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A NOTE ON TRUNCATED DEGENERATE EXPONENTIAL POLYNOMIALS

HYE KYUNG KIM[1], HUNKI BAEK[2], AND DAE SIK LEE[3*]

ABSTRACT. We discuss the properties of the degenerate truncated exponential polynomials and the degenerate Hermite's type truncated polynomials using the degenerate two variable Hermite polynomials. We derive a variety of formulas for these polynomials, including integral expressions, function generation, iterative relationships, and differential equations. Furthermore, as an application of the degenerate truncated exponential polynomials, we consider a partially degenerate flattened beams by referring to the flattened beams used in optics and we show the motion of the lambda value.

1. INTRODUCTION

Truncated polynomials have been shown to play a role of crucial importance in the evaluation of integrals involving products of special functions which possess a lot of importance in various fields of physics, mathematics, applied sciences, engineering and other related research areas, and so on. The truncated forms for special polynomials have been worked and investigated by several mathematicians [3, 4, 5, 14, 17] and see also the references cited therein. Dattoli et al. [3] introduced the higher-order truncated polynomials which plays a role of crucial importance in the evaluation of integrals involving products of special functions. Hassen et al. [7] and Komatsu et al. [14] considered the truncated Bernoulli polynomials and the truncated Euler polynomials respectively. Srivastava et al. [17] examined the truncated-exponential-based Apostol-type polynomials and derived their various properties covering some implicit summation formulas and symmetric identities. Duran et al. [4] considered the two-variable truncated Fubini polynomials and numbers and recently investigated degenerate truncated special polynomials. The properties of degenerate truncated exponential polynomials are relatively little known. Carlitz introduced the degenerate Bernoulli polynomials and the degenerate Euler polynomials [1]. Recently, many mathematicians have been studying various degenerate versions of special polynomials and numbers not only in some arithmetic and combinatorial aspects but also in applications to differential equations, identities of symmetry and probability theory [5, 6, 9-13].

In this paper, we study the properties of the degenerate truncated exponential polynomials and the form of the degenerate Hermite's type truncated polynomials which is constructed using a new type of the degenerate two variable Hermite polynomials. We derive a variety of formulas for these polynomials, including integral expressions, function generation, iterative relationships, and differential equations. Furthermore, as an application of the degenerate truncated exponential polynomials, we define a partially degenerate flattened beams by referring to the flattened beams used in optics [3] and we show the motion of the lambda value.

The truncated exponential polynomials are the first $(n+1)$ terms of the Mac Laurin series for e^x, namely

$$e_n(x)=\sum_{k=0}^{n}\frac{x^k}{k!},\qquad (\text{see } [3, 4, 5, 14, 17]).$$

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* is corresponding author.

The generating function of the truncated exponential polynomials is

$$\sum_{n=0}^{\infty} e_n(x)t^n = \frac{1}{1-t}e^{tx}, \qquad \text{(see [3, 4, 5, 14, 17])}.$$

The generating function of the two variable Hermite polynomials $H_n(x,y)$ is given by

$$e^{xt+yt^2} = \sum_{n=0}^{\infty} H_n(x,y)\frac{t^n}{n!}, \qquad \text{(see [2, 3, 8])}.$$

For $\lambda(\neq 0) \in \mathbb{R}$, the degenerate exponential functions are defined by

(1) $$e_\lambda^x(t) = \sum_{n=0}^{\infty} (x)_{n,\lambda}\frac{t^n}{n!}, \qquad \text{(see [1, 5, 8, 9, 12])},$$

where $(x)_{0,\lambda} = 1,\ (x)_{n,\lambda} = x(x-\lambda)\cdots(x-(n-1)\lambda),\ (n \geq 1)$.
When $x = 1$, $e_\lambda(t) = e_\lambda^1(t) = \sum_{n=0}^{\infty}(i)_{n,\lambda}\frac{t^n}{n!}$.

The gamma function $\Gamma(s)$ belongs to the category of the special transcendental functions given by

(2) $$\Gamma(s) = \int_0^\infty e^{-t}t^{s-1}dt = (s-1)!,\ \ (Re(s) > 0) \qquad \text{(see [12, 15])}.$$

We note that $\Gamma(s) = (s-1)!$ holds only for $s = k$ positive integers.

2. The truncated degenerate exponential polynomials

From (1), the truncated degenerate exponential polynomials is given by

(3) $$e_{n,\lambda}(x) = \sum_{k=0}^{n}(1)_{k,\lambda}\frac{x^k}{k!}, \qquad \text{(see [5])}.$$

When $\lambda \to 0$, $\lim_{\lambda\to 0} e_{n,\lambda}(x) = e_n(x)$ are the truncated exponential polynomials.

We observe that

(4) $$\begin{aligned}\sum_{n=0}^{\infty} e_{n,\lambda}(x)t^n &= \sum_{n=0}^{\infty}\sum_{k=0}^{n}(1)_{k,\lambda}\frac{x^k}{k!}t^n = \sum_{k=0}^{\infty}\left(\sum_{n=k}^{\infty}t^n\right)(1)_{k,\lambda}\frac{x^k}{k!}\\ &= \frac{1}{1-t}\sum_{k=0}^{\infty}(1)_{k,\lambda}\frac{(xt)^k}{k!} = \frac{1}{1-t}e_\lambda(xt).\end{aligned}$$

Therefor, we have the generating function of truncated degenerate exponential polynomials as follows:

(5) $$\sum_{n=0}^{\infty} e_{n,\lambda}(x)t^n = \frac{1}{1-t}e_\lambda(xt).$$

A derangement is a permutation with no fixed points. The number of derangements of an n-element set is called the nth derangement number and denoted by d_n. This number satisfies the following recurrences:

(6) $$d_n = n\cdot d_{n-1} + (-1)^n,\ \ (n \geq 1) \qquad \text{(see [2, 13])}.$$

By (6), we get

$$d_n = n!\sum_{k=0}^{n}\frac{(-1)^k}{k!},\ \ (n\geq 0)\qquad \text{(see [2, 13])}.$$

Kim et al. considered the derangement polynomials by the generating function

$$\frac{1}{1-xt}e^{-t} = \sum_{n=0}^{\infty} d_n(x)\frac{t^n}{n!},\quad (tx\neq 1),\qquad \text{(see [13])}. \tag{7}$$

When $x=1$, $d_n(1)=d_n$, $n\geq 0$.

