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# Certain novel results in relation with relationships between the confluent hypergeometric function and the confluent hypergeometric equation

**Hüseyin Irmak**[1]
[1]Department of Mathematics, Faculty of Science, Çankırı Karatekin University, Uluyazı Campus, Çankırı 18100, Turkey
e-mail: hirmak@karatekin.edu.tr *or* hisimya@yahoo.com

**Preveen Agarwal**[2,3,4,5]
[2]Department of Mathematics, Anand International College of Engineering, Jaipur 303012, India.
[3]International Center for Basic and Applied Sciences, Jaipur 302029, India.
[4]Department of Mathematics, Harish-Chandra Research Institute, Allahabad 211 019, India.
[5]Institute of Mathematics and Mathematical Modeling, Almaty, Kazakhstan.
e-mail: goyal.praveen2011@gmail.com

**Abstract**

The aim of this investigation is first to reveal certain novel-extensive results in relation with both the confluent hypergeometric function and the confluent hypergeometric equation in the complex plane and then to point out a number of the implications propounded by the related results.

*Keywords:* Complex plane, the second-order ordinary linear differential equation in the complex plane, the confluent hypergeometric equation, the confluent hypergeometric function, special functions with complex variable, functions defined by series, inequalities and equations in the complex plane.

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## 1. Introduction, definitions and motivation

In words of one syllable, any differential equation (DE) is an equation that involves a function and its derivatives. In other words, a DE relates to a statement connecting the value of a quantity to the rate at which that quantity is changing. Differential equations (DEs) frequently occur in a variety of contexts. To solve any DEs, it is helpful to group them into categories that can be solved with similar techniques, which is expressed with a variety of words describing them. Basically, there are two types of DEs. The first is the ordinary differential equation (ODE) and this contains one variable and its derivatives. The other also is the partial differential equation (PDE) and it contains different variables and also their partial derivatives. There is an order of a DE. It is the highest order derivative that appears in the given equation. A DE is linear if it includes only linear combinations of the derivatives of the unknown function and non-linear otherwise. Given an arbitrary DE is not usually very easy to solve. However, when DEs have one of several forms, it is possible they can be exactly solved. Indeed, when DEs can not be exactly solved, they are also solved by using numerical approximations, like Euler's method and the Runge-Kutta methods etc. Unfortunately, there are some DEs that we can not solve, and so one of the most important questions concerning DEs is which equations we can find solutions to, and when these solutions are unique. In such problems, the Picard-Lindelof theorem also play important role and it says that DEs can have solutions that are unique as long as certain conditions. In short, to determine existence and uniqueness is not easy work.

With this present study, our purpose is to reveal some novel results relating to certain relationships between the second order linear differential equations and their possible solutions in the complex plane and then to point some possible implications of them out. In doing this research, it is to reveal some possible geometrical or analytical results between the given equation and its unique solution(s), without finding a solution of a complex differential equation in the given general form.

After reminding basic information about DEs and also emphasizing the main goal of this work, we now present some additional information which will be related to our investigation.

The main theme of this research consists of the second-order ordinary linear differential equation:

$$z\,\frac{d^2\omega}{dz^2} + \big(\gamma - z\big)\,\frac{d\omega}{dz} - \lambda\,\omega = 0\,, \tag{1}$$

*or,* in its self-adjoint form:

$$\frac{d\omega}{dz}\left(z^{\gamma}\,e^{-z}\,\frac{d\omega}{dz}\right) - \lambda\,z^{\gamma-1}\,e^{-z}\,\omega = 0\,, \tag{2}$$

*which* is frequently encountered in the literature. Of course, it is possible that the above-indicated variables $z$ and $\omega$ and the parameters $\gamma$ and $\lambda$ may be also considered as their complex values.

The equation, which is well known as the Whittaker equation in the literature, is also reduced form of the equation presented in (1) and this equation in (1) is closely related to the well-known hypergeometric equation. Concurrently, the confluent hypergeometric equation can be also regarded as an equation obtained from the Riemann differential equation as a result of the merging of two singular points. The point $z = 0$ is a regular singular point for the related equation, while the point $z = \infty$ is a strong singular point. Kummer [14] was the first investigator to undertake a systematic study of the solutions of the equation given in (1). The possible solutions of the equation in (1) are usually expressed through the confluent hypergeometric function, denoted by $\Phi(\gamma;\lambda;z)$, as it was indicated just above. In short, in the literature, we usually come across any of the following notations for which express the confluent hypergeometric function of the first kind:

