Proceedings of the Jangjeon Mathematical Society
24 (2021), No. 1, pp. 91 - 98

www.jangjeon.or.kr
http://dx.doi.org/10.17777/pjms2021.24.1.91

INDEFINITE SASAKIAN MANIFOLD WITH QUARTER-SYMMETRIC METRIC CONNECTION

G. SOMASHEKHARA, S. GIRISH BABU, AND P. SIVA KOTA REDDY

Abstract. The object of the present paper is to study some results on indefinite Sasakian manifold admitting quarter-symmetric metric connection and η-Ricci solitons of some curvature tensors.

2010 Mathematics Subject Classification. 53C25, 53D10.

Keywords and phrases. Indefinite Sasakian manifold, η-Ricci soliton, Ricci flow, Concircular curvature tensor, Pseudo-projective curvature tensor.

1. Introduction

A linear connection $\tilde{\nabla}$ in a Riemannian manifold M is said to be a quarter-symmetric metric connection (See [2]) if the torsion tensor T of the connection $\tilde{\nabla}$

$$T(X_1, Y_1) = \tilde{\nabla}_{X_1} Y_1 - \tilde{\nabla}_{Y_1} X_1 - \tilde{\nabla}_{[X_1 Y_1]}, \tag{1}$$

satisfies

$$T(X_1, Y_1) = \eta(Y_1)\phi X_1 - \eta(X_1)\phi Y_1, \tag{2}$$

where η is the 1-form and ϕ is a tensor field of the type $(1, 1)$. Further, a quarter-symmetric connection $\tilde{\nabla}$ satisfies the condition

$$(\tilde{\nabla}_{X_1} g)(Y_1, Z_1) = 0, \tag{3}$$

where $X_1, Y_1, Z_1 \in \chi(M)$, where $\chi(M)$ is the Lie algebra of vector fields of the manifold M, then $\tilde{\nabla}$ is said to be a quarter-symmetric metric connection, otherwise it is said to be a quarter-symmetric non-metric connection.

If we put $\phi(X_1) = X_1$, then the quarter-symmetric connection reduces to the semi-symmetric connection (See [3]). Thus the notion of quarter-symmetric connection generalizes the idea of the semi-symmetric connection.

Let $(M, \varphi, \xi, \eta, g)$ be an indefinite Sasakian manifold with quarter-symmetric metric connection. We now consider the η-Ricci soliton equation as:

$$L_\xi g + 2S + 2\lambda g + 2\mu\eta(X_1)\eta(Y_1) = 0, \tag{4}$$

where L_ξ is the Lie derivative operator along the vector field ξ, S is the Ricci tensor field of the metric g, and λ, μ are the real constants.

Somashekhara et al. [5], studied some results on invariant submanifolds of LP-Sasakian manifolds endowed with semi-symmetric metric conection.

[1]Corresponding author:pskreddy@jssstuniv.in

Also, they have obtained a condition for totally geodesic by using certain geometrical conditions. In [6], we studied the C-Bochner curvature tensor under D-homothetic deformation in LP-Sasakian manifolds. Angadi et al. [1], studied the Ricci-Yamabe soliton on invariant and anti-invariant submanifolds of indefinite Sasakian manifolds, indefinite Kenmotsu manifolds and indefinite trans-Sasakian manifolds concerning Riemannian connection and quarter symmetric metric connection.

2. Preliminaries

An n-dimensional smooth manifold (M^n, g) is said to be an indefinite almost contact metric manifold, if it admits a $(1,1)$ tensor field φ, a structure vector field ξ, a 1-form η and an indefinite metric g such that

$$\phi^2 X_1 = -X_1 + \eta(X_1)\xi, \;\; \phi\xi = 0, \;\; \eta o \phi = 0, \tag{5}$$

