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ONE-DIMENSIONAL QUATERNION MELLIN TRANSFORM AND ITS APPLICATIONS

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Abstract. In this study, one-dimensional quaternion Mellin transform with properties like linearity, scaling, shifting, differentiation and convolution type property are demonstrated. The Parseval-type property and inversion theorem are also established. To support the study, applications from mathematical physics are given in the concluding section.

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1. Introduction

In 1853, quaternions were developed by W. R. Hamilton [15]. In 1993, Ell [7] introduced quaternion Fourier transform for application to two-dimensional linear time-invariant systems of partial differential equations. In [11], the author introduced the quaternionic Fourier-Mellin transform. In [17] author studied some properties of fractional order Mellin transform. Authors in [16] introduced a new type of q-Mellin transform called q-finite Mellin transform and studied an inversion formula and q-convolution product. Authors in [14, 18] studied fractional derivative and composition of operators associated with fractional transformations. Two dimensional Mellin transform in quantum calculus were studied in [4]. Mellin transform and conformable fractional operator and its applications were presented in 2019 [12]. Authors established some results on the generalized Mellin transform and its applications in [8].

The operations on three-dimensional vectors include multiplication and division leading to introduce the four-dimensional algebra of quaternions. Therefore, it is important to study one-dimensional quaternion Mellin transforms. To transfer signals from a real-valued time domain to the quaternion-valued frequency domain efficiently, a one-dimensional quaternion Mellin transform is derived in this study. One-dimensional quaternion Mellin transform is applicable in solving boundary value problems and Euler-Cauchy differential equations of quaternion-valued functions. Quaternion operations have extended applications in electrodynamics, instrumentation, potential wedge and general relativity.

In this study, the authors introduce one-dimensional quaternion Mellin transforms. The authors analyzed the operational properties of the transforms. Convolution type property, Parseval-type theorem and inversion formula

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are developed for one-dimensional quaternion Mellin transforms. In the concluding section, applications in mathematical physics are demonstrated.

2. Preliminary results

In quaternions, every element is a linear combination of a real scalar and three imaginary units $\mathbf{i}, \mathbf{j}$ and $\mathbf{k}$ with real coefficients.
Let q be a quaternion defined in

$$\mathbb{H} = \{q = x_0 + \mathbf{i}x_1 + \mathbf{j}x_2 + \mathbf{k}x_3 : x_0, x_1, x_2, x_3 \in \mathbb{R}\} \tag{1}$$

be the division ring of quaternions, where $\mathbf{i}, \mathbf{j}, \mathbf{k}$ satisfy Hamilton's multiplication rules (see, e.g. [10])

$$\mathbf{ij} = -\mathbf{ji} = \mathbf{k}, \mathbf{jk} = -\mathbf{kj} = \mathbf{i}, \mathbf{ki} = -\mathbf{ik} = \mathbf{j}, \mathbf{i}^2 = \mathbf{j}^2 = \mathbf{k}^2 = \mathbf{ijk} = -1. \tag{2}$$

The quaternion conjugate of q is defined by

$$\bar{q} = x_0 - \mathbf{i}x_1 - \mathbf{j}x_2 - \mathbf{k}x_3;\ x_0, x_1, x_2, x_3 \in \mathbb{R}. \tag{3}$$

The norm of $q \in \mathbb{H}$ is defined as

$$|q| = \sqrt{q\bar{q}} = \sqrt{x_0^2 + x_1^2 + x_2^2 + x_3^2}. \tag{4}$$

From [19] $f \in L^1(\mathbb{R}; \mathbb{H})$, then the function is expressed as

$$f(x) = f_0(x) + \mathbf{i}f_1(x) + \mathbf{j}f_2(x) + \mathbf{k}f_3(x). \tag{5}$$

Mawardi et al. in [3] defined one-dimensional quaternion Fourier transform of $g \in L^1(\mathbb{R}; \mathbb{H})$ as:

$$\mathcal{F}_l\{g\}(w) = \int_{\mathbb{R}} g(y) e^{-\mathbf{j}2\pi\omega y} dy, \tag{6}$$

and the inverse of one-dimensional quaternion Fourier transform is expressed as:

$$g(y) = \mathcal{F}_l^{-1}[\mathcal{F}_l\{g\}](y) = \int_{\mathbb{R}} \mathcal{F}_l\{g\}(w) e^{\mathbf{j}2\pi\omega y} d\omega. \tag{7}$$

Alternatively, in [19] the quaternions are defined as

$$\mathbb{H} = \{q = q_1 + jq_2 : q_1, q_2 \in \mathbb{C}\} \tag{8}$$

where j is the imaginary number satisfying following conditions:
$j^2 = -1,\ jr = rj,\ \forall\, r \in \mathbb{R},\ ji = -ij$, where i is the imaginary number.
Every quaternion number can be uniquely expressed as

$$q = q_1 + jq_2 = (q_1 - iq_2)\, e_1 + (q_1 + iq_2)\, e_2 \tag{9}$$

where $e_1 = \dfrac{1+k}{2}, e_2 = \dfrac{1-k}{2}$, $e_1 + e_2 = 1$, $e_1 e_2 = 1/2$ and $e_2 e_1 = -1/2$.
The Auxiliary complex spaces B_1 and B_2 are defined as follows:

$$B_1 = \{w_1 = q_1 - iq_2, \forall\, q_1, q_2 \in \mathbb{C}\}; B_2 = \{w_2 = q_1 + iq_2, \forall\, q_1, q_2 \in \mathbb{C}\}. \tag{10}$$

A cartesian set $X_1 \times_q X_2$ determined by $X_1 \subseteq B_1$ and $X_2 \subseteq B_2$ and is defined as:

$$X_1 \times_q X_2 = \{q_1 + jq_2 \in \mathbb{H} : q_1 + jq_2 = w_1 e_1 + w_2 e_2, w_1 \in X_1, w_2 \in X_2\}. \tag{11}$$