$$ {}_1F_1\big(\gamma;\lambda;z\big) \quad , \quad M\big(\gamma,\lambda,z\big) \quad \textit{or} \quad \Phi\big(\gamma;\lambda;z\big) \ , $$

and it is also known as Kummer's Function of the first kind. Indeed, it is a degenerate form of the hypergeometric function ${}_2F_1\big(a;b;c;z\big)$ which arises as a solution of the confluent (*or* Kummer) hypergeometric differential equation given in (1), as we pointed out above. The confluent (*or* Kummer) hypergeometric function (of the first kind) is an analytic function in $\mathbb{C}$ and also has the convergent power series as in the following form:

$$ \Phi\big(\gamma;\lambda;z\big) = 1 + \frac{\gamma}{\lambda}z + \frac{\gamma(\gamma+1)}{\lambda(\lambda+1)}\frac{z^2}{2!} + \ \ldots \ = \sum_{k=0}^{\infty}\frac{(\gamma)_k}{(\lambda)_k}\frac{z^k}{k!} \ , \tag{3} $$

*where* $\gamma \in \mathbb{C} - \mathbb{Z}_0^-$, $\lambda \in \mathbb{C} - \mathbb{Z}_0^-$ and $z$ is any complex number, and, in terms of the Gamma function $\Gamma(z)$, the Pochhammer symbol $(\lambda)_n$ is also defined by

$$ (\lambda)_n = \begin{cases} 1 & if \quad n = 0 \\ \lambda(\lambda+1)\cdots(\lambda+n-1) & if \quad n \in \mathbb{N} \end{cases} $$

$$ = \frac{\Gamma(\lambda+n)}{\Gamma(\lambda)} \ , $$

*where* the notations $\mathbb{C}$ , $\mathbb{N}$ and $\mathbb{Z}_0^-$ denote the sets of complex numbers, natural numbers and nonpositive integers, respectively.

If the parameter $\lambda$ is not an integer, the general solution of the equation in (1) may be then written in the form:

$$\omega = A\,\Phi\big(\gamma;\lambda;z\big) + B\,z^{1-\gamma}\,\Phi\big(\gamma+1-\lambda;2-\gamma;z\big),$$

*where* $A$ and $B$ are arbitrary complex numbers. If $\gamma$ is an integer, its general solution has a more complicated form since it may contain logarithmic terms. The functions other than those in (2) (*e.g.,* Whittaker functions [3], [14]) can also be selected as a fundamental system of solutions of the equation in (1). The solutions of the equation in (1) can also be represented by contour integrals in the complex plane. Many second-order ordinary linear differential equations (*e.g.,* the Bessel equation) can be reduced to the equation in (1) by a transformation of the unknown function and of the independent variable [4-17]. Particularly, for given complex numbers $a, b, c, d,$ and $d$, the equations in the following type:

$$\big(az+b\big)\,\frac{d^2\omega}{dz^2} + \big(cz+d\big)\,\frac{d\omega}{dz} + \big(dz+e\big)\,\omega = 0\;, \tag{4}$$

can be integrated by using the confluent hypergeometric function.

We note that hypergeometric functions, hypergeometric differential equations, a large of special functions and also equations associated with the information emphasized as in (1)-(4) play an important role in many fields of pure and applied mathematics as well as science and technology. For their details and also different investigations, one may refer to the earlier works given in [1, 3, 19, 21-28]. So, in this investigation, a wide range of relations between the themes, indicated just above, will be established without actually determining any solutions of certain linear (differential) equations to be given in the next section.

## 2. Certain extensive Results and Implications

In order to prove our main results, there is a need to present one of the special forms of the well-known assertion proven by S.S. Miller and P.T. Mocanu in the reference [18]. At the same time, for instance, one may refer to the works in [10] and [20].

**Lemma 2.1.** *Let the function* $p(z)$ *in the form:*

$$p(z) = 1 + a_n z^n + a_{n+1} z^{n+1} + a_{n+2} z^{n+2} + \cdots \;\big(a_n \neq 0\big) \tag{5}$$

*be analytic in the open unit disk:*

$$\mathbb{U} = \big\{z : z \in \mathbb{C} \text{ and } |z| < 1\big\}\,,$$

*where* $n \in \mathbb{N}$ *and* $a_n \in \mathbb{C}$.