From (8), H.K. Kim [10] considered the degenerate derangement polynomials by

$$\frac{1}{1-xt}e_\lambda(-t) = \sum_{n=0}^{\infty} d_{n,\lambda}(x)\frac{t^n}{n!},\quad (tx\neq 1). \tag{8}$$

When $x=1$, $d_{n,\lambda}(1) := d_{n,\lambda}$ is called the degenerate derangement numbers and $d_{0,\lambda}=1$.

We note that $\lim_{\lambda\to 0} d_{n,\lambda}(1) = d_n$, $n\geq 0$.

By (8), we get

$$d_{n,\lambda} = n!\sum_{k=0}^{n}\frac{(1)_{k,\lambda}(-1)^k}{k!}.$$

Thus, we get the degenerate derangement numbers as follows:
$d_{1,\lambda}=0,\ d_{2,\lambda}=1-\lambda,\ d_{3,\lambda}=2-2\lambda^2,\ d_{4,\lambda}=9-3\lambda((\lambda-\frac{1}{2})^2+\frac{7}{4}),\ \cdots$.

Remark 1. *From* (5) *when $x=-1$, and* (8) *when $x=1$,, we have*

$$\sum_{n=0}^{\infty} n!e_{n,\lambda}(-1)\frac{t^n}{n!} = e_\lambda(-t)\frac{1}{1-t} = \sum_{n=0}^{\infty} d_{n,\lambda}\frac{t^n}{n!}$$

Therefore, we get $d_{n,\lambda} = n!\ e_{n,\lambda}(-1)$.

Theorem 2. *For $n\geq 1$, we have*

$$\frac{d}{dx}e_{n,\lambda}(x) = \left(1-\lambda x\frac{d}{dx}\right)e_{n-1,\lambda}(x). \tag{9}$$

Furthermore, we get

$$\frac{d}{dx}e_{n,\lambda}(x) = \sum_{k=1}^{n}(-\lambda x)^{k-1}e_{n-k,\lambda}(x). \tag{10}$$

Proof. Differentiating both sides with respect to x, the right side of (5) is

$$\begin{aligned}\frac{d}{dx}e_\lambda(xt)\frac{1}{1-t} &= \frac{t}{1-t}e_\lambda^{1-\lambda}(xt) = te_\lambda^{-\lambda}(xt)\frac{1}{1-t}e_\lambda(xt)\\ &= \frac{1}{1+\lambda xt}\sum_{n=0}^{\infty}e_{n,\lambda}(x)t^{n+1} = \frac{1}{1+\lambda xt}\sum_{n=1}^{\infty}e_{n-1,\lambda}(x)t^n.\end{aligned} \tag{11}$$

On the other hand, the left side of (5) is

$$\frac{d}{dx}\sum_{n=0}^{\infty}e_{n,\lambda}(x)t^n = \sum_{n=1}^{\infty}\left(\frac{d}{dx}e_{n,\lambda}(x)\right)t^n. \tag{12}$$

Thus, from (11) and (12), we get

$$\sum_{n=1}^{\infty}\left(\frac{d}{dx}e_{n,\lambda}(x)\right)t^n = \frac{1}{1+\lambda xt}\sum_{n=1}^{\infty}e_{n-1,\lambda}(x)t^n. \tag{13}$$

Multiplying $(1+\lambda xt)$ of both side of (13),

$$(1+\lambda xt)\sum_{n=1}^{\infty}\left(\frac{d}{dx}e_{n,\lambda}(x)\right)t^n=\sum_{n=1}^{\infty}e_{n-1,\lambda}(x)t^n \tag{14}$$

Thus, we have

$$\sum_{n=1}^{\infty}\left(\frac{d}{dx}e_{n,\lambda}(x)\right)t^n+\lambda x\sum_{n=1}^{\infty}\left(\frac{d}{dx}e_{n,\lambda}(x)\right)t^{n+1}=\sum_{n=1}^{\infty}e_{n-1,\lambda}(x)t^n \tag{15}$$

Since $\frac{d}{dx}\big(e_{0,\lambda}(x)\big)=0$, from (15), we get

$$\sum_{n=1}^{\infty}\left(\frac{d}{dx}e_{n,\lambda}(x)\right)t^n=\sum_{n=1}^{\infty}\left(e_{n-1,\lambda}(x)-\lambda x\left(\frac{d}{dx}e_{n-1,\lambda}(x)\right)\right)t^n. \tag{16}$$

By comparing the coefficients of (16), we have the identity of (9).

In addition, we have

$$\begin{aligned}\frac{d}{dx}e_{n,\lambda}(x)&=e_{n-1,\lambda}(x)-\lambda x\left(e_{n-2,\lambda}(x)-\lambda x\left(\frac{d}{dx}e_{n-2,\lambda}(x)\right)\right)\\&=\cdots\cdots=\sum_{k=1}^{n}e_{n-k,\lambda}(x)(-\lambda x)^{k-1}.\end{aligned}$$

Therefore, we get the identity of (10). □

From Theorem 2, we easily get the second-order differential equation as follows.

Corollary 3. *For $n\geq 1$, we have*

$$\lambda x\frac{d^2}{dx^2}e_{n,\lambda}(x)-\big(\lambda x+1\big)\frac{d}{dx}e_{n,\lambda}(x)+e_{n,\lambda}(x)=0. \tag{17}$$

Proof. From (9), we have

$$\frac{d}{dx}\left((1-\lambda x\frac{d}{dx})e_{n,\lambda}(x)\right)=(1-\lambda x\frac{d}{dx})e_{n,\lambda}(x). \tag{18}$$

Therefore, from (18), we arrive at the desired result. □

Next, we have the recurrence formula of $e_{n,\lambda}(x)$ as follows.

