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SOME IDENTITIES OF DEGENERATE r-EXTENDED LAH-BELL POLYNOMIALS

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ABSTRACT. In recent years, many mathematicians have studied special numbers and polynomials in a variety of ways, using several means, including generating functions, combinatorial methods, umbral calculus, p-adic analysis, differential equations, and probabilities and so on. Umbral calculus was based on modern concepts such as linear functionals, linear operators, adjoints, and so on. Recently, Kim-Kim (J. Math. Anal. Appl. (2020), 23, 124521) introduced the degenerate Sheffer sequence and the λ-Sheffer sequence. In addition, many mathematicians have studied special polynomials and various degenerate versions of numbers in some arithmetic and combinatorial aspects, as well as applications to differential equations, identity of symmetry, and probability theory. For these point of view, this paper is divided in two parts. In the first part, we introduce the degenerate r-extended Lah-Bell polynomials and and derive several combinatorial identities related to those polynomials and numbers without using degenerate Sheffer sequences. In the second part, we also derive several combinatorial identities related to the degenerate r-extended Lah-Bell polynomials by using degenerate Sheffer sequences. Some of them include other special polynomials and numbers such as Lah numbers, r-extended Lah numbers, the degenerate Lah-Bell polynomials, the degenerate Bernoulli polynomials and numbers, the degenerate Euler polynomials and numbers, the degenerate Daehee polynomials and numbers, the degenerate Bell polynomials, the degenerate Stirling numbers of the first and second kind, and etc.

1. INTRODUCTION

Many mathematicians have been studying various degenerate versions of special polynomials and numbers not only in some arithmetic and combinatorial aspects but also in applications to differential equations, identities of symmetry and probability theory [11-21]. These degenerate versions began when Carlitz introduced the degenerate Bernoulli polynomials and the degenerate Euler polynomials [2]. Moreover, umbral calculus, established by Rota in the 1970s, was based on modern concepts such as linear functions, linear operators, and adjoints. Umbral calculus is one of the important methods for obtaining the symmetric identities for the (degenerate) version of special numbers and polynomials [4,5, 22-24, 27, 28]. In addition, Kim-Kim [12] introduced the λ-Sheffer sequence and the degenerate Sheffer sequence. They defined the λ- linear functional and λ- differential operator respectively, instead of the linear functional and the differential operator used by Rota. Furthermore, Kim et al. introduced the Lah-Bell polynomials [7] and the r-extended Lah-Bell polynomials [8].

For these point of view, we We focus on the degenerate version of r-extended Lah-Bell polynomials [8]. parts. In the first part, we introduce the degenerate r-extended Lah-Bell polynomials and derive several combinatorial identities related to those polynomials and numbers. In the second part, we also derive various combinatorial identities related to the degenerate r-extended Lah-Bell polynomials by using degenerate Sheffer sequences. Some of them include other special polynomials and numbers such as Lah numbers, r-extended Lah numbers, the degenerate Lah-Bell polynomials,

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the degenerate Bernoulli polynomials and numbers, the degenerate Euler polynomials and numbers, the degenerate Daehee polynomials and numbers, the degenerate Bell polynomials, the degenerate Stirling numbers of the first and second kind and etc.

Now, we give some definitions and properties needed in this paper.
Throughout this paper, we consider that $n,\ k,\ r$ be nonnegative integers, with $n \geq k$. The unsigned Lah number $L(n,k)$ counts the number of ways of all distributions of n balls, labelled $1,\cdots,n$, among k unlabelled, contents-ordered boxes, with no box left empty. An explicit formula and the generating function of $L(n,k)$ respectively are given by

$$(1) \qquad L(n,k) = \binom{n-1}{k-1}\frac{n!}{k!},\ (k \geq 0), \qquad \text{(see [7, 8, 24, 29])},$$

and

$$(2) \qquad \frac{1}{k!}\left(\frac{t}{1-t}\right)^k = \sum_{n=k}^{\infty} L(n,k)\frac{t^n}{n!},\ (k \geq 0), \qquad \text{(see [7, 8, 24, 29])}.$$