If g is a semi-symmetric metric connection with (ϕ, ξ, η), then we have:

$$g(\phi X_1, \phi Y_1) = g(X_1, Y_1) - \varepsilon\eta(X_1)\eta(Y_1), \tag{6}$$

$$g(X_1, \xi) = \varepsilon\eta(X_1), \;\; g(\xi, \xi) = \varepsilon, \;\; \eta(\xi) = 1, \tag{7}$$

$$g(X_1, \phi Y_1) = -g(\phi X_1, Y_1), \tag{8}$$

where $\varepsilon = \pm 1$ and for all vector fields X_1,Y_1 on M^n becomes an indefinite almost contact metric manifold equipped with an indefinite contact structure (ϕ, ξ, η). An indefinite almost contact metric manifold M^n is called normal if

$$N_\phi + d\eta \otimes \xi = 0, \tag{9}$$

where N_ϕ is the Nijenhuis tensor field. An indefinite normal contact metric manifold M^n is called an indefinite Sasakian manifold (See [7]) if

$$(\nabla_{X_1}\eta)(Y_1) = \varepsilon g(X_1, \phi Y_1), \tag{10}$$

$$(\nabla_{X_1}\xi) = \varepsilon\phi X_1, \tag{11}$$

$$(\nabla_{X_1}\phi)(Y_1) = -g(X_1, Y_1)\xi + \varepsilon\eta(Y_1)X_1. \tag{12}$$

Also, in an indefinite Sasakian Manifold M^n the following relations hold (See [8]):

$$\eta(R(X_1, Y_1)Z_1) = \varepsilon\left[g(X_1, Z_1)\eta(Y_1) - g(Y_1, Z_1)\eta(X_1)\right], \tag{13}$$

$$R(X_1, Y_1)\xi = \varepsilon\left[\eta(Y_1)X_1 - \eta(X_1)Y_1\right], \tag{14}$$

$$R(\xi, X_1)Y_1 = \varepsilon\left[g(X_1, Y_1)\xi - \eta(X_1)Y_1\right], \tag{15}$$

$$S(X_1, \xi) = -(n - \varepsilon)\eta(X_1), \tag{16}$$

$$Q\xi = -(n - \varepsilon)\xi, \tag{17}$$

$$S(\phi X_1, \phi Y_1) = S(X_1, Y_1) + \varepsilon(n - \varepsilon)\eta(X_1)\eta(Y_1), \tag{18}$$

for all the vector fields X_1, Y_1 and Z_1, where R is the semi-Riemannnian curvature tensor.

In [4], the authors studied quarter-symmetric metric connection $\tilde{\nabla}$ in an indefinite Sasakian manifold of dimension M^n and the following relations hold:

$$\tilde{R}(X_1,Y_1)Z_1 = R(X_1,Y_1)Z_1+\varepsilon\eta(Z_1)[\eta(Y_1)X_1-\eta(X_1)Y_1]+ \varepsilon[g(Y_1,Z_1)\eta(X_1)-g(X_1,Z_1)\eta(Y_1)]\xi-\varepsilon[g(\phi Y_1,Z_1)\phi X_1-g(\phi X_1,Z_1)\phi Y_1], \tag{19}$$

$$\tilde{S}(X_1,Y_1)=S(X_1,Y_1)+\varepsilon(n-\varepsilon)\eta(X_1)\eta(Y_1), \tag{20}$$

$$\tilde{Q}X_1=QX_1+\varepsilon(n-\varepsilon)\eta(X_1)\xi, \tag{21}$$

$$\tilde{r}=r+\varepsilon(n-\varepsilon), \tag{22}$$

where $\tilde{R}$ is the Riemannian curvature tensor with respect to quarter symmetric metric connection and Q is the Ricci operator given by $S(X_1,Y_1)=g(QX_1,Y_1)\forall X_1,Y_1\in\chi(M)$.

Using the notion η-Ricci soliton on indefinite Sasakian manifold, then the following relations are holds:

$$S(X_1,Y_1)=-\lambda g(X_1,Y_1)-\mu\eta(X_1)\eta(Y_1), \tag{23}$$

$$QX_1=-\lambda X_1-\mu\eta(X_1)\xi, \tag{24}$$

$$S(X_1,\xi)=A_1\eta(X_1), \tag{25}$$

where

$$A_1=-(\lambda\varepsilon+\mu).$$

Again by using the notion η-Ricci soliton on indefinite Sasakian manifold with quarter-symmetric metric connection, we have the following:

$$\tilde{S}(X_1,Y_1)=-\lambda g(X_1,Y_1)+(\varepsilon n-1-\mu)\eta(X_1)\eta(Y_1), \tag{26}$$

$$\tilde{Q}X_1=-\lambda X_1+(\varepsilon n-1-\mu)\eta(X_1)\xi, \tag{27}$$

$$\tilde{S}(X_1,\xi)=A_2\eta(X_1), \tag{28}$$

where

$$A_2=[\varepsilon(n-\lambda)-\mu-1]\eta(X_1).$$

3. Indefinite Sasakian Manifold with quarter-symmetric metric connection satisfying $\tilde{R}.(\xi,U).\tilde{C}=0$

Let M be a n-dimensional indefinite Sasakian manifold with quarter-symmetric metric connection. Then the concircular curvature tensor $\tilde{C}$ on M is defined by

$$\tilde{C}(X_1,Y_1)Z_1=\tilde{R}(X_1,Y_1)Z_1-\frac{\tilde{r}}{n(n-1)}[g(Y_1,Z_1)X_1-g(X_1,Z_1)Y_1], \tag{29}$$

where $\tilde{r}$ is scalar curvature. Substitute $Z_1=\xi$ and using (19) in (29), we get:

$$\tilde{C}(X_1,Y_1)\xi=A_3[\eta(Y_1)X_1-\eta(X_1)Y_1], \tag{30}$$

where

$$A_3=2\varepsilon-\varepsilon\frac{\tilde{r}}{n(n-1)}.$$

Taking inner product with ξ, we get

(31) $$\eta(\tilde{C}(X_1, Y_1)\xi) = 0.$$

We assume that the condition $\tilde{R}.(\xi, U).\tilde{C} = 0$, then we have

(32) $$\tilde{R}(\xi, U).\tilde{C}(X_1, Y_1)\xi - \tilde{C}(\tilde{R}(\xi, U)X_1, Y_1)\xi - \tilde{C}(X_1, \tilde{R}(\xi, U)Y_1)\xi - \tilde{C}(X_1, Y_1)\tilde{R}(\xi, U)\xi = 0.$$

In view of equation (19), the equation (32) reduces to

(33)
$$2\varepsilon g(U, \tilde{C}(X_1, Y_1)\xi)\xi - 2\varepsilon g(U, X_1)\tilde{C}(\xi, Y_1)\xi + 2\varepsilon\eta(X_1)\tilde{C}(U, Y_1)\xi - (\varepsilon - 1)\eta(X_1)\eta(U)\tilde{C}(\xi, Y_1)\xi - 2\varepsilon g(U, Y_1)\tilde{C}(X_1, \xi)\xi + 2\varepsilon\eta(Y_1)\tilde{C}(X_1, U)\xi - (\varepsilon - 1)\eta(U)\eta(Y_1)\tilde{C}(X_1, \xi)\xi - 2\varepsilon\eta(U)\tilde{C}(X_1, Y_1)\xi + 2\varepsilon\tilde{C}(X_1, Y_1)U = 0,$$

for all vector fields X_1, Y_1 *and* U on M.

Taking inner product with ξ and using (31) in (33), we have

(34) $$2g(U, \tilde{C}(X_1, Y_1)\xi) + 2\varepsilon g(\tilde{C}(X_1, Y_1)U, \xi) = 0,$$

replacing $U = Z_1$ and simplifying we get

(35) $$\tilde{C}(X_1, Y_1)Z_1 = A_3\,[g(Y_1, Z_1)X_1 - g(X_1, Z_1)Y_1].$$

In the view of (29), we get

(36) $$\tilde{R}(X_1, Y_1)Z_1 = A_4\,[g(Y_1, Z_1)X_1 - g(X_1, Z_1)Y_1],$$

where

$$A_4 = A_3 + \frac{\tilde{r}}{n(n-1)}.$$

Taking inner product with U and using (19), by assuming $X_1 = U = e_i$ and summing over $i = 1, 2, \cdots, n$, we get