Theorem 4. *For $n\geq 0$, we have*

$$e_{n+1,\lambda}(x)=\begin{cases}\dfrac{(n+x)+(1-\lambda xn)}{n+1}e_{n,\lambda}(x)+\dfrac{(\lambda n-1)x}{n+1}e_{n-1,\lambda}(x), & \text{if } n\geq 1,\\ x+1, & \text{if } n=0.\end{cases}$$

Proof. Differentiating both sides with respect to t, the right side of (5) is

$$\frac{d}{dt}\left(e_\lambda(xt)\frac{1}{1-t}\right)=xe_\lambda^{1-\lambda}(xt)\frac{1}{1-t}+e_\lambda(xt)\frac{1}{(1-t)^2}. \tag{19}$$

On the other hand, the left side of (5) is

$$\frac{d}{dt}\left(\sum_{n=0}^{\infty}e_{n,\lambda}(x)t^n\right)=\sum_{n=1}^{\infty}ne_{n,\lambda}(x)t^{n-1}=\sum_{n=0}^{\infty}(n+1)e_{n+1,\lambda}(x)t^n. \tag{20}$$

Therefore, from (19) and (20), we have

$$\sum_{n=0}^{\infty}(n+1)e_{n+1,\lambda}(x)t^n=\left(\frac{x}{1+\lambda xt}+\frac{1}{1-t}\right)\sum_{n=0}^{\infty}e_{n,\lambda}(x)t^n. \tag{21}$$

Multiplying $(1+\lambda xt)(1-t)$ of both sides of (21), we have

$$\sum_{n=0}^{\infty}(n+1)e_{n+1,\lambda}(x)t^n+(\lambda x-1)\sum_{n=0}^{\infty}(n+1)e_{n+1,\lambda}(x)t^{n+1}-\lambda x\sum_{n=0}^{\infty}(n+1)e_{n+1,\lambda}(x)t^{n+2}$$
$$=(x+1)\sum_{n=0}^{\infty}e_{n,\lambda}(x)t^n+x(\lambda-1)\sum_{n=0}^{\infty}e_{n,\lambda}(x)t^{n+1}. \tag{22}$$

From (22), we get

$$\begin{aligned}\sum_{n=0}^{\infty}(n+1)e_{n+1,\lambda}(x)t^n&=-(\lambda x-1)\sum_{n=1}^{\infty}ne_{n,\lambda}(x)t^n+\lambda x\sum_{n=2}^{\infty}(n-1)e_{n-1,\lambda}(x)t^n\\&\quad+(x+1)\sum_{n=0}^{\infty}e_{n,\lambda}(x)t^n+x(\lambda-1)\sum_{n=1}^{\infty}e_{n-1,\lambda}(x)t^n\\&=\sum_{n=0}^{\infty}(-\lambda xn+n+x+1)e_{n,\lambda}(x)t^n+\sum_{n=1}^{\infty}(\lambda xn-x)e_{n-1,\lambda}(x)t^n.\end{aligned} \tag{23}$$

Therefore by comparing coefficients of both sides of (23), we arrive at the desired result. □

We consider the straightforward extension as follows:
From (1) and (2), we observe that

$$\begin{aligned}e_{n,\lambda}(x)&=\sum_{k=0}^{n}(1)_{k,\lambda}\frac{x^k}{k!}=\sum_{k=0}^{n}(1)_{k,\lambda}\frac{x^k(n-k)!}{k!(n-k)!}\frac{n!}{n!}\\&=\frac{1}{n!}\sum_{k=0}^{n}\binom{n}{k}(1)_{k,\lambda}x^k\,\Gamma(n-k+1)\\&=\frac{1}{n!}\int_0^{\infty}\sum_{k=0}^{n}\binom{n}{k}(1)_{k,\lambda}x^k\varepsilon^{n-k}e^{-\varepsilon}d\varepsilon.\end{aligned} \tag{24}$$

From (24), for $\alpha\in\mathbb{N}$, we consider

$$e^{(\alpha)}_{n,\lambda}(x)=\frac{1}{n!}\sum_{k=0}^{n}\binom{n}{k}(1)_{k,\lambda}x^k\int_0^{\infty}e^{-\varepsilon}\varepsilon^{\alpha}\varepsilon^{n-k}d\varepsilon. \tag{25}$$

When $\alpha=0$, $e^{(0)}_{n,\lambda}=e_{n,\lambda}(x)$.

Theorem 5. *For $n\geq 0$ and $\alpha\in\mathbb{N}$, the generating function of $e^{(\alpha)}_{n,\lambda}(x)$ is*

$$\sum_{n=0}^{\infty}e^{(\alpha)}_{n,\lambda}(x)t^n=\frac{e_\lambda(xt)\alpha!}{(1-t)^{\alpha+1}}.$$

Proof. From (25), we have

$$\begin{aligned}\sum_{n=0}^{\infty}e^{(\alpha)}_{n,\lambda}(x)t^n&=\int_0^{\infty}\left(\sum_{n=0}^{\infty}\sum_{k=0}^{n}\binom{n}{k}(1)_{k,\lambda}x^k\varepsilon^{n-k}\right)\frac{t^n}{n!}e^{-\varepsilon}\varepsilon^{\alpha}d\varepsilon\\&=\int_0^{\infty}e_\lambda(xt)e^{\varepsilon t}e^{-\varepsilon}\varepsilon^{\alpha}d\varepsilon\\&=e_\lambda(xt)\int_0^{\infty}e^{-\varepsilon(1-t)}\varepsilon^{\alpha}d\varepsilon.\end{aligned} \tag{26}$$

Put $y=\varepsilon(1-t)$, $0\le\varepsilon<\infty$, we have $0\le y<\infty$, $\varepsilon=\frac{y}{1-t}$ and $d\varepsilon=\frac{1}{1-t}dy$.