The projection mappings (11) $\mathcal{P}_1 : \mathbb{H} \to B_1 \subseteq \mathbb{C}, \mathcal{P}_2 : \mathbb{H} \to B_2 \subseteq \mathbb{C}$ is represented as follows:

$\mathcal{P}_1(q_1+jq_2) = \mathcal{P}_1[(q_1 - iq_2)\, e_1+(q_1 + iq_2)\, e_2] = (q_1-iq_2) \in B_1, \forall\, q_1+jq_2 \in \mathbb{H}$,

$\mathcal{P}_2(q_1+jq_2) = \mathcal{P}_2[(q_1 - iq_2)\, e_1+(q_1 + iq_2)\, e_2] = (q_1+iq_2) \in B_2, \forall\, q_1+jq_2 \in \mathbb{H}$.

Analogus to [1, Theorem 1.1, p. 219], the convergence of quaternion function with respect to it's complex component functions can be represented by the following theorem:

Theorem 2.1. *$F(\xi) = F_{e_1}(\xi_1)e_1+F_{e_2}(\xi_2)e_2$ is convergent in domain $\mathcal{D} \subseteq \mathbb{C}$ iff $F_{e_1}(\xi_1)$ and $F_{e_2}(\xi_2)$, the projections under the functions $\mathcal{P}_1 : \mathcal{D} \to \mathcal{D}_1 \subseteq \mathbb{C}$ and $\mathcal{P}_2 : \mathcal{D} \to \mathcal{D}_2 \subseteq \mathbb{C}$, are convergent in domains $\mathcal{D}_1$ and $\mathcal{D}_2$ respectively.*

3. Main results

Introducing $(s-p) = \mathbf{j}2\pi\omega$ where p is a constant, $t = e^y$, $f(t) = t^{-p}g(\log t)$ in (6) as in [5], we define one-dimensional quaternion Mellin transform.

Definition 3.1. One-dimensional quaternion Mellin transform (QMT) of $f \in L^1(\mathbb{R};\mathbb{H})$ (5) exists within the strip $a_1 < Re(s) < a_2$ is defined as

$$\mathcal{M}_q\{f\}(s) = \tilde{f}(s) = \int_0^\infty f(t)t^{-s-1}dt, \tag{12}$$

where $\tilde{f}(s) = \tilde{f}_0(s) + \mathbf{i}\tilde{f}_1(s) + \mathbf{j}\tilde{f}_2(s) + \mathbf{k}\tilde{f}_3(s)$.
Existence and Convergence: $\tilde{f}_0(s)$, $\tilde{f}_1(s)$,$\tilde{f}_3(s)$ and $\tilde{f}_3(s)$ are analytic and convergent in the strip D_1, D_2, D_3 and D_4 respectively;
$D_1 = \{s \in \mathbb{H} : a < Re(s) < b\}$, $D_2 = \{s \in \mathbb{H} : c < Re(s) < d\}$,
$D_3 = \{s \in \mathbb{H} : e < Re(s) < f\}$ and $D_4 = \{s \in \mathbb{H} : g < Re(s) < h\}$
respectively. Thus $\tilde{f}(s)$ is linear combination of $\tilde{f}_0(s), \tilde{f}_1(s), \tilde{f}_2(s)$ and $\tilde{f}_3(s)$ w.r.t. 1, $\mathbf{i}$, $\mathbf{j}$ and $\mathbf{k}$ respectively.
Hence by theorem 2.1, $\tilde{f}(s)$ is analytic and convergent in the strip

$D = \{s \in \mathbb{H} : a_1 < Re(s) < a_2; a_1 = max(a, c, e, g) \text{ and } a_2 = min(b, d, f, h)\}$.

If $f \in L^1(\mathbb{R};\mathbb{H})$ is such that

$$f(t) = \begin{cases} \mathcal{O}(t^{-a_1}) \text{ as } t \to 0 \\ \mathcal{O}(t^{-a_2}) \text{ as } t \to \infty, \end{cases}$$

then $\mathcal{M}_q(f)$ exists for $a_1 < Re(s) < a_2$.

The inverse of one-dimensional QMT is represented by

$$f(t) = \frac{1}{2\pi\mathbf{j}} \int_{p-\mathbf{j}\infty}^{p+\mathbf{j}\infty} \mathcal{M}_q\{f\}(s)t^s ds. \tag{13}$$

If $f \in L^1(\mathbb{R};\mathbb{H})$ and $\mathcal{M}_q\{f\}(s) = \tilde{f}(s)$, then the following operational properties of one-dimensional QMT holds as follows:

Property 3.2 (Linearity)**.** *For $f_1, f_2 \in L^1(\mathbb{R};\mathbb{H})$ and $\lambda_1, \lambda_2 \in \mathbb{R}$,*

$$\mathcal{M}_q\{\lambda_1 f_1 + \lambda_2 f_2\}(s) = \lambda_1\mathcal{M}_q\{f_1\}(s) + \lambda_2\mathcal{M}_q\{f_2\}(s). \tag{14}$$

Property 3.3 (Scaling)**.** *For $f \in L^1(\mathbb{R};\mathbb{H})$ and $a > 0$,*

$$\mathcal{M}_q\{f(at)\}(s) = a^s\mathcal{M}_q\{f(t)\}(s). \tag{15}$$

Property 3.4 (Shifting)**.** *For* $f \in L^1(\mathbb{R};\mathbb{H})$,

$$\mathcal{M}_q\left\{t^a f(t)\right\}(s) = \mathcal{M}_q\left\{f(t)\right\}(s-a). \tag{16}$$