(37) $$S(Y_1, Z_1) = A_5 g(Y_1, Z_1) + A_6\eta(Y_1)\eta(Z_1),$$

where

$$A_5 = A_2(n-1) - \varepsilon\,,\ A_6 = 1 - \varepsilon(n-1).$$

Hence we can state the following theorem,

Theorem 3.1. *A n-dimensional indefinite Sasakian manifold with quarter-symmetric metric connection satisfying $\tilde{R}.(\xi, U).\tilde{C} = 0$ is a η-Einstein manifold, where $\tilde{C}$ and $\tilde{R}$ are concircular curvature tensor and Riemann curvature tensor respectively with quarter-symmetric metric connection.*

4. η-Ricci soliton in an Indefinite Sasakian Manifold with quarter-symmetric metric connection satisfying certain semi-symmetry conditions

Concircular curvature tensor $\tilde{C}$ satisfying $\tilde{C}.(X_1, Y_1).\tilde{S} = 0$. Let M be an n-dimensional indefinite Sasakian manifold admitting η-Ricci soliton (g, V, λ) with quarter-symmetric metric connection satisfying $\tilde{C}.(X_1, Y_1).\tilde{S} = 0$. It implies that

(38) $$\tilde{S}(\tilde{C}(X_1, Y_1)U, V) + \tilde{S}(U, \tilde{C}(X_1, Y_1)V) = 0,$$

where $\tilde{C}$ is concircular curvature tensor defined by

$$\tilde{C}(X_1,Y_1)Z_1 = \tilde{R}(X_1,Y_1)Z_1 - \frac{\tilde{r}}{n(n-1)}[g(Y_1,Z_1)X_1 - g(X_1,Z_1)Y_1]. \tag{39}$$

Substituting $X_1 = \xi$ and using equation (28), (39) in (38), we get

$$\begin{aligned}(40)\quad & A_2\left[2\varepsilon - \frac{\tilde{r}}{n(n-1)}\right] g(Y_1,U)\eta(V) + 2A_2(\varepsilon-1)\eta(U)\eta(V)\eta(Y_1) + \\ & \left[\frac{\varepsilon\tilde{r}}{n(n-1)} - 2\varepsilon\right]\eta(U)\tilde{S}(Y_1,V) + A_2\left[2\varepsilon - \frac{\tilde{r}}{n(n-1)}\right] g(Y_1,V)\eta(U) + \\ & \left[\frac{\varepsilon\tilde{r}}{n(n-1)} - 2\varepsilon\right]\eta(V)\tilde{S}(Y_1,U) = 0,\end{aligned}$$

Substituting $U = \xi$ and in the view of (20), we get

$$S(V,Y_1) = A_7 g(V,Y_1) + A_8\eta(V)\eta(Y_1), \tag{41}$$

where

$$A_7 = -A_2\left[\frac{2\varepsilon - \frac{\tilde{r}}{n(n-1)}}{\frac{\varepsilon\tilde{r}}{n(n-1)} - 2\varepsilon}\right],$$

$$A_8 = -\frac{A_2\left(\frac{\tilde{r}(\varepsilon-1)}{n(n-1)} - 2\right) + (\varepsilon n - 1)\left(\frac{\varepsilon\tilde{r}}{n(n-1)} - 2\varepsilon\right)}{\left(\frac{\varepsilon\tilde{r}}{n(n-1)} - 2\varepsilon\right)}.$$

Hence we can state the following theorem

Theorem 4.1. *A n-dimensional η-Ricci soliton in an indefinite Sasakian manifold with quarter-symmetric metric connection satisfying $\tilde{C}.(X_1,Y_1).\tilde{S} = 0$ is a η-Einstein manifold, where $\tilde{C}$ and $\tilde{S}$ are concircular curvature tensor and Riemann curvature tensor respectively with quarter-symmetric metric connection.*