*If*

$$\Re e\big[p(z_0)\big] = \min\Big\{\Re e\big[p(z)\big] \ : \ |z| \leq |z_0|\Big\}, \tag{6}$$

*then*

$$z_0 p'(z_0) \leq -\frac{n}{2}\,\frac{\big|1-p(z_0)\big|^2}{\Re e\big[1-p(z_0)\big]}, \tag{7}$$

*where* $n \in \mathbb{N}$, $z_0 \in \mathbb{U} - \{0\}$ *and* $z \in \mathbb{U}$.

We can now state and then prove some comprehensive theorem consisting of various results in association with certain novel relationships between the second-order nonhomogeneous linear differential equations with variable coefficients in the complex plane, which are given by the following theorems.

**Theorem 2.1.** *Let a function* $\kappa \equiv \kappa(z)$ *be a solution of the (differential) equation given in* (1) *and also let the function* $\Psi(z)$ *satisfy any one of the cases of the following inequalities:*

$$\Re e\Big(\Psi(z)\Big)\begin{cases} < 0 \quad if \quad \Re e\big(\lambda^2\big) \geq 0 \\ > 0 \quad if \quad \Re e\big(\lambda^2\big) \leq 0 \end{cases}, \tag{8}$$

*or, equivalently,*

$$\Im m\Big(i\,\Psi(z)\Big)\begin{cases} < 0 \quad if \quad \Im m\big(\lambda^2\big) \geq 0 \\ > 0 \quad if \quad \Im m\big(\lambda^2\big) \leq 0 \end{cases}, \tag{9}$$

*where* $z \in \mathbb{U}$ *and* $\lambda \in \mathbb{C} - \mathbb{Z}_0^-$.

*If the function* $\kappa$ *is an unique solution of the following second-order nonhomogeneous linear differential equation with variable coefficients:*

$$\left(z\frac{d^2\kappa}{dz^2} + \gamma\frac{d\kappa}{dz}\right)\left(z\frac{d\kappa}{dz} - \lambda\kappa\right) \;=\; \Psi(z)\,, \tag{10}$$

*then*

$$\Re e\Big(\kappa(z)\Big) > 0\,, \tag{11}$$

*where* $z \in \mathbb{U}$, $\gamma \in \mathbb{C} - \mathbb{Z}_0^-$ *and* $\lambda \in \mathbb{C} - \mathbb{Z}_0^-$.

*Proof.* Since $\kappa(z)$ is a solutions for the differential equation given in (1), it is equivalent to the function $\Phi(\gamma;\lambda;z)$ both satisfying the equation in (1) and having the series form as in (3).

Let us take $p(z)$ as

$$p(z) = \kappa(z)\,\Big(\equiv \Phi(\gamma;\lambda;z)\Big) \quad \big(z \in \mathbb{U}\big). \tag{12}$$

Clearly, the function $\kappa(z)$ both has the series form in (5) and is analytic in the open unit disk $\mathbb{U}$. Therefore, the function $p(z)$ is both analytic in $\mathbb{U}$ and satisfies the conditions of Lemma 2.1, which are $p(0) = 1$ and $n = 1$ there.

It follows from (12) that

$$\frac{d}{dz}\Big(p(z)\Big) = \frac{d\kappa}{dz} \tag{13}$$

and also by taking in consideration (12), the following relationship:

$$z\frac{d}{dz}\Big(p(z)\Big) - \lambda p(z) = z\frac{d\kappa}{dz} - \lambda\kappa \tag{14}$$

is easily obtained. In addition, in view of the equation in (1) and by combining (13) and (14), the following statement:

$$\begin{aligned}\Psi(\gamma;\lambda;z) &:= \left(z\frac{d^2\kappa}{dz^2} + \gamma\frac{d\kappa}{dz}\right)\left(z\frac{d\kappa}{dz} - \lambda\kappa\right)\\ &= \left(z\frac{d\kappa}{dz} + \lambda\kappa\right)\left(z\frac{d\kappa}{dz} - \lambda\kappa\right),\end{aligned}$$

*or,* equivalently,

$$\Psi(\gamma;\lambda;z) \;:=\; \left(z\frac{dP}{dz} + \lambda P\right)\left(z\frac{dP}{dz} - \lambda P\right) \tag{15}$$

is also obtained, *where*

$$z \in \mathbb{U} \;\;,\;\; \lambda \in \mathbb{C} - \mathbb{Z}_0^- \;\;,\;\; \gamma \in \mathbb{C} - \mathbb{Z}_0^- \quad \text{and} \quad P := p(z).$$