Property 3.5. *For* $f \in L^1(\mathbb{R};\mathbb{H})$ *is a continuous differential function, then*

$$\mathcal{M}_q\left\{\frac{d^n f}{dt^n}\right\}(s) = (s+1)(s+2).....(s+n)\mathcal{M}_q\left\{f\right\}(s+n). \tag{17}$$

Proof. By mathematical induction, for $n=1$ in (17) gives

$$\begin{aligned}
&\mathcal{M}_q\left\{\frac{df}{dt}\right\}(s) = \mathcal{M}_q\left\{\frac{df_0}{dt}\right\} + \mathbf{i}\mathcal{M}_q\left\{\frac{df_1}{dt}\right\} + \mathbf{j}\mathcal{M}_q\left\{\frac{df_2}{dt}\right\} + \mathbf{k}\mathcal{M}_q\left\{\frac{df_3}{dt}\right\}\\
&= \int_0^\infty \frac{df_0}{dt}t^{-s-1}dt + \mathbf{i}\int_0^\infty \frac{df_1}{dt}t^{-s-1}dt + \mathbf{j}\int_0^\infty \frac{df_2}{dt}t^{-s-1}dt + \mathbf{k}\int_0^\infty \frac{df_3}{dt}t^{-s-1}dt\\
&= (s+1)\mathcal{M}_q\left\{f_0\right\}(s+1) + \mathbf{i}(s+1)\mathcal{M}_q\left\{f_0\right\}(s+1)\\
&+ \mathbf{j}(s+1)\mathcal{M}_q\left\{f_0\right\}(s+1) + \mathbf{k}(s+1)\mathcal{M}_q\left\{f_0\right\}(s+1)\\
&= (s+1)\mathcal{M}_q\left\{f\right\}(s+1)
\end{aligned}$$

as $\lim_{t\to\infty} t^{-s-1}f(t) \to 0$ and $\lim_{t\to 0} t^{-s-1}f(t) \to 0$.
For $n=2$ in (17) gives

$$\begin{aligned}
&\mathcal{M}_q\left\{\frac{d^2 f}{dt^2}\right\}(s) = \mathcal{M}_q\left\{\frac{d^2 f_0}{dt^2}\right\} + \mathbf{i}\mathcal{M}_q\left\{\frac{d^2 f_1}{dt^2}\right\} + \mathbf{j}\mathcal{M}_q\left\{\frac{d^2 f_2}{dt^2}\right\} + \mathbf{k}\mathcal{M}_q\left\{\frac{d^2 f_3}{dt^2}\right\}\\
&= (s+1)(s+2)\mathcal{M}_q\left\{f_0\right\}(s+2) + \mathbf{i}(s+1)(s+2)\mathcal{M}_q\left\{f_0\right\}(s+2)\\
&+ \mathbf{j}(s+1)(s+2)\mathcal{M}_q\left\{f_0\right\}(s+2) + \mathbf{k}(s+1)(s+2)\mathcal{M}_q\left\{f_0\right\}(s+2)\\
&= (s+1)(s+2)\mathcal{M}_q\left\{f\right\}(s+2)
\end{aligned}$$

as $\lim_{t\to\infty} t^{-s-1}\frac{df}{dt} \to 0$ and $\lim_{t\to 0} t^{-s-1}\frac{df}{dt} \to 0$.
For $n=k-1$ in (17) gives

$$\mathcal{M}_q\left\{\frac{d^{k-1} f}{dt^{k-1}}\right\}(s) = (s+1)(s+2)\cdots(s+(k-1))\mathcal{M}_q\left\{f\right\}(s+(k-1))$$

as $\lim_{t\to\infty} t^{-s-1}\frac{d^{k-2}f}{dt^{k-2}} \to 0$ and $\lim_{t\to 0} t^{-s-1}\frac{d^{k-2}f}{dt^{k-2}} \to 0$.
By mathemtical induction: for $n=k$, we get

$$\mathcal{M}_q\left\{\frac{d^k f}{dt^k}\right\}(s) = (s+1)(s+2).....(s+k)\mathcal{M}_q\left\{f\right\}(s+k)$$

as $\lim_{t\to\infty} t^{-s-1}\frac{d^{k-1}f}{dt^{k-1}} \to 0$ and $\lim_{t\to 0} t^{-s-1}\frac{d^{k-1}f}{dt^{k-1}} \to 0$.

Hence by mathematical induction it is true $\forall n$. Thus the proof. □

Now by combining the operators of property 3.4 and 3.5, we define the following property.

Property 3.6. *For* $f \in L^1(\mathbb{R};\mathbb{H})$,

$$\mathcal{M}_q\left\{t^n f^{(n)}(t)\right\}(s) = s(s-1).....(s-(n-1))\mathcal{M}_q\left\{f(t)\right\}(s). \tag{18}$$

Proof. By mathematical induction, for $n=1$ in (18) gives

$$\mathcal{M}_q\left\{tf'(t)\right\} = \int_0^\infty tf'(t)t^{-s-1}dt = s\mathcal{M}_q\left\{f(t)\right\}(s)$$

as $\lim_{t\to\infty} t^{-s}f(t)\to 0$ and $\lim_{t\to 0} t^{-s}f(t)\to 0$.
For $n=2$ in (18) gives

$$\mathcal{M}_q\left\{t^2f''(t)\right\} = \int_0^\infty t^2f''(t)t^{-s-1}dt = s(s-1)\mathcal{M}_q\left\{f(t)\right\}(s)$$

as $\lim_{t\to\infty} t^{-s+1}f'(t)\to 0$ and as $\lim_{t\to 0} t^{-s+1}f'(t)\to 0$.
For $n=k-1; k\in\mathbb{N}$ in (18) gives

$$\mathcal{M}_q\left\{t^{(k-1)}f^{(k-1)}(t)\right\} = \int_0^\infty t^{(k-1)}f^{(k-1)}(t)t^{-s-1}dt.$$