Thus, from (26), we have

$$\begin{aligned}\sum_{n=0}^{\infty}e_{n,\lambda}^{(\alpha)}(x)t^n&=e_\lambda(xt)\int_0^\infty\left(\frac{y}{1-t}\right)^\alpha e^{-y}\frac{y}{1-t}dy\\&=\frac{e_\lambda(xt)}{(1-t)^{\alpha+1}}\int_0^\infty e^{-y}y^\alpha dy=\frac{e_\lambda(xt)}{(1-t)^{\alpha+1}}\,\Gamma(\alpha+1).\end{aligned}\tag{27}$$

Thus, from (27), we arrive at the desired result. □

In this paper, we consider a new type of the degenerate two variable Hermite polynomials defined by

$$e_\lambda(xt)e_\lambda(yt^2)=\sum_{n=0}^{\infty}h_{n,\lambda}(x,y)\frac{t^n}{n!},\qquad(\text{see [8]}).\tag{28}$$

when $\lambda\to0$, $e^{xt+yt^2}=\sum_{n=0}^{\infty}H_n(x,y)\frac{t^n}{n!}$ are the two variable Hermite polynomials.

For $n\ge0$, we observe that

$$\begin{aligned}e_\lambda(xt)e_\lambda(yt^2)&=\left(\sum_{l=0}^{\infty}\frac{(1)_{l,\lambda}}{l!}x^lt^l\right)\left(\sum_{r=0}^{\infty}\frac{(1)_{r,\lambda}}{r!}y^rt^{2r}\right)\\&=\sum_{n=0}^{\infty}n!\left(\sum_{r=0}^{[\frac{n}{2}]}\frac{(1)_{n-2r,\lambda}(1)_{r,\lambda}}{(n-2r)!r!}x^{n-2r}y^r\right)\frac{t^n}{n!}.\end{aligned}\tag{29}$$

Therefore, from (29), we have

$$h_{n,\lambda}(x,y)=n!\sum_{r=0}^{[\frac{n}{2}]}\frac{(1)_{n-2r,\lambda}(1)_{r,\lambda}}{(n-2r)!r!}x^{n-2r}y^r.\tag{30}$$

Now, we define the degenerate Hermite's type truncated polynomials by

$$e_{n,\lambda}^{[2]}(x)=\frac{1}{n!}\int_0^\infty e^{-y}h_{n,\lambda}(x,y)dy.\tag{31}$$

Theorem 6. *For $n\ge0$, we have*

$$e_{n,\lambda}^{[2]}(x)=\sum_{r=0}^{[\frac{n}{2}]}\frac{(1)_{n-2r,\lambda}(1)_{r,\lambda}}{(n-2r)!}x^{n-2r}.\tag{32}$$

Proof. From (30) and (31), we observe that

$$\begin{aligned}e_{n,\lambda}^{[2]}(x)&=\frac{1}{n!}\int_0^\infty e^{-y}h_{n,\lambda}(x,y)dy\\&=\frac{1}{n!}n!\sum_{r=0}^{[\frac{n}{2}]}\frac{(1)_{n-2r,\lambda}(1)_{r,\lambda}}{(n-2r)!r!}x^{n-2r}\int_0^\infty e^{-y}y^rdy\\&=\sum_{r=0}^{[\frac{n}{2}]}\frac{(1)_{n-2r,\lambda}(1)_{r,\lambda}}{(n-2r)!}x^{n-2r}.\end{aligned}\tag{33}$$

Therefore, from (33), we get the desired result. □

Theorem 7. *For $n \geq 2$, we have*

$$\frac{\partial}{\partial x} h_{n,\lambda}(x,y) = n\left(1 - \lambda x \frac{\partial}{\partial x}\right) h_{n-1,\lambda}(x,y). \tag{34}$$

Furthermore, we get

$$\frac{\partial}{\partial x} h_{n,\lambda}(x,y) = \sum_{k=1}^{n-1} (n)_k (-\lambda x)^{k-1} h_{n-k,\lambda}(x,y). \tag{35}$$

Proof. For $n \geq 1$, differentiating both sides with respect to x, the left side of (28) is

$$\begin{aligned} \frac{\partial}{\partial x} e_\lambda(xt) e_\lambda(yt^2) &= t e_\lambda^{1-\lambda}(xt) e_\lambda(yt^2) \\ &= \frac{1}{1+\lambda xt} \sum_{n=0}^{\infty} h_{n,\lambda}(x,y) \frac{t^{n+1}}{n!} \\ &= \frac{1}{1+\lambda xt} \sum_{n=1}^{\infty} n h_{n-1,\lambda}(x,y) \frac{t^n}{n!}. \end{aligned} \tag{36}$$

On the other hand, the right side of (28) is

$$\frac{\partial}{\partial x} \sum_{n=0}^{\infty} h_{n,\lambda}(x,y) \frac{t^n}{n!} = \sum_{n=1}^{\infty} \left(\frac{\partial}{\partial x} h_{n,\lambda}(x,y)\right) \frac{t^n}{n!}. \tag{37}$$

Therefore, from (36) and (37), we have

$$\sum_{n=1}^{\infty} \left(\frac{\partial}{\partial x} h_{n,\lambda}(x,y)\right) \frac{t^n}{n!} = \sum_{n=1}^{\infty} n h_{n-1,\lambda}(x,y) \frac{t^n}{n!} - \lambda x\, t \sum_{n=1}^{\infty} \left(\frac{\partial}{\partial x} h_{n,\lambda}(x,y)\right) \frac{t^n}{n!}. \tag{38}$$