Furthermore, the r-Lah number $L_r(n,k)$ counts the number of partitions of a set with $n+r$ elements into $k+r$ ordered blocks such that r distinguished elements have to be in distinct ordered blocks and an explicit formula of $L_r(n,k)$ (see [8, 25, 26]) given by

$$(3) \qquad L_r(n,k) = \binom{n+2r-1}{k+2r-1}\frac{n!}{k!}\ (k \geq 0), \qquad \text{(see [8, 25, 26])}.$$

From (3), we have the generating function of $L_r(n,k)$ given by

$$(4) \qquad \frac{1}{k!}\left(\frac{1}{1-t}\right)^{2r}\left(\frac{t}{1-t}\right)^k = \sum_{n=k}^{\infty} L_r(n,k)\frac{t^n}{n!}, (k \geq 0), \qquad \text{(see [8, 25, 26])}.$$

Recently, Kim-Kim introduced the Lah-Bell polynomials and the r-extended Lah-Bell polynomials respectively as follows:

$$(5) \qquad e^{x\left(\frac{1}{1-t}-1\right)} = \sum_{n=0}^{\infty} \mathbf{B}_n^L(x)\frac{t^n}{n!}, \quad \text{(see [7])},$$

and

$$(6) \qquad \left(\frac{1}{1-t}\right)^{2r} e^{x\left(\frac{1}{1-t}-1\right)} = \sum_{n=k}^{\infty} \mathbf{B}_{r,n}^L(x)\frac{t^n}{n!},\ (k \geq 0), \qquad \text{(see [8])}.$$

When $x=1$, $\mathbf{B}_n^L = \mathbf{B}_n^L(1)$ and $\mathbf{B}_{r,n}^L = \mathbf{B}_{r,n}^L(1)$ are called the Lah-Bell numbers and r-extended Lah-Bell numbers respectively.

Recently, Kim [7] studied the degenerate Lah-Bell polynomials defined by

$$(7) \qquad e_\lambda^x\left(\frac{t}{1-t}\right) = \sum_{n=0}^{\infty} \mathbf{B}_{n,\lambda}^L(x)\frac{t^n}{n!} \quad \text{(see [7, 12])}.$$

When $x=1$, $\mathbf{B}_{n,\lambda}^L := \mathbf{B}_{n,\lambda}^L(1)$ is called the n-th degenerate Lah-Bell number.
When $\lambda \to 0$, $\lim_{\lambda\to 0}\mathbf{B}_{n,\lambda}^L = \mathbf{B}_n^L$ is the n-th Lah-Bell number.

For any nonzero $\lambda \in \mathbb{R}$, the degenerate exponential function is defined by

$$e_\lambda^x(t) = (1+\lambda t)^{\frac{x}{\lambda}}, \quad e_\lambda(t) = (1+\lambda t)^{\frac{1}{\lambda}}, \qquad \text{(see [2, 10-17])}.$$

By Taylor expansion, we get

$$(8) \qquad e_\lambda^x(t) = \sum_{n=0}^{\infty} (x)_{n,\lambda}\frac{t^n}{n!}, \qquad \text{(see [10-17])},$$

where $(x)_{0,\lambda}=1,\ (x)_{n,\lambda}=x(x-\lambda)(x-2\lambda)\cdots(x-(n-1)\lambda),\ (n\geq 1)$.

We note that

$$(1-t)^{-m}=\sum_{l=0}^{\infty}\binom{-m}{l}(-1)^l t^l=\sum_{l=0}^{\infty}<m>_l\frac{t^l}{l!},\quad \text{(see [3])}. \tag{9}$$

where $<x>_0=1,\ <x>_n=x(x+1)(x+2)\cdots(x+n-1),\ (n\geq 1)$.