Thus we get

$$\mathcal{M}_q\left\{t^{(k-1)}f^{(k-1)}(t)\right\} = s(s-1)\cdots(s-(k-2))\mathcal{M}_q\left\{f(t)\right\}(s)$$

as $\lim_{t\to\infty} t^{-s+k-2}f^{(k-2)}\to 0$ and $\lim_{t\to 0} t^{-s+k-2}f^{(k-2)}\to 0$.
By mathemtical induction: for $n=k$, we get

$$\mathcal{M}_q\left\{t^kf^{(k)}(t)\right\}(s) = s(s-1)\cdots(s-(k-1))\mathcal{M}_q\left\{f(t)\right\}(s)$$

as $\lim_{t\to\infty} t^{-s+k-1}f^{(k-1)}\to 0$ and $\lim_{t\to 0} t^{-s+k-1}f^{(k-1)}\to 0$.
Hence by mathematical induction it is true $\forall n$. Thus the proof. □

Now we will study the effect of differential operator on the one-dimensional quaternion Mellin transformable function.

Lemma 3.7. *For $f\in L^1(\mathbb{R};\mathbb{H})$ is a continuous differential function, then*

$$\mathcal{M}_q\left\{\left(t\frac{d}{dt}\right)^n f(t)\right\} = s^n\mathcal{M}_q\left\{f(t)\right\}. \tag{19}$$

Proof. For $n=1$ in (19) gives,

$$\mathcal{M}_q\left\{\left(t\frac{d}{dt}\right) f(t)\right\} = s\mathcal{M}_q\left\{f(t)\right\}(s) \text{ by property 3.6}$$

For $n=2$,

$$\begin{aligned}\mathcal{M}_q\left\{\left(t\frac{d}{dt}\right)^2 f(t)\right\} &= \mathcal{M}_q\left\{t^2f''(t)+tf'(t)\right\}\\ &= \mathcal{M}_q\left\{t^2f''(t)\right\}+\mathcal{M}_q\left\{tf'(t)\right\}\\ &= s(s-1)\mathcal{M}_q\left\{f(t)\right\}+s\mathcal{M}_q\left\{f(t)\right\} \text{ by property 3.6}\\ &= s^2\mathcal{M}_q\left\{f(t)\right\}.\end{aligned}$$

For $n=k-1$,

$$\mathcal{M}_q\left\{\left(t\frac{d}{dt}\right)^{k-1} f(t)\right\} = s^{k-1}\mathcal{M}_q\left\{f(t)\right\}.$$

By mathematical induction: for $n = k$, we get

$$\mathcal{M}_q\left\{\left(t\frac{d}{dt}\right)^k f(t)\right\} = s^k \mathcal{M}_q\{f(t)\}.$$

Hence, by mathematical induction it is true $\forall n$. Thus the proof. □

Similar to differentiation, integration also has importance in various applications. So we will study the effect of integration on one-dimensional quaternion Mellin transformable function.

Lemma 3.8. *For $f \in L^1(\mathbb{R};\mathbb{H})$, then*

(20)

$$\mathcal{M}_q\left\{\int_0^t \cdots \int_0^t f(x)(dx)^n\right\}(s) = \frac{1}{s}\cdot\frac{1}{s-1}\cdots\frac{1}{s-(n-1)}\mathcal{M}_q\{f(t)\}(s-n).$$

Proof. For $n = 1$, let $F(t) = \int_0^t f(x)dx$. Using property 3.5,

$$\mathcal{M}_q\{f(t) = F'(t)\}(s) = (s+1)\mathcal{M}_q\left\{\int_0^t f(x)dx = F(t)\right\}(s+1).$$

Replacing s by $s-1$,

$$\mathcal{M}_q\left\{\int_0^t f(x)dx\right\}(s) = \frac{1}{s}\mathcal{M}_q\{f(t)\}(s-1).$$

For $n = 2$,

$$\mathcal{M}_q\left\{\int_0^t\int_0^t f(x)dxdx\right\}(s) = \frac{1}{s}\mathcal{M}_q\left\{\int_0^t f(x)dx\right\}(s-1)$$

$$\mathcal{M}_q\left\{\int_0^t\int_0^t f(x)dxdx\right\}(s) = \frac{1}{s}\cdot\frac{1}{s-1}\mathcal{M}_q\{f(t)\}(s-2).$$

For $n = k-1$,

$$\mathcal{M}_q\left\{\int_0^t \cdots \int_0^t f(x)(dx)^{k-1}\right\}(s)$$
$$= \frac{1}{s}\cdot\frac{1}{s-1}\cdots\frac{1}{s-(k-2)}\mathcal{M}_q\{f(t)\}(s-(k-1)).$$

By mathematical induction: for $n = k$

$$\mathcal{M}_q\left\{\int_0^t \cdots \int_0^t f(x)(dx)^k\right\}(s) = \frac{1}{s}\cdot\frac{1}{s-1}\cdots\frac{1}{s-(k-1)}\mathcal{M}_q\{f(t)\}(s-k).$$

Hence, by mathematical induction it is true $\forall n$. Thus the proof. □

Theorem 3.9. *For any $g, h \in L^1(\mathbb{R};\mathbb{H})$,*

(21)
$$\int_{p-j\infty}^{p+j\infty} m(t,s)\mathcal{M}_q\{g\}(s)\overline{\mathcal{M}_q\{h\}(s)}ds = \int_0^\infty g(t)\overline{h(t)}dt$$

where $m(t,s) = \dfrac{t^{2s+1}}{2\pi \boldsymbol{j}}$.

