{n} \lambda_i = 2[n_1a^2 + n_2b^2 + n_3c^2]$

(2) $\sum_{i=1}^{n} \lambda_i^2 = \sum_{i=1}^{n} \left(\sum_{j=1}^{m} b_{ij}^2 \right)^2 + 2[n_1a^4 + n_2b^4 + n_3c^4]$,

where n_1, n_2, n_3 denote number of 0 labeled edges, 1 labeled edges and unlabeled edges incident on 0 labeled vertex, 1 labeled vertex and 0 or 1 labeled vertex respectively.

Proof. (1) Sum of squares of singular values of B_l is equal to trace of $B_lB_l^T$,

$$\sum_{i=1}^{n} \lambda_i = \sum_{i=1}^{n} \left(\sum_{j=1}^{m} b_{ij}^2 \right) = 2[n_1a^2 + n_2b^2 + n_3c^2].$$

(2) We have

$$\begin{aligned} \sum_{i=1}^{n} \lambda_i^2 &= \sum_{i=1}^{n} \sum_{i=j}^{n} b_{ij}b_{ji} \\ &= \sum_{i=1}^{n} b_{ii}^2 + \sum_{i \neq j} b_{ij}b_{ji} \\ &= \sum_{i=1}^{n} \left(\sum_{j=1}^{m} b_{ij}^2 \right)^2 + 2[n_1a^4 + n_2b^4 + n_3c^4] \end{aligned}$$

where n_1, n_2, n_3 are number of 0 labeled edges, 1 labeled edges and unlabeled edges incident on 0 labeled vertex, 1 labeled vertex and 0 or 1 labeled vertex respectively. □

Theorem 2.2. *For a partial edge labeled graph G,*

$$\sqrt{2(n_1a^2 + n_2b^2 + n_3c^2)} \leq IE_l(G) \leq \sqrt{2n(n_1a^2 + n_2b^2 + n_3c^2)}.$$

Proof. Taking $a_i = 1, b_i = \sqrt{\lambda_i}$ in Cauchy Schwarz inequality we get,

$$\begin{aligned}\left(\sum_{i=1}^{n}\sqrt{\lambda_i}\right)^2 &\leq \left(\sum_{i=1}^{n}1\right)\left(\sum_{i=1}^{n}(\sqrt{\lambda_i})^2\right)\\ (IE_l(G))^2 &\leq n\sum_{i=1}^{n}\lambda_i\\ IE_l(G) &\leq \sqrt{2n[n_1a^2 + n_2b^2 + n_3c^2]}\end{aligned}$$

Also

$$\begin{aligned}\left[\sum_{i=1}^{n}\sqrt{\lambda_i}\right]^2 &\geq \sum_{i=1}^{n}(\sqrt{\lambda_i})^2\\ [IE_l(G)]^2 &\geq \sum_{i=1}^{n}\lambda_i\\ IE_l(G) &\geq \sqrt{2(n_1a^2 + n_2b^2 + n_3c^2)}\end{aligned}$$

Hence, $\sqrt{2(n_1a^2 + n_2b^2 + n_3c^2)} \leq IE_l(G) \leq \sqrt{2n(n_1a^2 + n_2b^2 + n_3c^2)}$. □

Theorem 2.3. *If G is a partial edge labeled graph, then $IE_l(G) \geq \sqrt{2(n_1a^2 + n_2b^2 + n_3c^2) + n(n-1)D^{2/n}}$ where $D = |B_l(G)B_l(G)^T|$.*

Proof. We know that

$$\begin{aligned}\frac{1}{n(n-1)}\sum_{i\neq j}\sqrt{\lambda_i}\sqrt{\lambda_j} &\geq \left[\prod_{i\neq j}\sqrt{\lambda_i}\sqrt{\lambda_j}\right]^{\frac{1}{n(n-1)}}\\ &= \left[\prod_{i=1}^{n}(\sqrt{\lambda_i})^{2(n-1)}\right]^{\frac{1}{n(n-1)}}\\ &= \left[\prod_{i=1}^{n}\lambda_i\right]^{\frac{1}{n}}\\ &= [\det(B_l(G)B_l(G)^T]^{2/n}\\ &= D^{2/n}.\end{aligned}$$