The degenerate Bernoulli polynomials and degenerate Euler polynomials respectively are given by the generating function to be

$$\frac{t}{e_\lambda(t)-1}e_\lambda^x(t)=\sum_{n=0}^{\infty}\beta_{n,\lambda}(x)\frac{t^n}{n!},\quad \text{(see [2,10-15])}, \tag{10}$$

and

$$\frac{2}{e_\lambda(t)+1}e_\lambda^x(t)=\sum_{n=0}^{\infty}E_{n,\lambda}(x)\frac{t^n}{n!},\quad \text{(see [2, 11, 12, 22, 29])}. \tag{11}$$

We note that $\beta_{n,\lambda}=\beta_{n,\lambda}(0)$ and $E_{n,\lambda}=E_{n,\lambda}(0)$ $(n\geq 0)$, are called the degenerate Bernoulli numbers and degenerate Euler numbers respectively.

The degenerate Daehee polynomials are defined by as follows

$$\frac{\log_\lambda(1+t)}{t}(1+t)^x=\sum_{n=0}^{\infty}D_{n,\lambda}(x)\frac{t^n}{n!},\quad \text{(see [17])}, \tag{12}$$

where $\log_\lambda(t)=\frac{1}{\lambda}(t^\lambda-1)$. When $x=0$, $D_{n,\lambda}=D_{n,\lambda}(0)$ $(n\geq 0)$ are called the degenerate Daehee numbers.

The Bell polynomials are defined by the generating function

$$e^{x(e^t-1)}=\sum_{n=0}^{\infty}bel_n(x)\frac{t^n}{n!},\quad \text{(see [9, 18])}.$$

The generating function of the degenerate Bell polynomials is given by

$$e_\lambda^x(e_\lambda(t)-1)=\sum_{l=0}^{\infty}Bel_{l,\lambda}(x)\frac{t^l}{l!},\quad \text{(see [18])}. \tag{13}$$

The degenerate Stirling numbers of the first and second kind respectively are given by

(14)

$$(x)_n=\sum_{l=0}^{n}S_{1,\lambda}(n,l)(x)_{l,\lambda}\ \text{ and }\ \frac{1}{k!}\big(\log_\lambda(1+t)\big)^k=\sum_{n=k}^{\infty}S_{1,\lambda}(n,k)\frac{t^n}{n!}\quad (k\geq 0),\quad \text{(see [14, 16])},$$

and

$$(x)_{n,\lambda}=\sum_{l=0}^{n}S_{2,\lambda}(n,l)(x)_l\ \text{ and }\ \frac{1}{k!}\big(e_\lambda(t)-1\big)^k=\sum_{n=k}^{\infty}S_{2,\lambda}(n,k)\frac{t^n}{n!}\quad (k\geq 0),\quad \text{(see [14, 16])}. \tag{15}$$

Let $\mathbb{C}$ be the field of complex numbers and let $\mathscr{F}$ be the set of all formal power series in the variable t over $\mathbb{C}$ with

$$\mathscr{F}=\left\{f(t)=\sum_{k=0}^{\infty}a_k\frac{t^k}{k!}\ \middle|\ a_k\in\mathbb{C}\right\}.$$

Let $\mathbb{P}=\mathbb{C}[x]$ and $\mathbb{P}^*$ be the vector space all linear functional on $\mathbb{P}$.

$$\mathbb{P}_n=\{\ P(x)\in\mathbb{C}[x]\mid degP(x)\leq n\},\ \ (n\geq 0).$$

Then $\mathbb{P}_n$ is an $(n+1)$-dimensional vector space over $\mathbb{C}$.

Recently, Kim-Kim [11] considered λ-linear functionals and λ-differential operators as follows:

For $f(t)=\sum_{k=0}^{\infty} a_k \frac{t^k}{k!} \in \mathscr{F}$ and a fixed nonzero real number λ, each λ gives rise to the linear functional $\langle f(t) \mid \cdot \rangle_\lambda$ on $\mathbb{P}$, called λ-linear functional given by $f(t)$, which is defined by

$$\langle f(t) \mid (x)_{n,\lambda} \rangle_\lambda = a_n, \quad \text{for all } n \geq 0 \quad \text{(see [11])}, \tag{16}$$

and by (16), $\langle t^k \mid (x)_{n,\lambda} \rangle_\lambda = n!\delta_{n,k}, \quad (n,\ k \geq 0)$, where $\delta_{n,k}$ is the Kronecker's symbol.