Pseudo-projective curvature tensor $\tilde{P}$ satisfying $\tilde{R}.(\xi,X_1).\tilde{P} = 0$.
Let M be a n-dimensional indefinite Sasakian manifold admitting η-Ricci soliton (g,V,λ) with quarter-symmetric metric connection satisfying $\tilde{R}.(\xi,X_1).\tilde{P} = 0$. Then the pseudo-projective curvature tensor $\tilde{P}$ on M is defined by

$$\begin{aligned}\tilde{P}(X_1,Y_1)Z_1 = & a\tilde{R}(X_1,Y_1)Z_1 + b[\tilde{S}(Y_1,Z_1)X_1 - \tilde{S}(X_1,Z_1)Y_1] - \\ & \frac{\tilde{r}}{n}\left[\frac{a}{n-1} + b\right][g(Y_1,Z_1)X_1 - g(X_1,Z_1)Y_1],\end{aligned} \tag{42}$$

where $\tilde{r}$ is scalar curvature. Substituting $Z_1 = \xi$ and using equation (19), we get

$$\tilde{P}(X_1,Y_1)\xi = A_9[\eta(Y_1)X_1 - \eta(X_1)Y_1]. \tag{43}$$

where

$$A_9 = 2a\varepsilon + A_2 b - \frac{\varepsilon\tilde{r}}{n}\left(\frac{a}{n-1} + b\right)$$

Taking inner product with ξ, we get

$$\eta(\tilde{P}(X_1,Y_1)\xi) = 0 \tag{44}$$

We assume that the condition $\tilde{R}.(\xi, X_1).\tilde{P} = 0$, then we have

(45)
$$\tilde{R}(\xi, X_1).\tilde{P}(U, V)\xi - \tilde{P}.(\tilde{R}(\xi, X_1)U, V)\xi - \tilde{P}(U, \tilde{R}(\xi, X_1)V)\xi - \tilde{P}(U, V)\tilde{R}(\xi, X_1)\xi = 0.$$

In view of the equations (19) and (44), we get

(46)
$$\begin{aligned} &2\varepsilon g(X_1, \tilde{P}(U, V)\xi)\xi - 2\varepsilon g(X_1, U)\tilde{P}(\xi, V)\xi + 2\varepsilon\eta(U)\tilde{P}(X_1, V)\xi - (\varepsilon - 1) \\ &\eta(X_1)\eta(U)\tilde{P}(\xi, V)\xi - 2\varepsilon g(X_1, V)\tilde{P}(U, \xi)\xi + 2\varepsilon\eta(V)\tilde{P}(U, X_1)\xi - (\varepsilon - 1) \\ &\eta(V)\eta(X_1)\tilde{P}(U, \xi)\xi - (\varepsilon - 1)\eta(X_1)\tilde{P}(U, V)\xi + 2\varepsilon\tilde{P}(U, V)X_1 = 0, \end{aligned}$$

for all vector fields X_1, U *and* V on M.

Taking inner product with ξ and using (44), we get

$$2g(X_1, \tilde{P}(U, V)\xi) + 2\varepsilon g(\tilde{P}(U, V)X_1, \xi) = 0. \tag{47}$$

Substituting $U = \xi$ and using (42), we get

(48)
$$\begin{aligned} &a[\varepsilon(g(V, X_1)\xi - \eta(X_1)V)] + \varepsilon\eta(X_1)[\eta(V)\xi - V] + \varepsilon[g(V, X_1) - \varepsilon\eta(V)\eta(X_1)]\xi + \\ &b[(S(V, X_1) + (\varepsilon n - 1)\eta(V)\eta(X_1))\xi - A_2\eta(X_1)V] - \frac{\tilde{r}}{n}\left[\frac{a}{n-1} + b\right] \\ &[g(V, X_1)\xi - \varepsilon\eta(X_1)V] = A_9[g(V, X_1)\xi - \varepsilon\eta(X_1)V]. \end{aligned}$$

Now taking inner product with ξ and using (7), we get

$$S(V, X_1) = A_{10}g(V, X_1) + A_{11}\eta(V)\eta(X_1) \tag{49}$$

where

$$A_{10} = -\left(\frac{2a\varepsilon - A_9\varepsilon - \frac{\tilde{r}}{n}\left[\frac{a}{n-1} + b\right]}{b\varepsilon}\right)$$
$$A_{11} = -\left(\frac{a(\varepsilon - 1) + b(n - \varepsilon) - A_2\varepsilon b + A_9\varepsilon + \frac{\tilde{r}}{n}\left[\frac{a}{n-1} + b\right]}{b\varepsilon}\right).$$