Hence, $\sum_{i\neq j}\sqrt{\lambda_i}\sqrt{\lambda_j} \geq n(n-1)D^{2/n}$.
Now consider

$$
\begin{aligned}
[IE_l(G)]^2 &= \left(\sum_{i=1}^{n}\sqrt{\lambda_i}\right)^2 \\
&= \sum_{i=1}^{n}(\sqrt{\lambda_i})^2 + 2\sum_{i<j}\sqrt{\lambda_i}\sqrt{\lambda_j} \\
&= \sum_{i=1}^{n}\lambda_i + 2\sum_{i<j}\sqrt{\lambda_i\lambda_j} \\
&\geq \sum_{i=1}^{n}\lambda_i + \sum_{i<j}\sqrt{\lambda_i\lambda_j} \\
IE_l(G) &\geq \sqrt{2(n_1a^2+n_2b^2+n_3c^2)+n(n-1)D^{2/n}}.
\end{aligned}
$$

□

Theorem 2.4. *Let $G(n,m)$ be a partial edge labeled graph. Suppose that $\sqrt{\lambda_1} \geq \sqrt{\lambda_2} \geq \cdots \geq \sqrt{\lambda_n}$ are the singular values of $B_l(G)$, then, $IE_l(G) \leq \sqrt{\lambda_1} + \sqrt{(n-1)[2(n_1a^2+n_2b^2+n_3c^2)-\lambda_1]}$.*

Proof. Applying Cauchy Schwarz inequality for $(n-1)$ terms by taking $a_i = 1$ and $b_i = \sqrt{\lambda_i}$,

$$
\begin{aligned}
\left(\sum_{i=2}^{n}\sqrt{\lambda_i}\right)^2 &\leq \left(\sum_{i=2}^{n}1\right)\left(\sum_{i=2}^{n}(\sqrt{\lambda_i})^2\right) \\
[IE_l(G)-\lambda_1]^2 &\leq (n-1)(2(n_1a^2+n_2b^2+n_3c^2)-\lambda_1) \\
IE_l(G) &\leq \sqrt{\lambda_1} + \sqrt{(n-1)[2(n_1a^2+n_2b^2+n_3c^2)-\lambda_1]}.
\end{aligned}
$$

□

Theorem 2.5. *Let G be a partial edge labeled graph with $2(n_1a^2+n_2b^2+n_3c^2) \geq n$. Then*

$$IE_l(G) \leq \frac{2(n_1a^2+n_2b^2+n_3c^2)}{n} + \sqrt{(n-1)\left[2(n_1a^2+n_2b^2+n_3c^2)\right]}.$$

Proof. We have $IE_l(G) \leq \sqrt{\lambda_1} + \sqrt{(n-1)[2(n_1a^2+n_2b^2+n_3c^2)-\lambda_1]}$. Let $f(x) = \sqrt{x} + \sqrt{(n-1)[2(n_1a^2+n_2b^2+n_3c^2)-x]}$. For decreasing function

$$
\begin{aligned}
f'(x) \leq 0 &\Rightarrow \frac{1}{2\sqrt{x}} - \frac{(n-1)}{2\sqrt{(n-1)[2(n_1a^2+n_2b^2+n_3c^2)-x]}} \leq 0 \\
&\Rightarrow \frac{n-1}{2\sqrt{(n-1)[2(n_1a^2+n_2b^2+n_3c^2)-x]}} \geq \frac{1}{2\sqrt{x}} \\
&\Rightarrow x \geq \frac{2(n_1a^2+n_2b^2+n_3c^2)}{n}
\end{aligned}
$$

Since $2(n_1a^2+n_2b^2+n_3c^2) \geq n$,

we have, $\sqrt{\dfrac{2(n_1a^2+n_2b^2+n_3c^2)}{n}} \leq \dfrac{2(n_1a^2+n_2b^2+n_3c^2)}{n} \leq \lambda_1$

$$f(\lambda_1) \leq f\left(\frac{2(n_1a^2+n_2b^2+n_3c^2)}{n}\right)$$

Thus, $IE_l(G) \leq \sqrt{\dfrac{2(n_1a^2+n_2b^2+n_3c^2)}{n}} + (n-1)\sqrt{2(n_1a^2+n_2b^2+n_3c^2)}$.

□

Lemma 2.6. [14] *Let $a, a_1, a_2, \ldots, a_n, A$ and $b, b_1, b_2, \ldots b_n, B$ be real numbers such that $a \leq a_i \leq A$ and $b \leq b_i \leq B$ for all $i = 1, 2, \ldots, n$. Then following inequality is valid.*

$$\left| n\sum_{i=1}^{n} a_ib_i - \sum_{i=1}^{n} a_i \sum_{i=1}^{n} b_i \right| \leq \alpha(n)(A-a)(B-b)$$

where $\alpha(n) = n\left[\frac{n}{2}\right]\left(1 - \frac{1}{n}\left[\frac{n}{2}\right]\right)$ and $[x]$ denote the integral part of a real number x and equality holds if and only if $a_1 = a_2 = \cdots = a_n$ and $b_1 = b_2 = \cdots = b_n$.