Proof. Using (12), we get

$$\begin{aligned}
&\int_{p-\mathbf{j}\infty}^{p+\mathbf{j}\infty} m(t,s)\mathcal{M}_q\{g\}(s)\overline{\mathcal{M}_q\{h\}(s)}ds \\
&= \int_{p-\mathbf{j}\infty}^{p+\mathbf{j}\infty} m(t,s)\mathcal{M}_q\{g\}(s)\int_0^\infty \overline{h(t)t^{-s-1}}dt\ ds \\
&= \int_0^\infty \left(\frac{1}{2\pi\mathbf{j}}\int_{p-\mathbf{j}\infty}^{p+\mathbf{j}\infty}\mathcal{M}_q\{g\}(s)t^s ds\right)\overline{h(t)}dt \\
&= \int_0^\infty g(t)\overline{h(t)}dt.
\end{aligned}$$

□

Theorem 3.10. *Under the consideration of theorem 3.9,*

$$\begin{aligned}
\int_0^\infty w(t,\xi)g(\xi)\mathcal{M}_q\{h\}(\xi)d\xi &= \int_0^\infty \mathcal{M}_q\{g\}(t)\left(h_0(t)+\boldsymbol{j}h_1(t)\right)dt \\
&+ \int_0^\infty \mathcal{M}_q\{g\}(t)\left(\boldsymbol{i}h_2(t)+\boldsymbol{k}h_3(t)\right)dt
\end{aligned} \tag{22}$$

where, $w(t,\xi)=\xi^{-t-1}t^{\xi+1}$ *is the weight function.*

Proof.

$$\begin{aligned}
&\int_0^\infty w(t,\xi)g(\xi)\mathcal{M}_q\{h\}(\xi)d\xi \\
&= \int_0^\infty \left[\int_0^\infty g(\xi)\xi^{-t-1}d\xi\right]\left(h_0(t)+\mathbf{j}h_1(t)\right)dt \\
&+ \int_0^\infty \left[\int_0^\infty g(\xi)\xi^{-t-1}d\xi\right]\left(\mathbf{i}h_2(t)+\mathbf{k}h_3(t)\right)dt \\
&= \int_0^\infty \mathcal{M}_q\{g\}(t)\left(h_0(t)+\mathbf{j}h_1(t)\right)dt + \int_0^\infty \mathcal{M}_q\{g\}(t)\left(\mathbf{i}h_2(t)+\mathbf{k}h_3(t)\right)dt.
\end{aligned}$$

□

The one-dimensional quaternion Mellin-type convolution is defined analogous to [21, eq. (1), p. 116]. Introducing convolution type properties in one-dimensional QMT as follows:

Definition 3.11. Given two quaternion functions $f, g \in L^1(\mathbb{R};\mathbb{H})$, the convolution type properties are defined in [6] as

$$f(t)*g(t) = \int_0^\infty f(\xi)g\left(\frac{t}{\xi}\right)\frac{d\xi}{\xi}. \tag{23}$$

$$f(t)\circ g(t) = \int_0^\infty f(t\xi)g(\xi)d\xi. \tag{24}$$

Theorem 3.12. *For* $f, g \in L^1(\mathbb{R};\mathbb{H})$,

$$\begin{aligned}
\mathcal{M}_q\{f(t)*g(t)\}(s) &= \mathcal{M}_q\{f\}(s)\mathcal{M}_q\{g_0+\boldsymbol{j}g_2\}(s) \\
&+ \mathcal{M}_q\{f\}(s)\mathcal{M}_q\{\boldsymbol{i}g_1+\boldsymbol{k}g_3\}(s).
\end{aligned} \tag{25}$$

$$\begin{aligned}\mathcal{M}_q\left\{f(t)\circ g(t)\right\}(s) = {} & \mathcal{M}_q\left\{f\right\}(s)\mathcal{M}_q\left\{g_0+\boldsymbol{j}g_2\right\}(-s-1)\\ & +\mathcal{M}_q\left\{f\right\}(s)\mathcal{M}_q\left\{\boldsymbol{i}g_1+\boldsymbol{k}g_3\right\}(-s-1).\end{aligned} \tag{26}$$

Proof. Applying one-dimensional QMT on (23), we get

$$\mathcal{M}_q\left\{f(t)*g(t)\right\}(s) = \mathcal{M}_q\left\{\int_0^\infty f(\xi)g\left(\frac{t}{\xi}\right)\frac{d\xi}{\xi}\right\}$$

$$= \int_0^\infty\int_0^\infty f(\xi)g\left(\frac{t}{\xi}\right)t^{-s-1}\frac{d\xi}{\xi}dt.$$

Further substituting $\frac{t}{\xi}=\eta$, we get

$$\begin{aligned}&\mathcal{M}_q\left\{f(t)*g(t)\right\}(s)\\ &= \int_0^\infty\int_0^\infty f(\xi)g\left(\eta\right)\left(\eta\xi\right)^{-s-1}d\eta d\xi\\ &= \int_0^\infty f(\xi)\xi^{-s-1}d\xi\int_0^\infty\left(g_0(\eta)+\mathbf{j}g_2(\eta)\right)\eta^{-s-1}d\eta\\ &\quad+\int_0^\infty f(\xi)\xi^{-s-1}d\xi\int_0^\infty\left(\mathbf{i}g_1(\eta)+\mathbf{k}g_3(\eta)\right)\eta^{-s-1}d\eta\\ &= \mathcal{M}_q\left\{f\right\}(s)\mathcal{M}_q\left\{g_0+\mathbf{j}g_2\right\}(s)+\mathcal{M}_q\left\{f\right\}(s)\mathcal{M}_q\left\{\mathbf{i}g_1+\mathbf{k}g_3\right\}(s).\end{aligned}$$