By the help of (6), we now assume that there exists a point $z_0 \in \mathbb{U} - \{0\}$ such that

$$\Re e\big[p(z)\big] > 0 \;\; \big(0 < |z| < |z_0|\big) \quad \text{and} \quad p(z_0) = i\alpha\,,$$

*where* $z_0 \in \mathbb{U} - \{0\}$ and $\alpha \in \mathbb{R} - \{0\}$. In the present case, from (7), we get

$$\beta := z_0 p'(z_0) \leq -\frac{n}{2}\,\frac{\big|1 - p(z_0)\big|^2}{\Re e\big[1 - p(z_0)\big]} = -\frac{n}{2}\big(1 + \alpha^2\big) \leq -\frac{1 + \alpha^2}{2}\,,$$

In the light of Lemma 2.1, it follows from (15) that

$$\Psi\big(z_0\big) = \beta^2 + (\alpha\lambda)^2$$

and also

$$\Im m\Big(\Psi\big(z_0\big)\Big) = \alpha^2\,\Im m\big(\lambda^2\big) \;\text{ and }\; \Re e\Big(i\,\Psi\big(z_0\big)\Big) = \alpha^2\,\Re e\big(\lambda^2\big)\,, \tag{16}$$

*where* $\alpha \in \mathbb{R} - \{0\}$ and $\lambda \in \mathbb{C} - \mathbb{Z}_0^-$.

So, naturally, the expressions in (16) together with the conditions just above gives us the following inequalities:

$$\Im m\Big(\Psi\big(z_0\big)\Big) = \alpha^2\,\Im m\big(\lambda^2\big)\begin{cases} \geq 0 & if \quad \Im m\big(\lambda^2\big) \geq 0 \\ \leq 0 & if \quad \Im m\big(\lambda^2\big) \leq 0 \end{cases}$$

and

$$\Re e\Big(i\,\Psi\big(z_0\big)\Big) = \alpha^2\,\Re e\big(\lambda^2\big)\begin{cases} \geq 0 & if \quad \Re e\big(\lambda^2\big) \geq 0 \\ \leq 0 & if \quad \Re e\big(\lambda^2\big) \leq 0 \end{cases},$$

*where* $\alpha \in \mathbb{R} - \{0\}$ and $\lambda \in \mathbb{C} - \mathbb{Z}_0^-$. But, the cases of the results above are contradictions, respectively, with assertions given in (8) and (9). Therefore, the function $p(z)$, defined in (12), immediately yields the the related inequality given in (11). This completes the desired proof of Theorem 2.1.

**Theorem 2.2.** *Let a function* $\kappa \equiv \kappa(z)$ *be a solution of the equation given in* (1) *and also let the function* $\Phi(z)$ *satisfy any one of the cases of the following inequalities:*

$$\Re e\Big(\overline{\lambda}\,\Phi(z)\Big) > 0\,, \tag{17}$$

*or, equivalently,*

$$\Im m\Big(i\,\overline{\lambda}\,\Phi(z)\Big) > 0\,, \tag{18}$$

*where* $z \in \mathbb{U}$ and $\lambda \in \mathbb{C} - \mathbb{Z}_0^-$.

*If the function* $\kappa$ *is an unique solution of the following second-order nonhomogeneous (complex) linear differential equation with (complex) variable coefficients:*

$$\left(z\frac{d^2\kappa}{dz^2} + \big(\gamma - z\big)\frac{d\kappa}{dz}\right)\left(z\frac{d\kappa}{dz} + \kappa\right) \;=\; \Phi(z)\,, \tag{19}$$

then

$$\Re e\Big(\kappa(z)\Big) > 0 \quad \big(z \in \mathbb{U}\big),$$

*where* $z \in \mathbb{U}$ and $\gamma \in \mathbb{C} - \mathbb{Z}_0^-$.

*Prof.* In the light of the hypotheses of Theorem 2.2 and with the help of Lemma 2.1, by taking into account the function $p(z)$ defined as in (12) and then using similar steps used in the proof of Theorem 2.1, the desired proof can be easily obtained. Its details are also omitted.