For each $\lambda \in \mathbb{R}$, and each nonnegative integer k, they also defined the λ-differential operator on $\mathbb{P}$ by

$$(t^k)_\lambda (x)_{n,\lambda} = \begin{cases} (n)_k (x)_{n-k,\lambda}, & \text{if } k \leq n, \\ 0 & \text{if } k \geq n, \quad \text{(see [11])}. \end{cases} \tag{17}$$

Extending this linearly, any power series $f(t)=\sum_{k=0}^{\infty} a_k \frac{t^k}{k!} \in \mathscr{F}$ yields the differential operator on $\mathbb{P}$, called λ-differential operator given by $f(t)$, which is defined by

$$(f(t))_\lambda (x)_{n,\lambda} = \sum_{k=0}^{n} \binom{n}{k} a_k (x)_{n-k,\lambda}, \tag{18}$$

and by linear extension.

Note that we should note that different λ's give rise to different linear functionals on $\mathbb{P}$ (see [11]). We also observe that, for $\lambda=0$, the linear functional $\langle f(t) \mid \cdot \rangle$ agrees with the one in $\langle f(t) \mid x^n \rangle = a_k$, $(k \geq 0)$.

The order $o(f(t))$ of a power series $f(t)(\neq 0)$ is the smallest integer k for which the coefficient of t^k does not vanish. The series $f(t)$ is called invertible if $o(f(t))=0$ and such series has a multiplicative inverse $1/f(t)$ of $f(t)$. $f(t)$ is called a delta series if $o(f(t))=1$ and it has a compositional inverse $\overline{f}(t)$ of $f(t)$ with $\overline{f}(f(t)) = f(\overline{f}(t)) = t$.

Let $f(t)$ and $g(t)$ be a delta series and an invertible series, respectively. Then there exists a unique sequences $s_{n,\lambda}(x)$ such that the orthogonality conditions

$$\langle g(t)\big(f(t)\big)^k \mid s_{n,\lambda}(x) \rangle_\lambda = n!\delta_{n,k}, \quad (n,k \geq 0) \quad \text{(see [11])}. \tag{19}$$

The sequences $s_{n,\lambda}(x)$ are called the λ-Sheffer sequence for $(g(t), f(t))$, which are denoted by $s_{n,\lambda}(x) \sim (g(t), f(t))_\lambda$.

The sequence $s_{n,\lambda}(x) \sim (g(t), f(t))_\lambda$ if and only if

$$\frac{1}{g\big(\overline{f}(t)\big)} e_\lambda^x\big(\overline{f}(t)\big) = \sum_{k=0}^{\infty} \frac{s_{k,\lambda}(x)}{k!} t^k \quad (n,k \geq 0) \quad \text{(see [11])}, \tag{20}$$

for all $y \in \mathbb{C}$.

Assume that for each $\lambda \in \mathbb{R}^*$ of the set of nonzero real numbers, $s_{n,\lambda}(x)$ is λ-Sheffer for $(g_\lambda(t), f_\lambda(t))$. Assume also that $\lim_{\lambda\to 0} f_\lambda(t) = f(t)$ and $\lim_{\lambda\to 0} g_\lambda(t) = g(t)$, for some delta series $f(t)$ and an invertible series $g(t)$. Then $\lim_{\lambda\to 0} \overline{f}_\lambda(t) = \overline{f}(t)$, where $\overline{f}(t)$ is the compositional inverse of $f(t)$ with $\overline{f}(f(t)) = f(\overline{f}(t)) = t$. We note that $\lim_{\lambda\to 0} s_{k,\lambda}(x) = s_k(x)$.

In this case, Kim-Kim called that the family $\{s_{n,\lambda}(x)\}_{\lambda \in \mathscr{R}-\{0\}}$ of λ-Sheffer sequences $s_{n,\lambda}$ are the degenerate (Sheffer) sequences for the Sheffer polynomial $s_n(x)$.