Hence we can state the following theorem

Theorem 4.2. *A n-dimensional η-Ricci soliton in an indefinite Sasakian manifold with quarter-symmetric metric connection satisfying $\tilde{R}.(\xi, X_1).\tilde{P} = 0$ is a η-Einstein manifold, where $\tilde{P}$ and $\tilde{R}$ are pseudo-projective curvature tensor and Riemannian curvature respectively with quarter-symmetric metric connection.*

Pseudo-projective curvature tensor $\tilde{P}$ satisfying $\tilde{P}.(\xi, X_1).\tilde{S} = 0$. Let M be an n-dimensional indefinite Sasakian manifold admitting η-Ricci soliton (g, V, λ) with quarter-symmetric metric connection satisfying $\tilde{P}.(\xi, X_1).\tilde{S} = 0$. It implies that

$$\tilde{S}(\tilde{P}(\xi, X_1)Y_1, Z_1) + \tilde{S}(Y_1, \tilde{P}(\xi, X_1)Z_1) = 0, \tag{50}$$

where $\tilde{P}$ is pseudo-projective curvature tensor defined by

(51)

$$\tilde{P}(X_1,Y_1)Z_1 = a\tilde{R}(X_1,Y_1)Z_1 + b(\tilde{S}(Y_1,Z_1)X_1 - \tilde{S}(X_1,Z_1)Y_1) - \frac{\tilde{r}}{n}\left(\frac{a}{n-1}+b\right)$$
$$[g(Y_1,Z_1)X_1 - g(X_1,Z_1)Y_1],$$

using (28), (51), the equation (50) reduces to

(52)

$$2A_2a\varepsilon g(X_1,Y_1)\eta(Z_1) - a\varepsilon\tilde{S}(X_1,Z_1)\eta(Y_1) + A_2a\varepsilon\eta(X_1)\eta(Y_1)\eta(Z_1) - a\varepsilon\eta(Y_1)\tilde{S}(X_1,Z_1)$$
$$- A_2a\eta(X_1)\eta(Y_1)\eta(Z_1) + A_2b\tilde{S}(X_1,Y_1)\eta(Z_1) - A_2b\tilde{S}(X_1,Z_1)\eta(Y_1) - A_2\frac{\tilde{r}}{n}\left(\frac{a}{n-1}+b\right)$$
$$g(X_1,Y_1)\eta(Z_1) + \frac{\tilde{r}}{n}\left(\frac{a}{n-1}+b\right)\varepsilon\eta(Y_1)\tilde{S}(X_1,Z_1) + 2A_2a\varepsilon g(X_1,Z_1)\eta(Y_1) - a\varepsilon$$
$$\tilde{S}(X_1,Y_1)\eta(Z_1) + A_2a\varepsilon\eta(X_1)\eta(Z_1)\eta(Y_1) - a\varepsilon\eta(Z_1)\tilde{S}(X_1,Y_1) - A_2a\eta(X_1)\eta(Z_1)\eta(Y_1)$$
$$+ A_2b\tilde{S}(X_1,Z_1)\eta(Y_1) - A_2b\tilde{S}(X_1,Y_1)\eta(Z_1) - A_2\frac{\tilde{r}}{n}\left(\frac{a}{n-1}+b\right)g(X_1,Z_1)\eta(Y_1)+$$
$$\frac{\tilde{r}}{n}\left(\frac{a}{n-1}+b\right)\varepsilon\eta(Z_1)\tilde{S}(X_1,Y_1) = 0.$$

Substituting $Z_1 = \xi$ and using (7), (28) in (52), we get

(53)

$$\left[2aA_2\varepsilon - A_2\frac{\tilde{r}}{n}\left(\frac{a}{n-1}+b\right)\right]g(X_1,Y_1) + \tilde{S}(X_1,Y_1)\left[\varepsilon\frac{\tilde{r}}{n}\left(\frac{a}{n-1}+b-2a\varepsilon\right)\right] = 0.$$