Thus the proof.
Using (24), one-dimensional QMT follows:

$$\mathcal{M}_q\left\{f(t)\circ g(t)\right\}(s) = \mathcal{M}_q\left\{\int_0^\infty f(t\xi)g(\xi)d\xi\right\}$$

$$= \int_0^\infty\int_0^\infty f(t\xi)g(\xi)t^{-s-1}d\xi dt.$$

Further substituting $t\xi=\eta$, we get

$$\begin{aligned}&\mathcal{M}_q\left\{f(t)\circ g(t)\right\}(s)\\ &= \int_0^\infty\int_0^\infty f(\eta)g(\xi)\eta^{-s-1}\xi^s d\eta d\xi\\ &= \int_0^\infty f(\eta)\eta^{-s-1}d\eta\int_0^\infty\left(g_0(\xi)+\mathbf{j}g_2(\xi)\right)\xi^s d\xi\\ &\quad+\int_0^\infty f(\eta)\eta^{-s-1}d\eta\int_0^\infty\left(\mathbf{i}g_1(\xi)+\mathbf{k}g_3(\xi)\right)\xi^s d\xi\\ &= \mathcal{M}_q\left\{f\right\}(s)\mathcal{M}_q\left\{g_0+\mathbf{j}g_2\right\}(-s-1)+\mathcal{M}_q\left\{f\right\}(s)\mathcal{M}_q\left\{\mathbf{i}g_1+\mathbf{k}g_3\right\}(-s-1).\end{aligned}$$

Hence the proof of convolution property in one-dimensional QMT, namely convolution-type property of QMT. □

4. Operational property

Definition 4.1. Let $f, g \in L^1(\mathbb{R}; \mathbb{H})$, then we define a operator $\# : \mathbb{H} \times \mathbb{H} \to \mathbb{H}$ by

$$\begin{aligned} f\#g = &(f_0g_0 + f_1g_1 + f_2g_2 + f_3g_3) \\ &+ \mathbf{i}(-f_0g_1 + f_1g_0 - f_2g_3 + f_3g_2) \\ &+ \mathbf{j}(-f_0g_2 + f_1g_3 + f_2g_0 - f_3g_1) \\ &+ \mathbf{k}(-f_0g_3 - f_1g_2 + f_2g_1 + f_3g_0). \end{aligned} \tag{27}$$

Lemma 4.2. *Let $f, g, h, l, m \in L^1(\mathbb{R}; \mathbb{H})$ and $\gamma, \tau \in \mathbb{R}$, the mapping $\#$ defines an inner product on $\mathbb{H}$ as*
i) $(f\#g) = \overline{(g\#f)}$.
ii) $f\#(\gamma g + \tau h) = \gamma(f\#g) + \tau(f\#h)$.
iii) $f\#(gh + lm) = (f\#h)\bar{g} + (f\#m)\bar{l}$.
iv) $(gh + lm)\#f = g(h\#f) + l(m\#f)$.

Proof. i) Considering R.H.S. of lemma 4.2 and definition 4.1, we get

$$\begin{aligned} \overline{(g\#f)} = &(g_0f_0 + g_1f_1 + g_2f_2 + g_3f_3) \\ &- \mathbf{i}(-g_0f_1 + g_1f_0 - g_2f_3 + g_3f_2) \\ &- \mathbf{j}(-g_0f_2 + g_1f_3 + g_2f_0 - g_3f_1) \\ &- \mathbf{k}(-g_0f_3 - g_1f_2 + g_2f_1 + g_3f_0). \\ &= (f\#g). \end{aligned} \tag{28}$$

From definition 4.1, (27) and (28), the proof is established.
Lemma 4.2 ii), iii), iv) can be proved on similar lines as in (27), (28) and [2]
. □

Theorem 4.3. *$f, g \in L^1(\mathbb{R}; \mathbb{H})$, then*

$$\mathcal{M}_q\left\{(f\#g)\right\}(s) = \mathcal{M}_q\left\{\overline{(g\#f)}\right\}(s). \tag{29}$$

Proof. Using (28), we get

$$\begin{aligned} \mathcal{M}_q\left\{(f\#g)\right\}(s) = &\mathcal{M}_q\{(f_0g_0 + f_1g_1 + f_2g_2 + f_3g_3) \\ &+ \mathbf{i}(-f_0g_1 + f_1g_0 - f_2g_3 + f_3g_2) \\ &+ \mathbf{j}(-f_0g_2 + f_1g_3 + f_2g_0 - f_3g_1) \\ &+ \mathbf{k}(-f_0g_3 - f_1g_2 + f_2g_1 + f_3g_0)\}(s) \\ &= \mathcal{M}_q\{(g_0f_0 + g_1f_1 + g_2f_2 + g_3f_3) \\ &- \mathbf{i}(-g_0f_1 + g_1f_0 - g_2f_3 + g_3f_2) \\ &- \mathbf{j}(-g_0f_2 + g_1f_3 + g_2f_0 - g_3f_1) \\ &- \mathbf{k}(-g_0f_3 - g_1f_2 + g_2f_1 + g_3f_0)\}(s) \\ &= \mathcal{M}_q\left\{\overline{(g\#f)}\right\}(s). \end{aligned}$$

Hence the proof. □

Corollary 4.4. *The mapping $\#$ defines an inner product on $\mathbb{H}$ given by*

$$[i]\ \mathcal{M}_q\left\{f\#(\gamma g + \tau h)\right\}(s) = \gamma\mathcal{M}_q\left\{f\#g\right\}(s) + \tau\mathcal{M}_q\left\{f\#h\right\}(s). \tag{30}$$

(31) [*ii*] $\mathcal{M}_q\{f\#(gh+lm)\}(s) = \mathcal{M}_q\{(f\#h)\bar{g}\}(s) + \mathcal{M}_q\{(f\#m)\bar{l}\}(s)$.

(32) [*iii*] $\mathcal{M}_q\{(gh+lm)\#f\}(s) = \mathcal{M}_q\{g(h\#f)\}(s) + \mathcal{M}_q\{l(m\#f)\}(s)$.