Theorem 2.7. *For a partial edge labeled graph G, let $\sqrt{\lambda_1} \geq \sqrt{\lambda_2} \geq \cdots \geq \sqrt{\lambda_n}$ be singular values of G. Then*

$$IE_l(G) \geq \sqrt{2n\left(n_1a^2+n_2b^2+n_3c^2\right) - \alpha(n)\left(\sqrt{\lambda_1} - \sqrt{\lambda_n}\right)^2}.$$

Proof. Taking $a_i = b_i = \sqrt{\lambda_i}$, $\quad a = b = \sqrt{\lambda_n}$ and $A = B = \sqrt{\lambda_1}$ in Lemma 2.6,

$$\left| n\sum_{i=1}^{n} \lambda_i - \left(\sum_{i=1}^{n} \sqrt{\lambda_i}\right)^2 \right| \leq \alpha(n)\left(\sqrt{\lambda_1} - \sqrt{\lambda_n}\right)^2 \tag{1}$$

Using Theorem 2.1, inequality (1) becomes

$$n\left[2\left(n_1a^2+n_2b^2+n_3c^2\right)\right] - (IE_l(G))^2 \leq \alpha(n)\left(\sqrt{\lambda_1} - \sqrt{\lambda_n}\right)^2$$

$$(IE_l(G))^2 \geq 2\left(n_1a^2+n_2b^2+n_3c^2\right) - \alpha(n)\left(\sqrt{\lambda_1} - \sqrt{\lambda_n}\right)^2.$$

□

Lemma 2.8. [14] *Let $a_i \neq 0, b_i, r$ and R be real numbers satisfying $ra_i \leq qb_i \leq Ra_i$. Then following inequality holds.*

$$\sum_{i=1}^{n} b_i^2 + rR\sum_{i=1}^{n} a_i \leq (r+R)\sum_{i=1}^{n} a_ib_i.$$

Theorem 2.9. *If $\sqrt{\lambda_1}, \sqrt{\lambda_2}, \ldots, \sqrt{\lambda_n}$ are non-zero positive singular values of partial edge labeled graph G, then*

$$IE_l(G) \geq \frac{n\left[2\left(n_1a^2+n_2b^2+n_3c^2\right) + \sqrt{\lambda_1\lambda_n}\right]}{\sqrt{\lambda_1} + \sqrt{\lambda_n}}.$$

Proof. Taking $b_i = \sqrt{\lambda_i}$, $a_i = 1$, $r = \sqrt{\lambda_n}$ and $R = \sqrt{\lambda_1}$ in Lemma 2.8 we obtain

$$\sum_{i=1}^{n} \lambda_i + \sqrt{\lambda_1\lambda_n}\sum_{i=1}^{n} 1 \leq \left(\sqrt{\lambda_1} + \sqrt{\lambda_n}\right)\sum_{i=1}^{n}\sqrt{\lambda_i}$$
$$\sum_{i=1}^{n} \lambda_i + \left(\sqrt{\lambda_1\lambda_n}\right) n \leq \left(\sqrt{\lambda_1} + \sqrt{\lambda_n}\right)\sum_{i=1}^{n}\sqrt{\lambda_i}$$
$$IE_l(G) \geq \frac{n\left[2\left(n_1a^2 + n_2b^2 + n_3c^2\right) + \sqrt{\lambda_1\lambda_n}\right]}{\sqrt{\lambda_1} + \sqrt{\lambda_n}}.$$

□

3. Incidence label energy of some classes of graphs

Lemma 3.1. [15] *Let M, N, P, Q be matrices, M invertible and*

$$S = \begin{pmatrix} M & N \\ P & Q \end{pmatrix}.$$

Then $\det(S) = \det(M)\det(Q - PM^{-1}N)$. *Furthermore, if M and P commute, then* $\det(S) = \det(MQ - PN)$.