In the view of (20), we get

$$S(X_1,Y_1) = A_{12}g(X_1,Y_1) + A_{13}\eta(X_1)\eta(Y_1), \quad (54)$$

where

$$A_{12} = -\frac{\left[2aA_2\varepsilon - A_2\frac{\tilde{r}}{n}\left(\frac{a}{n-1}+b\right)\right]}{\left[\varepsilon\frac{\tilde{r}}{n}\left(\frac{a}{n-1}+b\right)-2a\varepsilon\right]},$$
$$A_{13} = -\frac{\left[(\varepsilon n-1)\left(\varepsilon\frac{\tilde{r}}{n}\left(\frac{a}{n-1}+b\right)-2a\varepsilon\right)\right]}{\left[\varepsilon\frac{\tilde{r}}{n}\left(\frac{a}{n-1}+b\right)-2a\varepsilon\right]}.$$

Hence we can state the following theorem

Theorem 4.3. *A n-dimensional η-Ricci soliton in an indefinite Sasakian manifold with quarter-symmetric metric connection satisfying $\tilde{P}.(\xi,X_1).\tilde{S} = 0$ is a η-Einstein manifold, where $\tilde{P}$ and $\tilde{S}$ are pseudo-projective curvature tensor and Ricci tensor respectively with quarter-symmetric metric connection.*

Alternatively, the Theorems 4.2, 4.3 and 4.4 can be put together as follows:

Theorem 4.4. *A n-dimensional η-Ricci soliton in an indefinite Sasakian manifold admitting quarter-symmetric metric connection satisfying any of the following conditions is η-Einstein.*
a. $\tilde{C}.(X_1, Y_1).\tilde{S} = 0$,
b. $\tilde{R}.(\xi, X_1).\tilde{P} = 0$,
c. $\tilde{P}.(\xi, X_1).\tilde{S} = 0$.

Acknowledgements

The authors would like to thank the referees for their invaluable comments and suggestions which led to the improvement of the manuscript.

References

[1] P. G. Angadi, G. S. Shivaprasanna, G. Somashekhara and P. S. K. Reddy, *Ricci Ricci-Yamabe Solitons on Submanifolds of Some Indefinite Almost Contact Manifolds*, Adv. Math., Sci. J., 9(11) (2020), 10067–10080.
[2] S. Golab, *On semi-symmetric and quarter-symmetric linear connections*, Tensor N.S., 29 (1975), 249-254.
[3] A. Friedmann and J. A. Schouten, *Uber die Geometrie der halbsymmetrischen Ubertragung*, Math. Zeitschr., 21 (1924), 211-223.
[4] R. N. Singh, S. K. Pandey and Kiran Tiwari, *On Quarter-Symmetric Metric Connection in an Indefinite Sasakian Manifold*, J. Int. Acad. Phys. Sci., 17(3) (2013), 255-275.
[5] G. Somashekhara, N. Pavani and P. S. K. Reddy, *Invariant Sub-manifolds of LP-Sasakian Manifolds with Semi-Symmetric Connection*, Bull. Math. Anal. Appl., 12(2) (2020), 35-44.
[6] G. Somashekhara, S. Girish Babu and P. S. K. Reddy, *C-Bochner Curvature Tensor under D-Homothetic Deformation in LP-Sasakian Manifold*, Bull. Int. Math. Virtual Inst., 11(1) (2021), 91-98.
[7] S. Sular, C. Ozgur and U. C. De, *Quarter-symmetric metric connection in a Kenmotsu manifold*, SUT J. Math., 44(2) (2008), 297-306.
[8] T. Takahashi, *Sasakian ϕ-symmetric spaces*, Tohoku Math J., 29 (1977), 91-113.

Department of Mathematics and Statistics, M. S. Ramaiah University of Applied Sciences, Bangalore-560 054, INDIA
Email address: somashekhara.mt.mp@msruas.ac.in

Department of Mathematics, Acharya institute of Technology, Bangalore-560 107, INDIA
Email address: sgirishbabu84@gmail.com

Department of Mathematics, Sri Jayachamarajendra College of Engineering, JSS Science and Technology University, Mysuru-570 006, INDIA
Email address: pskreddy@jssstuniv.in