5. Inversion formula & Parseval-type property

Inversion formula: If the integral (12) converges absolutely on the line $Re(s) = p$ and if f is of bounded variation in a neighborhood of $t = x$ in (12) $\forall x > 0$, then

$$\lim_{r\to\infty} \frac{1}{2\pi\mathbf{j}} \int_{p-\mathbf{j}r}^{p+\mathbf{j}r} \tilde{f}(s)x^s ds = \frac{f(x^+)+f(x^-)}{2}. \tag{33}$$

In particular, if f is continuous at x, the left-hand side of (33) converges to $f(x)$. Conversely if $\tilde{f}(s)$ is absolutely integrable on the line $Re(s) = p$ and of bounded variation in a neighbourhood of the point $(p+\mathbf{j}t)$ and if (13) holds [20], then

$$\left[\tilde{f}(p+\mathbf{j}t^+) + \tilde{f}(p+\mathbf{j}t^-)\right]/2 = \lim_{r\to\infty} \int_{1/r}^{r} f(x)x^{-(p+\mathbf{j}t)-1}dx. \tag{34}$$

Property 5.1 (Paseval-type property)**.** *If* $\mathcal{M}_q\{f(t)\} = \tilde{f}(s)$ *and* $\mathcal{M}_q\{g(t)\} = \tilde{g}(s)$*, then*

$$\mathcal{M}_q\{f(t)g(t)\} = \int_0^\infty t^{-s-1}f(t)g(t)dt = \frac{1}{2\pi\boldsymbol{j}} \int_{p-\boldsymbol{j}\infty}^{p+\boldsymbol{j}\infty} \tilde{f}(u)\tilde{g}(s-u)du. \tag{35}$$

In particular, when $s = -1$

$$\int_0^\infty f(t)g(t)dt = \frac{1}{2\pi\boldsymbol{j}} \int_{p-\boldsymbol{j}\infty}^{p+\boldsymbol{j}\infty} \tilde{f}(u)\tilde{g}(-u-1)du. \tag{36}$$

Proof. Using definition 3.1,

$$\begin{aligned}
\mathcal{M}_q\{f(t)g(t)\} &= \int_0^\infty t^{-s-1}f(t)g(t)dt \\
&= \frac{1}{2\pi\mathbf{j}} \int_0^\infty t^{-s-1}g(t)dt \int_{p-\mathbf{j}\infty}^{p+\mathbf{j}\infty} t^u\tilde{f}(u)du \\
&= \frac{1}{2\pi\mathbf{j}} \int_{p-\mathbf{j}\infty}^{p+\mathbf{j}\infty} \tilde{f}(u)du \int_0^\infty t^{-s+u-1}g(t)dt \\
&= \frac{1}{2\pi\mathbf{j}} \int_{p-\mathbf{j}\infty}^{p+\mathbf{j}\infty} \tilde{f}(u)\tilde{g}(s-u)du.
\end{aligned}$$

In particular when $s = -1$, then

$$\int_0^\infty f(t)g(t)dt = \frac{1}{2\pi\mathbf{j}} \int_{p-\mathbf{j}\infty}^{p+\mathbf{j}\infty} \tilde{f}(u)\tilde{g}(-u-1)du.$$

□

Theorem 5.2 (Uniqueness)**.** *Let* $\mathcal{M}_q\{f\}$ *and* $\mathcal{M}_q\{g\}$ *denote the one-dimensional QMT of* f *and* g *respectively. If* $s \in D_f \bigcap D_g \neq \emptyset$ *(non-empty) and* $\mathcal{M}_q\{f\}(s) = \mathcal{M}_q\{g\}(s)$*, then* $f = g$ ***a.e.*** *on* $\mathbb{R}$.

Certain type of linear system gives rise to Euler-Cauchy differential equations. Application of one-dimensional QMT will yield an algebraic equation which is more convenient in generating suitable solution.

Example 5.3. The Euler-Cauchy differential equation is of the form

$$\sum_{i=0}^{n} C_i t^i \frac{d^{(i)}y(t)}{dt^{(i)}} = g(t) \tag{37}$$

where C_i's are constant.

Solution: Applying one-dimensional QMT to (37) as in [9],

$$\sum_{i=0}^{n} C_i \mathcal{M}_q \left\{ t^i \frac{d^{(i)}y(t)}{dt^{(i)}} \right\} = \mathcal{M}_q \left\{ g(t) \right\}.$$

By applying property 3.6, we get

$$\sum_{i=0}^{n} C_i (s)_i \mathcal{M}_q \left\{ y(t) \right\} = \mathcal{M}_q \left\{ g(t) \right\}$$

where $(s)_n = s(s-1)....(s-(n-1))$.

$$\mathcal{M}_q \left\{ y(t) \right\} = \frac{\mathcal{M}_q \left\{ g(t) \right\}}{\sum_{i=0}^{n} C_i (s)_i}. \tag{38}$$

On applying inverse of one-dimensional QMT to (38), the desired solution is obtained.

6. Distributions

One-dimensional QMT is developed over quaternion manifold $\mathbb{H}^n$ in this section analogous to [13, Definition A.3, p. 32].