Theorem 3.2. *The label incidence energy of a star S_n on n vertices is $(m-2)a^2 + (n-m-1)c^2 + \alpha$ where $m-1$ vertices including apex vertex labeled zero, $n-m+1$ vertices are labeled one, $\alpha = ma^2 + (n-m+1)c^2$ and $m \leq n$.*

Proof. Let $m-1$ vertices of S_n including apex vertex be labeled zero and remaining $n-m+1$ vertices be labeled one. We have

$$B_l(S_n)B_l\left(S_n\right)^T = \left[\begin{array}{c|c} ((m-1)a^2 + (n-m)c^2I)_{1\times 1} & B_{1\times(n-1)} \\ \hline B^T_{(n-1)\times 1} & C_{(n-1)\times(n-1)} \end{array}\right]$$

with $C = \left[\begin{array}{c|c} a^2I_{(m-1)\times(m-1)} & 0_{(m-1)\times(n-m)} \\ \hline 0_{(n-m)\times 1} & c^2I_{(n-m)\times(n-m)} \end{array}\right]$,

$$B = \begin{array}{c} \scriptstyle 1 \quad\; 2 \quad\; \dots \quad m-1 \quad m \quad m+1 \quad \dots \quad n-1 \\ \left[\begin{array}{cccccccc} a^2 & a^2 & \dots & a^2 & c^2 & c^2 & \cdots & c^2 \end{array}\right]. \end{array}$$

By using Lemma 3.1, $\det(\lambda I - B_l(S_n)B_l\left(S_n\right)^T) = |C||((m-1)a^2 + (n-m)c^2)I_1 - BC^{-1}B^T|$.

Note that, $|C| = (\lambda - a^2)^{m-1}(\lambda - c^2)^{n-m}$, $BC^{-1}B^T = \dfrac{(m-1)a^4}{\lambda - a^2} + \dfrac{(n-m)c^4}{\lambda - c^2}$.

Thus, $\det(\lambda I - B_l(S_n)B_l\left(S_n\right)^T) = \lambda(\lambda - a^2)^{m-2}(\lambda - c^2)^{n-m-1}(\lambda^2 - (ma^2 + (n-m+1)c^2)\lambda + na^2c^2)$. Label incidence spectrum of S_n is given by $\begin{Bmatrix} 0 & \frac{\alpha+\beta}{2} & c^2 & \frac{\alpha-\beta}{2} & a^2 \\ 1 & 1 & n-m-1 & 1 & m-2 \end{Bmatrix}$, where $\alpha = ma^2 + (n-m+1)c^2$ and $\beta = \sqrt{(ma^2 + (n-m+1)c^2)^2 - 4na^2c^2}$.

Hence, label incidence energy of S_n is $(m-2)a^2 + (n-m-1)c^2 + \alpha$. □

Theorem 3.3. *For $n \geq 2$, label incidence energy of complete graph K_n is $((m-2)a^2 + (n-m)c^2)(m-1) + (mc^2 + (n-m-2)b^2)(n-m-1) + \alpha + \beta$, where α, β are roots of characteristic equation $(\lambda^2 + (2b^2(m-n+1) - 2a^2(m-1) - c^2n)\lambda - 2m^2(2a^2b2 + a^2c^2 + b^2c^2) + 2m(2a^2b^2n - a^2c^2 - 2b^2c^2n + 2b^2c^2)) = 0$.*

Proof. For complete graph K_n with m vertices labeled zero and $n-m$ vertices labeled one, we have

$$BB_l^T(K_n) = \left[\begin{array}{c|c} (((m-2)a^2+(n-m)c^2)I+a^2J)_{m\times m} & c^2J_{m\times(n-m)} \\ \hline c^2J_{(n-m)\times m} & ((mc^2+(n-m-2)b^2)I+b^2J)_{(n-m)\times(n-m)} \end{array}\right]$$

Consider det $\left(\lambda I - B_l(K_n)B_l(K_n)^T\right)$.

Step 1: Replace R_i by $R_i' = R_i - R_{i+1}$, for $i = 1, 2, \ldots, m-1, m+1, m+2, \ldots, n-1$. Then, det $\left(\lambda I - B_l(K_n)B_l(K_n)^T\right)$ reduces to new determinant, say det(E).

Step 2: In det(E), replace C_i by $C_i' = C_i + C_{i-1} + \cdots + C_1$, for $i = m, m-1, \ldots, 2$ and C_i by $C_i' = C_i + C_{i-1} + \cdots + C_{m+2}$, for $i = n, n-1, \ldots, m+2$. A new determinant, say $det(F)$ is found.