As certain implications and concluding remarks, this investigation is important from two perspectives. The first relates to the main results that we obtained above, namely, Theorems 2.1 and 2.2, and it can be extended to certain new *or* equivalent results by taking into consideration the relationships indicated in (2) and/or (3). Furthermore, several possible results *or* some special results, for example, by considering certain elementary functions like

$$e^z = \Phi\big(\alpha; \alpha; z\big) \quad , \quad sin(-z) = e^{iz}\Phi\big(1; 2; -2iz\big) \quad etc\,,$$

can be revealed and also each one of them can be exemplified for the main results. They and their possible applications are here left to the researchers.

The second is appertaining to extensive implications of the relationships between confluent hypergeometric functions and certain special functions. More particularly, a number of representations of certain functions in reference to confluent hypergeometric functions have important roles in mathematics, other sciences and also most engineering. In particular, some of these important and comprehensive ones are well known as the functions: The Bessel function, the Laguerre polynomials, the error function, the exponential (integral) function, the logarithmic (integral) function and the Gamma functions, which have the following relationships:

$$J_v(z) = \frac{1}{\Gamma(1+v)}\left(\frac{z}{2}\right)^v e^{-z}\Phi\big(v+1/2; 2v+1; 2z\big)\,, \tag{20}$$

$$L_n^{(\alpha)}(z) = \frac{(\alpha+1)_n}{n!}\Phi\big(-n; \alpha+1; z\big)\,, \tag{21}$$

$$\mathrm{Erf}(z) = z\Phi\big(1/2; 3/2; -z^2\big) = ze^{-z^2}\Phi\big(1; 3/2; z^2\big)\,, \tag{22}$$

$$-\mathrm{Ei}(-z) = e^{-z}\Phi\big(1; 1; z\big)\,, \tag{23}$$

$$\mathrm{li}(-z) = z\Phi\big(1; 1; -lnz\big) \tag{24}$$

and

$$\Gamma(\alpha, z) = e^{-z}\Phi\big(1-\alpha; 1-\alpha; z\big) \ \left(or\ \gamma(\alpha, z) = \frac{z^\alpha}{\alpha}\Phi\big(\alpha; 1+\alpha; -z\big)\right), \tag{25}$$

respectively. For more comprehensive relations, definitions *or* other possible demonstrations regarding to those, see the works given in [1, 4-6, 14-28]. Before we pointed out, the hypergeometric functions are associated with a large number of special functions. Specially, one of their most comprehensive is Bessel functions and their scope is more extensive than the others. Because of this reason, primarily, we want to present some extra information about these functions, which have more applications, and then highlight some of the implications of our main results.

The Bessel functions are the solutions of the following equation:

$$\frac{d^2w}{dz^2} + \frac{1}{z}\frac{dw}{dz} + \Big(1 - \frac{v^2}{z^2}\Big)w = 0\,, \tag{26}$$

which is known as Bessel Differential Equation. The Bessel function (of the first kind) is an analytic function around the point $z = 0$ and is also defined by its Taylor-Maclaurin series expansion:

$$J_v(z) = \Big(\frac{z}{2}\Big)^v \sum_{n=0}^{\infty} \frac{(-1)^n}{2^{2n} n! \Gamma(n+v+1)} z^{2n} \quad \Big(|\arg(z)| < \pi\Big)\,. \tag{27}$$

For non-integer $v$, the Bessel functions $J_v(z)$ annd $J_{-v}(z)$ are linearly independent, so that $J_v(z)$ may serve for a second solution, which has a singular point at $z = 0$. The well known Bessel functions of the second kind, which is denoted as the operator $N_v(z)$ and, occasionally, called as the *Neumann functions,* are related to the notation $J_{-v}(z)$ by

$$N_v(z) = csc(\pi v)\Big(cos(\pi v)J_v(z) - J_{-v}(z)\Big) \quad \Big(|\arg(z)| < \pi\Big)\,. \tag{28}$$

For integer $v = n$, this solution is defined by taking the limit: $\lim_{v\to n} N_v(z)$. For a special case of a purely imaginary argument, the Bessel equation reduces to to the following form:

$$\frac{d^2w}{dz^2} + \frac{1}{z}\frac{dw}{dz} - \Big(1 + \frac{v^2}{z^2}\Big)w = 0\,. \tag{29}$$

The respective solutions, the modified Bessel functions of the first and second kind, are then defined by

$$I_v(z) = i^{-v} J_v(iz) \tag{30}$$

and

$$K_v(z) = \frac{\pi}{2} csc(\pi v)\Big(cos(\pi v)J_v(z) - J_{-v}(z)\Big)\,, \tag{31}$$

respectively. Indeed, the function $K_v(z)$ is known also as the *Macdonald* function. As functions of a real argument at large values of $|z|$,

$$I_v(z) \;\sim\; z^{-1/2} e^{|z|} \quad \text{and} \quad K_v(z) \;\sim\; z^{-1/2} e^{|z|}$$

are exponentially increasing and decreasing functions, respectively.