Definition 6.1. Let $f_J : \mathbb{R}_+^n := \{y \in \mathbb{R}^n : 0 < y < \infty\} \to \mathbb{H}$ be a function with support $J = \left\{x \in \mathbb{R}_+^n : 0 < x < x_0 \text{ for some } x_0 \in \mathbb{R}_+^n\right\}$.
For some $\alpha, \beta \in \mathbb{R}^n$, $f_J \in L^1\left(\mathbb{R}_+^n, \mathbb{H}\right)$; $\lim_{x\to 0^+} f_J(x) = \mathcal{O}(x^{-\alpha})$ and $\lim_{x\to\infty} f_J(x) = \mathcal{O}(x^{-\beta})$ holds true.
The QMT $\tilde{f}(s)$ with $s \in \langle \alpha, \beta \rangle = (\alpha, \beta) \times \mathbf{j}\mathbb{R}^n \subset \mathbb{H}^n$ is defined by

$$\tilde{f}(s) = \int_{\mathbb{R}_+^n} f_J(x) x^{-s-1} dx. \tag{39}$$

The notation (α, β) denotes a poly-interval $\{y \in \mathbb{R}^n : \alpha < y < \beta\}$ and $x^s = x_1^{s_1}, x_2^{s_2} \cdots x_n^{s_n}$.

Definition 6.2. Let $q \in \mathbb{R}^n$ and define

$$W_q(J) = \left\{ \phi \in C^\infty(J) : \sup_{x\in J} \left| x^{q+1} \left(x \frac{\partial}{\partial x} \right)^l \phi(x) \right| < \infty \right\}$$

where $l \in \mathbb{N}_0^n$; $\mathbb{N}_0^n$ is the set of non-negative multi-indices.
$W_q(J)$ is associated with the topology defined by sequence of seminorms

$$\rho_{q,l}(\phi) = \sup_{x\in J} \left| x^{q+1} \left(x \frac{\partial}{\partial x} \right)^l \phi(x) \right|.$$

Then $W_{(p)}(J)$ for $p \in \mathbb{R}^n_\infty := (\mathbb{R} \cup \{\infty\})^n$ is defined to be inductive limit of $W_q(J)$.
The dual space $W'_{(p)}(J)$ is comprised of quaternion Mellin distributions and the total space of quaternion Mellin distributions is given by

$$W'(J) = \cup_{p \in \mathbb{R}^n_\infty} W'_{(p)}(J).$$

Finally, the QMT of a distribution $G \in W'_{(p)}(J)$ is defined by

$$\tilde{G}(s) = \left\langle G, x^{-s-1} \right\rangle.$$

Suppose $V(t) = Vt$ (product of V and t), where $V \in \mathbb{H}_+ := \mathbb{R}_+ \times \mathbf{j}\mathbb{R}$, then the exponential QMT is given by

$$\int_0^\infty e^{-Vt} t^{-s-1} dt = V^s \Gamma(-s), \ s \in \langle 0, \infty \rangle. \tag{40}$$

7. Applications

(a) Boundary value Problem:
The following boundary value problem for quaternion-valued function u is solved using A as a constant [6].

$$x^2 u_{xx} + x u_x + u_{yy} = 0, \ 0 < x < \infty, \ 0 < y < 1. \tag{41}$$

$$u(x,0) = 0, \quad u(x,1) = \begin{cases} A, \ 0 \le x \le 1 \\ 0, \ x > 1. \end{cases} \tag{42}$$

Applying one-dimensional QMT with respect to x defined by

$$\tilde{u}(s,y) = \int_0^\infty x^{-s-1} u(x,y) dx. \tag{43}$$

We reduce the given system into

$$\frac{d^2 \tilde{u}(s,y)}{dy^2} + s^2 \tilde{u}(s,y) = 0, \ 0 < y < 1, \tag{44}$$

$$\tilde{u}(s,0) = 0, \ \tilde{u}(s,1) = \int_0^\infty x^{-s-1} A dx = \frac{-A}{s}. \tag{45}$$

The solution of the (44) is

$$\tilde{u}(s,y) = \frac{-A}{s \sin(s)} \sin(sy), \ 0 < Res(s) < 1. \tag{46}$$

The inverse one-dimensional QMT is given as:

$$u(x,y) = \frac{1}{2\pi \mathbf{j}} \int_{p-\mathbf{j}\infty}^{p+\mathbf{j}\infty} \frac{-Ax^s}{s \sin(s)} \sin(sy) ds \tag{47}$$

which is the required solution.
(b) Instrumentation:
If the current in a circuit is given by

$$i(t) = 1/t + 1/t^2 \tag{48}$$

for resistance $R = R_0/t$, find the driving voltage of the network[9].
To determine the extent of output voltage e and current i using the concept

of quaternions from input values (48), the differential equation for such network is given in [9] as

$$e(t) = L\frac{di}{dt} + \frac{R_0}{t}i. \tag{49}$$

Applying one-dimensional QMT, we get

$$\mathcal{M}_q\{te(t)\} = Ls\mathcal{M}_q\{i\} + R_0\mathcal{M}_q\{i\} = (Ls + R_0)\mathcal{M}_q\{i\} \tag{50}$$

for $i(0) = 0$.
If the current meter reads (48), then by applying one-dimensional QMT, we get

$$\mathcal{M}_q\{i\} = \frac{-1}{-s-1} + \frac{-1}{-s-2} = \frac{2s+3}{(-s-1)(-s-2)}. \tag{51}$$

Thus (50) becomes

$$\mathcal{M}_q\{te(t)\} = L\left(s + \frac{R_0}{L}\right)\frac{2s+3}{(-s-1)(-s-2)}.$$

$$\mathcal{M}_q\{te(t)\} = \frac{L(\kappa-1)}{-s-1} + \frac{L(2-\kappa)}{-s-2}. \tag{52}$$

where $\kappa = \dfrac{R_0}{L}$.
Taking inverse one-dimensional QMT on both sides of (52), we get

$$e(t) = -\frac{L}{t^2}\left[\kappa - 1 + \frac{2-\kappa}{t}\right]$$

which is the required voltage.

8. Conclusion

In this paper, the authors have presented one-dimensional QMT and its inversion. Some basic properties of one-dimensional QMT are derived. Convolution type theorem and Parseval-type property are also developed. In the end, one-dimensional QMT is applied to mathematical physics and the engineering field.

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