Step 3: On expanding det(F) along the rows R_i, for $i = 1, 2, \ldots, m-1, m+1, \ldots, n-2, n-1$, we obtain
det$(F) = (\lambda - ((m-2)a^2+(n-m)c^2))^{m-1}(\lambda - (mc^2+(n-m-2)b^2))^{n-m-1}$

$$\begin{vmatrix} \lambda - ((m-1)2a^2 - c^2(m-n)+2) & (m-n)c^2 \\ -mc^2 & \lambda - ((n-m)2b^2 + +mc^2 - b^2) \end{vmatrix}$$

Thus det $\left(\lambda I - B_l(K_n)B_l(K_n)^T\right) = (\lambda - ((m-2)a^2+(n-m)c^2))^{m-1}(\lambda - (mc^2+(n-m-2)b^2))^{n-m-1}(\lambda^2 + (2b^2(m-n+1) - 2a^2(m-1) - c^2n)\lambda - 2m^2(2a^2b2 + a^2c^2 + b^2c^2) + 2m(2a^2b^2n - a^2c^2 - 2b^2c^2n + 2b^2c^2))$. Label incidence spectrum of K_n is given by
$\begin{Bmatrix} (m-2)a^2+(n-m)c^2 & mc^2+(n-m-2)b^2 & \alpha & \beta \\ m-1 & n-m-1 & 1 & 1 \end{Bmatrix}$, where α, β are roots of characteristic equation $(\lambda^2 + (2b^2(m-n+1) - 2a^2(m-1) - c^2n)\lambda - 2m^2(2a^2b2 + a^2c^2 + b^2c^2) + 2m(2a^2b^2n - a^2c^2 - 2b^2c^2n + 2b^2c^2)) = 0$. Therefore, label incidence energy of complete graph K_n, is $((m-2)a^2+(n-m)c^2)(m-1) + (mc^2+(n-m-2)b^2)(n-m-1) + \alpha + \beta$. □

Theorem 3.4. *For $m_1 \le r$, $m_2 \le s$, characteristic polynomial of label incidence matrix of complete bipartite graph $K(r,s)$ is* $\lambda(\lambda - m_2a^2 - (s-m_2)c^2))^{m_1-1}(\lambda - m_2c^2 - (s-m_2)b^2))^{r-m_1-1}(\lambda - m_1a^2 - (r-m_1)c^2)^{m_2-1}(\lambda - m_1c^2 - (r-m_1)b^2)^{s-m_2-1}(\lambda^3 - ((a^2+c^2)(m_2+m_1) + (b^2+c^2)(r+s-m_1-m_2))\lambda^2 + (a^2(c^2(m_2(m_2+r) + m_1(m_1+s)) - b^2(m_1+m_2)(m_1+m_2-r-s))) + c^2(b^2(m_1^2+m_2^2+(r+s)^2 - m_1(2r+s) - m_2(r+2s)) - c^2(m_1-m_2-r)(m_1-m_2+s)) + c^2(r+s)(b^2c^2(m_1-r)(m_2-s) + a^2(c^2m_1m_2 + b^2(m_1s+m_2r-2m_1m_2))))$.

Proof. Let $\underbrace{000\ldots0}_{m_1}\underbrace{111\ldots1}_{r-m_1}$ and $\underbrace{000\ldots0}_{m_2}\underbrace{111\ldots1}_{s-m_2}$ be the labels of vertices of $K(r,s)$. We have $B_l(K(r,s))B_l(K(r,s))^T = \left[\begin{array}{c|c} A_{r\times r} & B_{r\times s} \\ \hline B^T_{s\times r} & C_{s\times s} \end{array}\right]$ with

$$A = \left[\begin{array}{c|c} (m_2a^2+(s-m_2)c^2)I_{m_1\times m_1} & 0_{m_1\times(r-m_1)} \\ \hline 0_{(r-m_1)\times m_1} & (m_2c^2+(s-m_2)b^2)I_{(r-m_1)\times(r-m_1)} \end{array}\right],$$

$$B = \left[\begin{array}{c|c} a^2J_{m_1\times m_2} & c^2J_{m_1\times(s-m_2)} \\ \hline c^2J_{(r-m_1)\times m_2} & b^2J_{(r-m_1)\times(s-m_2)} \end{array}\right],$$

$$C = \left[\begin{array}{c|c} (m_1a^2+(r-m_1)c^2)I_{m_2\times m_2} & 0_{m_2\times(s-m_2)} \\ \hline 0_{(s-m_2)\times m_2} & (m_1c^2+(r-m_1)c^2)I_{(s-m_2)\times(s-m_2)} \end{array}\right].$$

Consider

(2) $\det(\lambda I - B_l(K(r,s))B_l(K(r,s))^T) = |\lambda I - A||(\lambda I - C) - B^TA^{-1}B|$.