Since $\frac{\partial}{\partial x} H_{0,\lambda}(x,y) = 0$, from (38),

$$\sum_{n=1}^{\infty} \left(\frac{\partial}{\partial x} h_{n,\lambda}(x,y)\right) \frac{t^n}{n!} = \sum_{n=1}^{\infty} n h_{n-1,\lambda}(x,y) \frac{t^n}{n!} - n\lambda x \sum_{n=1}^{\infty} \left(\frac{\partial}{\partial x} h_{n-1,\lambda}(x,y)\right) \frac{t^n}{n!}. \tag{39}$$

Thus, by comparing coefficients of both sides of (39), we have

$$\frac{\partial}{\partial x} h_{n,\lambda}(x,y) = n h_{n-1,\lambda}(x,y) - n\lambda x \left(\frac{\partial}{\partial x} h_{n-1,\lambda}(x,y)\right), \quad (n \geq 1).$$

Furthermore, we get

$$\frac{\partial}{\partial x} h_{n,\lambda}(x,y) = \sum_{k=1}^{n-1} (n)_k (-\lambda x)^{k-1} h_{n-k,\lambda}(x,y).$$

□

From Theorem 7, we get easily the second-order partial differential equation of $h_{n,\lambda}(x,t)$ with respect to x as follows.

Corollary 8. *For $n \geq 1$, we have*

$$\lambda x \frac{\partial^2}{\partial x^2} h_{n,\lambda}(x,y) - \big(\lambda x + 1\big) \frac{\partial}{\partial x} h_{n,\lambda}(x,y) + h_{n,\lambda}(x,y) = 0. \tag{40}$$

Proof. From (34), we get

$$\frac{\partial}{\partial x}\left(n(1 - \lambda x \frac{\partial}{\partial x}) h_{n,\lambda}(x,y)\right) = n(1 - \lambda x \frac{\partial}{\partial x}) h_{n,\lambda}(x,y). \tag{41}$$

Therefore, from (41), we get the desired result. □

Theorem 9. *For $n \geq 1$, we have*

$$\frac{\partial}{\partial y}h_{n,\lambda}(x,y) = \begin{cases} n(n-1)\left(1-\lambda y\frac{\partial}{\partial y}\right)h_{n-2,\lambda}(x,y) & \text{if } n \geq 2 \\ 0 & \text{if } n = 1. \end{cases} \tag{42}$$

Furthermore, we get

$$\frac{\partial}{\partial y}h_{n,\lambda}(x,y) = \sum_{k=1}^{[\frac{n}{2}]}(n)_2 k(-\lambda y)^{k-1}h_{n-2k,\lambda}(x,y). \tag{43}$$

Proof. For $n \geq 1$, differentiating both sides with respect to y, the left side of (28) is

$$\begin{aligned} \frac{\partial}{\partial y}e_\lambda(xt)e_\lambda(yt^2) &= t^2 e_\lambda(xt)e_\lambda^{1-\lambda}(yt^2) \\ &= \frac{1}{1+\lambda yt^2}\sum_{n=0}^{\infty}h_{n,\lambda}(x,y)\frac{t^{n+2}}{n!} \\ &= \frac{1}{1+\lambda yt^2}\sum_{n=2}^{\infty}n(n-1)h_{n-2,\lambda}(x,y)\frac{t^n}{n!}. \end{aligned} \tag{44}$$

On the other hand, the right side of (28) is

$$\frac{\partial}{\partial y}\sum_{n=0}^{\infty}h_{n,\lambda}(x,y)\frac{t^n}{n!} = \sum_{n=1}^{\infty}\left(\frac{\partial}{\partial y}h_{n,\lambda}(x,y)\right)\frac{t^n}{n!}. \tag{45}$$

Therefore, from (44) and (45), we have

$$\sum_{n=1}^{\infty}\left(\frac{\partial}{\partial y}h_{n,\lambda}(x,y)\right)\frac{t^n}{n!} = \sum_{n=2}^{\infty}n(n-1)h_{n-2,\lambda}(x,y)\frac{t^n}{n!} - \lambda y\, t^2\sum_{n=1}^{\infty}\left(\frac{\partial}{\partial y}h_{n,\lambda}(x,y)\right)\frac{t^n}{n!}. \tag{46}$$

Since $\frac{\partial}{\partial y}H_{0,\lambda}(x,y) = 0$, from (46),

$$\sum_{n=1}^{\infty}\left(\frac{\partial}{\partial y}h_{n,\lambda}(x,y)\right)\frac{t^n}{n!} = \sum_{n=2}^{\infty}n(n-1)h_{n-2,\lambda}(x,y)\frac{t^n}{n!} - \lambda y\sum_{n=2}^{\infty}\left(n(n-1)\frac{\partial}{\partial y}h_{n-2,\lambda}(x,y)\right)\frac{t^n}{n!}. \tag{47}$$

Thus, by comparing coefficients of both sides of (47), we have the desired result. □

From Theorem 9, we easily get the second-order partial differential equation of $h_{n,\lambda}(x,y)$ with respect to y as follows.

Corollary 10. *For $n \geq 1$, we have*

$$\lambda y\frac{\partial^2}{\partial y^2}h_{n,\lambda}(x,y) - (\lambda y+1)\frac{\partial}{\partial y}h_{n,\lambda}(x,y) + h_{n,\lambda}(x,y) = 0. \tag{48}$$

Proof. From (42), we get

$$\frac{\partial}{\partial y}\left((n+2)(n+1)\left(1-\lambda y\frac{\partial}{\partial y}\right)h_{n,\lambda}(x,y)\right) = (n+2)(n+1)\left(1-\lambda y\frac{\partial}{\partial y}\right)h_{n,\lambda}(x,y). \tag{49}$$

Therefore, from (49), we get the desired result. □

Furthermore, we have the recurrence formula of $h_{n,\lambda}(x,y)$ as follows.