The following functional series, namely, the confluent hypergeometric series:

$$I_v(z) = \Big(\frac{z}{2}\Big)^v \sum_{n=0}^{\infty} \frac{z^{2n}}{2^{2n} n! \Gamma(n+v+1)} \tag{32}$$

arises in many problems of applied mathematics and, specially, applied physics. For $\Re e\big(v+1/2\big)>0$, it can be also related to the confluent hypergeometric function with the following relationship:

$$\Phi\big(v+1/2;1+2v;2z\big)=\Gamma(v+1)e^{z}\Big(\frac{2}{z}\Big)^{v}I_{v}(z)\,. \tag{33}$$

Since the information content of the Bessel functions is quite wide, for the details of these functions, and various examples, other special functions, equations and also inequalities related to them, see [1, 4-6, 14, 16, 19, 21, 22]. In view of the information stated by (20)-(33), there are too many (novel and/or non-linear) special results in relation with the Bessel functions as we both emphasized in this and first sections. We want to present only one of these section's results.

In consideration of (1) (*or* (26)) and by putting

$$\gamma:=v+\frac{1}{2}\quad,\quad \lambda:=2v+1\quad\text{and}\quad z:=2z$$

in (3), the series representation:

$$\Phi\big(v+1/2;1+2v;2z\big)=\sum_{k=0}^{\infty}\frac{(v+1/2)_k}{(1+2v)_k}\frac{(2z)^k}{k!}$$

is then obtained, *where* $z\in\mathbb{C}$, $1+2v\in\mathbb{C}-\mathbb{Z}_0^-$ and $1/2+v\in\mathbb{C}-\mathbb{Z}_0^-$. In the present case, by setting

$$\kappa\equiv\kappa(z):=J\equiv J_v(z)=\Gamma(1+v)\Big(\frac{2}{z}\Big)^{v}e^{z}J_v(z) \tag{34}$$

in Theorem 2.1, the following result can be then presented as an implication of our main results.

**Proposition 2.1.** *Let a function $J$ be a solution of the (differential) equation given in* (1) (*or* (26)) *and also let the function $\Psi(z)$ satisfy any one of the cases of the following inequalities:*

$$\Re e\Big(\Psi(z)\Big)\begin{cases}<0 & if\quad \Re e\big(v+v^2\big)\geq -1/4\\ >0 & if\quad \Re e\big(v+v^2\big)\leq -1/4\end{cases},$$

*or, equivalently,*

$$\Im m\Big(i\,\Psi(z)\Big)\begin{cases}<0 & if\quad \Im m\big(v+v^2\big)\geq -1/4\\ >0 & if\quad \Im m\big(v+v^2\big)\leq -1/4\end{cases},$$

*where* $z\in\mathbb{U}$ and $1+2v\in\mathbb{C}-\mathbb{Z}_0^-$.

*If the function J is a solution of the following second-order non-homogeneous linear differential equation with (complex) variable coefficients:*

$$\left[ z \frac{d^2J}{dz^2} + \left(v + \frac{1}{2}\right) \frac{dJ}{dz} \right] \left[ z \frac{dJ}{dz} - \left(1 + 2v\right) J \right] = \Psi(z) ,$$

*then*

$$\Re e \Big( J_v(z) \Big) > 0 \ ,$$

*where* $z \in \mathbb{U}$ and $1/2 + v \in \mathbb{C} - \mathbb{Z}_0^-$ and $1 + 2v \in \mathbb{C} - \mathbb{Z}_0^-$.

*Proof.* With the help of (34), by choosing the function $p(z)$, defined by (12), as $J \equiv J_v(z)$ and also letting

$$\gamma := v + 1/2 \quad \left(1/2 + v \in \mathbb{C} - \mathbb{Z}_0^-\right)$$

and

$$\lambda := 1 + 2v \quad \left(1 + 2v \in \mathbb{C} - \mathbb{Z}_0^-\right)$$

in the proof of Theorem 2.1 and then using the same techniques, one can can complete the proof of the related Proposition. Therefore, it is omitted here.

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