Let $X = m_2a^2+(s-m_2)c^2$, $Y = m_2c^2+(s-m_2)b^2$, $W = m_1a^2+(r-m_1)c^2$ and $Z = m_1c^2+(r-m_1)c^2$ be the label degree of m_1, $r-m_1$, m_2 and $s-m_2$ vertices respectively. Note that

$$B^TA^{-1}B = \left[\begin{array}{c|c} \left\{\frac{a^4m_1}{\lambda-X}+\frac{c^4(r-m_1)}{\lambda-Y}\right\}J_{m_2\times m_2} & \left\{\frac{a^2c^2m_1}{\lambda-X}+\frac{b^2c^2(r-m_1)}{\lambda-Y}\right\}J_{m_2\times(s-m_2)} \\ \hline \left\{\frac{a^2c^2m_1}{\lambda-X}+\frac{b^2c^2(r-m_1)}{\lambda-Y}\right\}J_{(s-m_2)\times m_2} & \left\{\frac{c^4m_1}{\lambda-X}+\frac{b^4(r-m_1)}{\lambda-Y}\right\}J_{(s-m_2)\times(s-m_2)} \end{array}\right]$$

By simplifying, $\det(C - B^TA^{-1}B)$ reduces to order m_2+1. Hence,

(3) $|(\lambda I - C) - B^TA^{-1}B| = (\lambda - W)^{m_2-1}(\lambda - Z)^{s-m_2-1}\det(E)$

$$\det(E) = \begin{vmatrix} (\lambda-W)-m_2E & -(m_2-1)E & -(m_2-2)E & \dots & -E & -(s-m_2)F \\ 0 & 1 & 0 & \dots & 0 & 0 \\ 0 & 0 & 1 & \dots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ -m_2F & -(m_2-1)F & -(m_2-2)F & \dots & -F & (\lambda-Z)-(s-m_2)G \end{vmatrix}$$

where

$$E = \frac{a^4m_1}{\lambda-X}+\frac{c^4(r-m_1)}{\lambda-Y}, F = \frac{a^2c^2m_1}{\lambda-X}+\frac{b^2c^2(r-m_1)}{\lambda-Y}, G = \frac{c^4m_1}{\lambda-X}+\frac{b^4(r-m_1)}{\lambda-Y}.$$

Now expression (2) becomes, $\det(\lambda - B_l(K(r,s))B_l(K(r,s))^T) = (\lambda - X)^{m_1-1}(\lambda-Y)^{r-m_1-1}(\lambda-W)^{m_2-1}(\lambda-Z)^{s-m_2-1}[(\lambda-X)(\lambda-Y)(\lambda-W)(\lambda-Z) - (\lambda-y)(\lambda-W)(s-m_2)m_1c^4 - (\lambda-X)(\lambda-W)(s-m_2)(r-m_1)b^4 - (\lambda-y)(\lambda-Y)(\lambda-Z)m_1m_2a^4 - (\lambda-X)(\lambda-Z)(r-m_1)m_2c^4 + m_1m_2(s-m_2)(r-m_1)(c^8+a^4b^4-2a^2b^2c^4)]$.

On simplification we get $\det(\lambda - B_l(K(r,s))B_l(K(r,s))^T) = \lambda(\lambda-X)^{m_1-1}(\lambda-Y)^{r-m_1-1}(\lambda-W)^{m_2-1}(\lambda-Z)^{s-m_2-1}[\lambda^3 - \lambda^2(X+Y+W+Z) + \lambda(WZ+ZX+XY+YW+WX+ZY) - (WZX+WZY+ZXY+XYW) - c^4m_1(s-m_2)(\lambda-(W+Y)) - b^4(r-m_1)(s-m_2)(\lambda-(W+X)) - a^4m_1m_2(\lambda-(Z+Y)) - (r-m_1)m_2c^4(\lambda-(Z+X))]$.

Substituting X,Y, W and Z and simplifying the terms, we get characteristic polynomial of incidence matrix of $K_{r,s}$ $\lambda(\lambda-m_2a^2-(s-m_2)c^2))^{m_1-1}(\lambda-m_2c^2-(s-m_2)b^2))^{r-m_1-1}(\lambda-m_1a^2-(r-m_1)c^2)^{m_2-1}(\lambda-m_1c^2-(r-m_1)b^2)^{s-m_2-1}(\lambda^3-((a^2+c^2)(m_2+m_1)+(b^2+c^2)(r+s-m_1-m_2))\lambda^2+(a^2(c^2(m_2(m_2+r)+m_1(m_1+s))-b^2(m_1+m_2)(m_1+m_2-r-s)))+c^2(b^2(m_1^2+m_2^2+(r+s)^2-m_1(2r+s)-m_2(r+2s))-c^2(m_1-m_2-r)(m_1-m_2+s))+c^2(r+s)(b^2c^2(m_1-r)(m_2-s)+a^2(c^2m_1m_2+b^2(m_1s+m_2r-2m_1m_2))))$. □