Theorem 11. *For $n \geq 0$, we have*

$$h_{n+1,\lambda}(x,y) = x(1-\lambda n)h_{n,\lambda}(x,y) + yn(2-\lambda n+\lambda)h_{n-1,\lambda}(x,y) + \lambda xyn(n-1)(3-\lambda n+2\lambda)h_{n-2,\lambda}(x,y). \tag{50}$$

Proof. For $n \geq 1$, differentiating both sides with respect to t, the left side of (28) is

$$\frac{d}{dt}e_\lambda(xt)e_\lambda(yt^2) = \frac{x}{1+\lambda xt}e_\lambda(xt)e_\lambda(yt^2) + e_\lambda(xt)\frac{2yt}{1+\lambda yt^2}e_\lambda(yt^2). \tag{51}$$

On the other hand, the right side of (28) is

$$\frac{d}{dt}\sum_{n=0}^{\infty} h_{n,\lambda}(x,y)\frac{t^n}{n!} = \sum_{n=1}^{\infty} h_n(x,y)\frac{t^{n-1}}{(n-1)!} = \sum_{n=0}^{\infty} h_{n+1}(x,y)\frac{t^n}{n!}. \tag{52}$$

Thus, from (51) and (52), we have

$$(1+\lambda xt)(1+\lambda yt^2)\sum_{n=0}^{\infty} h_{n+1,\lambda}(x,y)\frac{t^n}{n!} = \Big(x(1+\lambda yt^2)+2yt(1+\lambda xt)\Big)\sum_{n=0}^{\infty} h_{n,\lambda}(x,y)\frac{t^n}{n!}. \tag{53}$$

From (53), we observe

$$\begin{aligned}\sum_{n=0}^{\infty} h_{n+1,\lambda}(x,y)\frac{t^n}{n!} &+ \lambda x\sum_{n=1}^{\infty} nh_{n,\lambda}(x,y)\frac{t^n}{n!} + \lambda y\sum_{n=2}^{\infty}(n)_2 h_{n-1,\lambda}(x,y)\frac{t^n}{n!} \\ &+ \lambda^2 xy\sum_{n=3}^{\infty}(n)_3 h_{n-2,\lambda}(x,y)\frac{t^n}{n!} \\ = x\sum_{n=0}^{\infty} h_{n,\lambda}(x,y)\frac{t^n}{n!} &+ 2y\sum_{n=1}^{\infty} nh_{n-1,\lambda}(x,y)\frac{t^n}{n!} + 3\lambda xy\sum_{n=2}^{\infty}(n)_2 h_{n-2,\lambda}(x,y)\frac{t^n}{n!}.\end{aligned} \tag{54}$$

Since $(n)_k = n(n-1)\cdots n-((k-1))$, from (54), we get

$$\begin{aligned}\sum_{n=0}^{\infty} h_{n+1,\lambda}(x,y)\frac{t^n}{n!} &+ \lambda x\sum_{n=0}^{\infty} nh_{n,\lambda}(x,y)\frac{t^n}{n!} + \lambda y\sum_{n=0}^{\infty}(n)_2 h_{n-1,\lambda}(x,y)\frac{t^n}{n!} \\ &+ \lambda^2 xy\sum_{n=0}^{\infty}(n)_3 h_{n-2,\lambda}(x,y)\frac{t^n}{n!} \\ = x\sum_{n=0}^{\infty} h_{n,\lambda}(x,y)\frac{t^n}{n!} &+ 2y\sum_{n=0}^{\infty} nh_{n-1,\lambda}(x,y)\frac{t^n}{n!} + 3\lambda xy\sum_{n=0}^{\infty}(n)_2 h_{n-2,\lambda}(x,y)\frac{t^n}{n!}.\end{aligned} \tag{55}$$

Therefore, from (55), we have what we want.

□

Naturally, we consider a new type of the degenerate higher order two variable Hermite polynomials given by

$$e_\lambda(xt)e_\lambda(yt^m) = \sum_{n=0}^{\infty} h_{n\lambda}(x,y|m)\frac{t^n}{n!}. \tag{56}$$

Put $mr+l=n,\ mr \leq n$, we observe that

$$\begin{aligned}e_\lambda(xt)e_\lambda(yt^m) &= \left(\sum_{l=0}^{\infty}\frac{(1)_{l,\lambda}}{l!}x^l t^l\right)\left(\sum_{r=0}^{\infty}\frac{(1)_{r,\lambda}}{r!}y^r t^{mr}\right) \\ &= \sum_{n=0}^{\infty} n!\left(\sum_{r=0}^{[\frac{n}{m}]}\frac{(1)_{n-mr,\lambda}(1)_{r,\lambda}}{(n-mr)!r!}x^{n-mr}y^r\right)\frac{t^n}{n!}.\end{aligned}$$

Therefore, we have

(57) $$h_{n\lambda}(x,y|m)=n!\sum_{r=0}^{[\frac{n}{m}]}\frac{(1)_{n-mr,\lambda}(1)_{r,\lambda}}{(n-mr)!r!}x^{n-mr}y^{r},\ \ (n\geq 0).$$

We consider the degenerate higher order Hermite's type truncated polynomials by

(58) $$e_{n,\lambda}^{[m]}(x)=\frac{1}{n!}\int_{0}^{\infty}e^{-y}h_{n,\lambda}(x,y|m)dy.$$

Theorem 12. *For $n\geq 0$, we have*

$$e_{n,\lambda}^{[m]}(x)=\sum_{r=0}^{[\frac{n}{m}]}\frac{(1)_{n-mr,\lambda}(1)_{r,\lambda}}{(n-mr)!}x^{n-mr}\ \ (n\geq 0).$$

Proof. From (57) and (58), we observe that

(59) $$\begin{aligned}e_{n,\lambda}^{[m]}(x)&=\frac{1}{n!}\int_{0}^{\infty}e^{-y}h_{n,\lambda}^{(m-1)}(x,y|m)dy\\&=\sum_{r=0}^{[\frac{n}{m}]}\frac{(1)_{n-mr,\lambda}(1)_{r,\lambda}}{(n-mr)!r!}x^{n-mr}\Gamma(r+1)=\sum_{r=0}^{[\frac{n}{m}]}\frac{(1)_{n-mr,\lambda}(1)_{r,\lambda}}{(n-mr)!}x^{n-mr}.\end{aligned}$$