Definition 3.5. [16] *The double star $S_{m,n}$ is a tree of diameter three such that there are m pendant edges on one end of the path and n pendant edges on the other end.*

Theorem 3.6. *Let apex vertices of double star graph $S_{m,n}$ be labeled 0 and $m_1 \le m$, $m_2 \le n$ pendant vertices be labeled one and remaining vertices* *be labeled zero. Then characteristic polynomial of incidence label matrix of double star* $S_{m,n}$ *is* $\lambda(\lambda-a^2)^{m_1+m_2-3}(\lambda-c^2)^{n+m-m_1-m_2-3}(\lambda^5-(a^2(m_1+m_2+3)+(m+n-m_1-m_2+1))\lambda^4+(a^4(2m_2+(m_2+1)m_1+3)+c^4((n-m_2+1)(m-m_1))+a^2c^2(n(m_1+3)-2m_2(m_1+1)+m(m_2+2)+4))\lambda^3-(a^6(m_1+m_2+1)+a^2c^4((n+1)(2m+1-m_1)-m_2(1+m))+a^4c^2(m(2+m_2)+n(3+m_1)+4))\lambda^2+(a^4c^4(m(2+n)+2n-m_1-m_2+1)+a^6c^2(m+n+m_1+m_2+1))\lambda-a^6c^4(m+n))$.

Proof. Let $(v_1,v_2,\ldots,v_{m_1},v_{m_1+1},\ldots,v_{m-1},v_m)$ be labeled as $(0,0,\ldots,0,1,\ldots,1,0)$ and $(v_{m+1},v_{m+2},\ldots,v_{m_2},v_{m_2+1},\ldots,v_{m+n-1},v_{m+n})$ be labeled as $(0,0,\ldots,0,1,\ldots,1,1)$, where v_m and v_{m+1} be the apex vertices.

$$B_l(S_{m,n})B_l(S_{m,n})^T=\left[\begin{array}{c|c} B_l(S_m)B_l(S_m)^T & B_{m\times n} \\ \hline B^T_{n\times m} & B_l(S_n)B_l(S_n)^T\end{array}\right]$$

with $B_l(S_m)B_l(S_m)^T=\left[\begin{array}{c|c} C_{(m-1)\times(m-1)} & D_{(m-1)\times 1} \\ \hline D^T_{1\times(m-1)} & ((m-1)a^2+(n-m)c^2)I_{1\times 1},\end{array}\right]$

$C=\left[\begin{array}{c|c} a^2I_{m_1\times m_1} & 0_{m_1\times(m-m_1-1)} \\ \hline 0_{(m-m_1-1)\times m_1} & c^2I_{(n-m)\times(n-m)}\end{array}\right]$,

$D^T=\begin{array}{c} \begin{matrix} 1 & 2 & \ldots & m_1 & m_1+1 & m_1+2 & \ldots & m-1 \end{matrix} \\ \left[\begin{matrix} a^2 & a^2 & \ldots & a^2 & c^2 & c^2 & \cdots & c^2 \end{matrix}\right]\end{array}$ and

$B=\left[\begin{array}{c|c} 0_{(m-1)\times 1} & 0_{(m-1)\times(n-1)} \\ \hline a^2I_{1\times 1} & 0_{1\times(n-1)}\end{array}\right]$.

Also $B_l(S_n)B_l(S_n)^T$ follows from Theorem 3.2.

Consider $\det\left(\lambda I-B_l(S_{m,n})B_l(S_{m,n})^T\right)$.

Step 1: Replacing R_i by $R_i'=R_i-R_{i+1}$, for $i=1,2,\ldots,m_1-1,m_1+1,m_1+2,\ldots,m-1$ and R_i by $R_i'=R_i-R_{i-1}$, for $i=n,n-1,\ldots,m_2+2,m_2,m_2-1,\ldots,2$, $\det\left(\lambda I-BB_l^T(S_{m,n})\right)$ will reduce to new determinant, say $\det(E)$.

Step 2: On expanding the $\det(E)$ along the rows R_i, for $i=1,2,\ldots,m_1-1,m_1+1,m_1+2,\ldots,m-1,2,3,\ldots,m_2,m_2+2,m_2+3,\ldots,n$, we obtain $(\lambda-a^2)^{m_1+m_2-3}(\lambda-c^2)^{n+m-m_1-m_2-3}(\lambda-(mc^2+(n-m-2)b^2))^{n-m-1}\det(F)$.

Step 3: On simplifying $\det(F)$ we obtain a polynomial, $\lambda(\lambda^5-(a^2(m_1+m_2+3)+(m+n-m_1-m_2+1))\lambda^4+(a^4(2m_2+(m_2+1)m_1+3)+c^4((n-m_2+1)(m-m_1))+a^2c^2(n(m_1+3)-2m_2(m_1+1)+m(m_2+2)+4))\lambda^3-(a^6(m_1+m_2+1)+a^2c^4((n+1)(2m+1-m_1)-m_2(1+m))+a^4c^2(m(2+m_2)+n(3+m_1)+4))\lambda^2+(a^4c^4(m(2+n)+2n-m_1-m_2+1)+a^6c^2(m+n+m_1+m_2+1))\lambda-a^6c^4(m+n))$.

Thus $\det\left(\lambda I-B_l(S_{m,n})B_l(S_{m,n})^T\right)=\lambda(\lambda-a^2)^{m_1+m_2-3}$ $(\lambda-c^2)^{n+m-m_1-m_2-3}(\lambda^5-(a^2(m_1+m_2+3)+(m+n-m_1-m_2+1))\lambda^4+(a^4(2m_2+(m_2+1)m_1+3)+c^4((n-m_2+1)(m-m_1))+a^2c^2(n(m_1+3)-2m_2(m_1+1)+m(m_2+2)+4))\lambda^3-(a^6(m_1+m_2+1)+a^2c^4((n+1)(2m+1-m_1)-m_2(1+m))+a^4c^2(m(2+m_2)+n(3+m_1)+4))\lambda^2+(a^4c^4(m(2+n)+2n-m_1-m_2+1)+a^6c^2(m+n+m_1+m_2+1))\lambda-a^6c^4(m+n))$. □

References

[1] D. B. West, *Introduction to Graph Theory*, Prentice Hall (1996).

[2] J. A. Gallian, *A dynamic survey of graph labeling*, Linear Algebra Appl. DS6 (2019).
[3] I. Gutman, *The energy of a graph*, Ber. Math. Stat. Sekt. Forschungsz. Graz. 103 (1978), 1-22.
[4] V. Nikiforov, *The energy of graph and matrices*, J. Math. Anal. Appl. 326 (2007), 1472-1475.
[5] M. Jooyandesh, D. Kiani and M. Mirzakhah, *Incidence energy of a graph*, MATCH Commun. Math. Comput. Chem. 62 (2009), 561–572.
[6] S. B. Bozkurt and I. Gutman, *Estimating the incidence energy*, MATCH Commun. Math. Comput. Chem. 70 (2013), 143-156.
[7] I. Gutman, D. Kiani and M. Mirzakhah, *On incidence energy of graphs*, MATCH Commun. Math. Comput. Chem. 62 (2009), 573-580.
[8] Z. Tong and Y. Hou, *On incidence energy of trees*, MATCH Commun. Math. Comput. Chem. 66 (2011), 977-984.
[9] I. Gutman, D. Kiani, M. Mirzakhah and B. Zhou, *On incidence energy of a graph*, Linear Algebra Appl. 431 (2009), 1223-1233.
[10] B. Zhou, *More on energy and Laplacian energy*, MATCH Commun. Math. Comput. Chem. 64 (2010), 75-84.
[11] J. Zhang and J. Li, *New results on the incidence energy of graphs*, MATCH Commun. Math. Comput. Chem. 68 (2012), 777-803.
[12] P. G. Bhat and S. D'Souza, *Energy of binary labeled graph*, Trans. Comb. 2 (2013), 53-67.
[13] S. M. Lee, A. Liu and S. K. Tan, *On balanced graphs*, Cong. Numer. 87 (1992), 59-64.
[14] I. Z. Milovanovic, E. I. Milovanovic and A. Zakic, *A short note on graph energy*, MATH Commun. Math. Comput. Chem. 72 (2014), 179-182.
[15] G. Indulal, I. Gutman and A. Vijayakumar, *On distance energy of graphs*, MATCH Commun. Math. Comput. Chem. 60 (2008), 461-472.
[16] X. Liu and P. Lu, *One special double starlike graph is determined by its Laplacian spectrum*, Appl. Math. Lett. 22(4) (2009), 435-438.

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