Therefore, from (59), we get the desired result. □

Theorem 13. *For $n\geq 1$, we have*

(60) $$\frac{\partial}{\partial x}h_{n,\lambda}(x,y|m)=n\left(1-\lambda x\frac{\partial}{\partial x}\right)h_{n-1,\lambda}(x,y|m).$$

Furthermore, we get

(61) $$\frac{\partial}{\partial x}h_{n,\lambda}(x,y|m)=\sum_{k=1}^{n-1}(n)_{k}(-\lambda x)^{k-1}h_{n-k,\lambda}(x,y|m),\ (n\geq 1).$$

Proof. Differentiating both sides with respect to x, the left side of (56) is

(62) $$\begin{aligned}\frac{\partial}{\partial x}e_{\lambda}(xt)e_{\lambda}(yt^{m})&=te_{\lambda}^{1-\lambda}(xt)e_{\lambda}(yt^{m})\\&=\frac{1}{1+\lambda xt}\sum_{n=0}^{\infty}h_{n,\lambda}(x,y|m)\frac{t^{n+1}}{n!}\\&=\frac{1}{1+\lambda xt}\sum_{n=1}^{\infty}nh_{n-1,\lambda}(x,y|m)\frac{t^{n}}{n!}.\end{aligned}$$

On the other hand, the right side of (56) is

(63) $$\frac{\partial}{\partial x}\sum_{n=0}^{\infty}h_{n,\lambda}(x,y|m)\frac{t^{n}}{n!}=\sum_{n=1}^{\infty}\left(\frac{\partial}{\partial x}h_{n,\lambda}(x,y|m)\right)\frac{t^{n}}{n!}.$$

From (62) and (63), we have

(64) $$\sum_{n=1}^{\infty}\left(\frac{\partial}{\partial x}h_{n,\lambda}(x,y|m)\right)\frac{t^{n}}{n!}=\sum_{n=1}^{\infty}nh_{n-1,\lambda}(x,y|m)\frac{t^{n}}{n!}-\lambda x\sum_{n=1}^{\infty}n\left(\frac{\partial}{\partial x}h_{n-1,\lambda}(x,y|m)\right)\frac{t^{n}}{n!}$$

Thus, by comparing coefficients of both sides of (64), we get the desired result.

□

From Theorem 13, we get easily the second-order partial differential equations of $H_{n,\lambda}(x,y|m)$, as follows.

Corollary 14. *For $n \geq 1$, we have*

$$\lambda x \frac{\partial^2}{\partial x^2} h_{n,\lambda}(x,y|m) - \big(\lambda x + 1\big) \frac{\partial}{\partial x} h_{n,\lambda}(x,y|m) + h_{n,\lambda}(x,y|m) = 0. \tag{65}$$

Proof. From (60), we get

$$\frac{\partial}{\partial x}\left((n+1)\left(1-\lambda x \frac{\partial}{\partial x}\right) h_{n,\lambda}(x,y|m)\right) = (n+1)\left(1-\lambda x \frac{\partial}{\partial x}\right) h_{n,\lambda}(x,y|m). \tag{66}$$

Therefore, from (66), we get the desired result. □

Theorem 15. *For $n \geq m$, we have*

$$\frac{\partial}{\partial y} h_{n,\lambda}(x,y|m) = (n)_m \left(1 - \lambda y \frac{\partial}{\partial y}\right) h_{n-m,\lambda}(x,y|m). \tag{67}$$

Proof. Differentiating both sides with respect to y, the left side of (56) is

$$\begin{aligned} \frac{\partial}{\partial y} e_\lambda(xt) e_\lambda(yt^m) &= t^m e_\lambda^{1-\lambda}(yt^m) e_\lambda(xt) \\ &= \frac{1}{1+\lambda y t^m} \sum_{n=0}^{\infty} h_{n,\lambda}(x,y|m) \frac{t^{n+m}}{n!} \\ &= \frac{1}{1+\lambda y t^m} \sum_{n=m}^{\infty} (n)_m h_{n-m,\lambda}(x,y|m) \frac{t^n}{n!}. \end{aligned} \tag{68}$$

On the other hand, the right side of (56) is

$$\frac{\partial}{\partial y} \sum_{n=0}^{\infty} h_{n,\lambda}(x,y|m) \frac{t^n}{n!} = \sum_{n=1}^{\infty} \left(\frac{\partial}{\partial y} h_{n,\lambda}(x,y|m)\right) \frac{t^n}{n!}. \tag{69}$$

From (68) and (69), we have

$$\begin{aligned} \sum_{n=1}^{\infty} \left(\frac{\partial}{\partial y} h_{n,\lambda}(x,y|m)\right) \frac{t^n}{n!} &= \sum_{n=m}^{\infty} (n)_m h_{n-m,\lambda}(x,y|m) \frac{t^n}{n!} - \lambda y t^m \sum_{n=1}^{\infty} n \left(\frac{\partial}{\partial y} h_{n-m,\lambda}(x,y|m)\right) \frac{t^n}{n!} \\ &= \sum_{n=m}^{\infty} (n)_m h_{n-m,\lambda}(x,y|m) \frac{t^n}{n!} - \lambda y \sum_{n=m}^{\infty} (n)_m \left(\frac{\partial}{\partial y} h_{n-m,\lambda}(x,y|m)\right) \frac{t^n}{n!}. \end{aligned} \tag{70}$$

Thus, by comparing coefficients of both sides of (70), we get the desired result. □

From Theorem 15, we get easily the second-order partial differential equations of $h_{n,\lambda}(x,y|m)$, as follows.

Corollary 16. *For $n \geq m$, we have*

$$\lambda y \frac{\partial^2}{\partial y^2} h_{n,\lambda}(x,y|m) - \big(\lambda y + 1\big) \frac{\partial}{\partial y} h_{n,\lambda}(x,y|m) + h_{n,\lambda}(x,y|m) = 0. \tag{71}$$

Proof. From (67), we get

$$\frac{\partial}{\partial y}\left((n+m)_n \left(1-\lambda y \frac{\partial}{\partial y}\right) h_{n,\lambda}(x,y|m)\right) = (n+m)_n \left(1-\lambda y \frac{\partial}{\partial y}\right) h_{n,\lambda}(x,y|m). \tag{72}$$

Therefore, from (72), we get the desired result. □

3. Further remark

As an application of the truncated exponential polynomials, Dattoli and Migliorati (see [3]) introduced the flattened beams given by

$$F(x,m) = e_m(x^2)exp(-x^2), \tag{73}$$

which provide a good tool of approximation of a super-Gaussian, and their paraxial evolution can be treated using straightforward analytical tools. The flattened beams have been introduced to study optical systems employing the so-called super-Gaussian beams, namely, optical beams whose transverse shape is not reproduced by a simple Gaussian, but by a function exhibiting a quasi constant flat top. Unlike the ordinary Gaussian beams, the super-Gaussians do not have transparent propagation properties, which can easily be exploited in the design of an optical resonator (see [3]).

From (73), we consider a partially degenerate beam that takes into account noise or various ambient conditions as follows and show the motion of the lambda value.

$$F_\lambda(x,m) = e_{m,\lambda}(x^2)exp(-x^2). \tag{74}$$

Thus, we get

$$e_{m,\lambda}(x^2) = exp(x^2)F_\lambda(x,m). \tag{75}$$

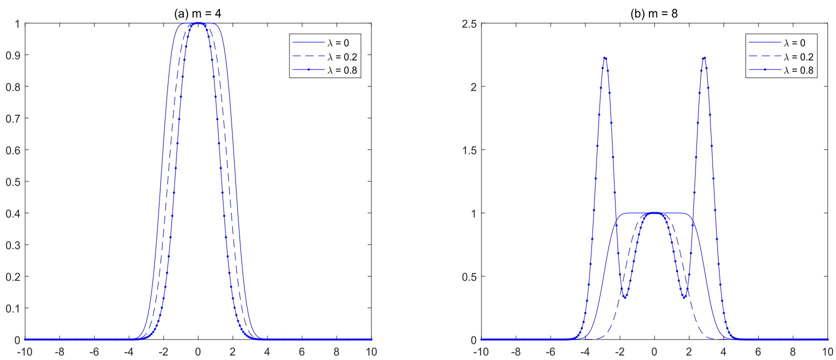

FIGURE 1. Partially degenerate flattened beam when $m = 4$ and $m = 8$, respectively.

In order to plot Figures 1, MATLAB R2017b software is used.

In addition, we can consider a differential operator from Corollary 3 that

$$\begin{aligned} &\widetilde{D}_m[exp(x^2)F_\lambda(x,m)] = 0, \\ &\widetilde{D}_m = \lambda w\frac{d^2}{dw^2} - \left(\lambda w + 1\right)\frac{d}{dw} + 1, \\ &w = x^2. \end{aligned} \tag{76}$$

We observe that

$$\frac{d}{dw}\left(e^wF_\lambda(x,m)\right) = e^wF_\lambda(x,m) + e^w\frac{d}{dx}F_\lambda(x,m)\frac{dx}{dw} = e^wF_\lambda(x,m) + e^w\frac{1}{2x}\frac{d}{dx}F_\lambda(x,m) \tag{77}$$

and

$$\frac{d^2}{dw^2}\left(e^u F_\lambda(x,m)\right) = \frac{d}{dw}\left(e^w F_\lambda(x,m)\right) + \frac{d}{dw}\left(e^w \frac{1}{2x}\frac{d}{dx}F_\lambda(x,m)\right) \tag{78}$$
$$= e^w F_\lambda(x,m) + e^w\left(\frac{1}{x} - \frac{1}{4x^3}\right)F'_\lambda(x,m) + e^w \frac{1}{4x^2}F''_\lambda(x,m).$$

From (77) and (78), we have

$$\widehat{D}_m[e^{x^2}F_\lambda(x,m)]$$
$$= \lambda w\left(e^w \frac{d^2}{dw^2}F_\lambda(x,m)\right) - (\lambda w+1)\left(e^w \frac{d}{dw}F_\lambda(x,m)\right) + e^w F_\lambda(x,m) \tag{79}$$
$$= \lambda x^2 e^{x^2}\left[\frac{1}{4x^2}F''(x,m) + \frac{4x^2-1}{4x^3}\frac{d}{dw}F'_\lambda(x,m) + F_\lambda(x,m)\right]$$
$$-(\lambda x^2+1)e^{x^2}\left[\frac{1}{2x}F'_\lambda(x,m) + F_\lambda(x,m)\right] + e^{x^2}F_\lambda(x,m) = 0$$

Thus, from (79), we get the second-order differential equations of $F_\lambda(x,m)$, as follows.

$$\frac{\lambda}{4}F''_\lambda(x,m) + \left(\frac{\lambda(3x^2-1)-1}{4x}\right)F'_\lambda(x,m) = 0. \tag{80}$$

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DEPARTMENT OF MATHEMATICS EDUCATION, DAEGU CATHOLIC UNIVERSITY, GYEONGSAN 38430, REPUBLIC OF KOREA

E-mail address: hkkim@cu.ac.kr

DEPARTMENT OF MATHEMATICS EDUCATION, DAEGU CATHOLIC UNIVERSITY, GYEONGSAN 38430, REPUBLIC OF KOREA

E-mail address: hkbaek@cu.ac.kr

SCHOOL OF ELECTRONIC AND ELECTRIC ENGINEERING, DAEGU UNIVERSITY, GYEONGSAN 38453, REPUBLIC OF KOREA

E-mail address: dslee@daegu.ac.kr