Let $s_{n,\lambda}(x) \sim (g(t), f(t))_\lambda$ and $r_{n,\lambda}(x) \sim (h(t), g(t))_\lambda$, $(n \geq 0)$. Then

$$s_{n,\lambda}(x) = \sum_{k=0}^{n} c_{n,k} r_{k,\lambda}(x), \quad (n \geq 0), \tag{21}$$
$$\text{where} \quad c_{n,k} = \frac{1}{k!}\left\langle \frac{h(\overline{f}(t))}{g(\overline{f}(t))}\big(l(\overline{f}(t))\big)^k \mid (x)_{n,\lambda} \right\rangle_\lambda, \quad (n,k \geq 0), \quad \text{(see [11])}.$$

2. Some identities of degenerate r-Lah-Bell polynomials

Naturally, from (6), we define a degenerate r-extended Lah-Bell polynomials by

$$\left(\frac{1}{1-t}\right)^{2r} e_\lambda^x\left(\frac{t}{1-t}\right) = \sum_{n=0}^{\infty} \mathbf{B}_{r,n,\lambda}^L(x)\frac{t^n}{n!}. \tag{22}$$

When $x = 1$, $\mathbf{B}_{r,n,\lambda}^L := \mathbf{B}_{r,n,\lambda}^L(1)$ is called the n-th degenerate r-extended Lah-Bell number.
As $\lambda \to 0$, $\lim_{\lambda\to 0} \mathbf{B}_{r,n,\lambda}^L = \mathbf{B}_{r,n}^L$ is the n-th r-extended Lah-Bell number.

Theorem 1. *For $n,\ r \geq 0$, we have*

$$\mathbf{B}_{r,n,\lambda}^L(x) = \sum_{l=0}^{n}\sum_{j=0}^{l}\binom{n}{l}\binom{l}{j}(-1)^{l-j} < j+2r >_{n-l} (x)_{l,\lambda}.$$

Proof. From (8), (9) and (22), wehave

$$\begin{aligned}
\sum_{n=0}^{\infty}\mathbf{B}_{r,n,\lambda}^L(x)\frac{t^n}{n!} &= \left(\frac{1}{1-t}\right)^{2r} e_\lambda^x\left(\frac{1}{1-t}-1\right) \\
&= \left(\frac{1}{1-t}\right)^{2r}\sum_{l=0}^{\infty}(x)_{l,\lambda}\frac{1}{l!}\sum_{j=0}^{l}\binom{l}{j}(-1)^{l-j}\left(\frac{1}{1-t}\right)^{j} \\
&= \sum_{l=0}^{\infty}\sum_{j=0}^{l}\binom{l}{j}(-1)^{l-j}(x)_{l,\lambda}\frac{1}{l!}\sum_{m=0}^{\infty} < j+2r >_m \frac{t^m}{m!} \\
&= \sum_{n=0}^{\infty}\left(\sum_{l=0}^{n}\sum_{j=0}^{l}\binom{n}{l}\binom{l}{j}(-1)^{l-j} < j+2r >_{n-l} (x)_{l,\lambda}\right)\frac{t^n}{n!}.
\end{aligned} \tag{23}$$

Therefore, by comparing coefficients of both sides of (23), we have the desired result. □

Theorem 2. *For $n, r \geq 0$, we have*

$$\sum_{l=0}^{n}\sum_{m=l}^{n}\binom{n}{m}(-l)_{n-m,\lambda}S_{2,\lambda}(m,l)\mathbf{B}_{r,l,\lambda}^L(x) = \sum_{l=0}^{n}\binom{n}{l}(2r)_{l,\lambda}Bel_{n-l,\lambda}(x).$$

Proof. Replacing t by $1-e_\lambda^{-1}(t)$ in (22), from (8) and (13), the left side of (22) is

$$\begin{aligned}
e_\lambda^{2r}(t)e_\lambda^x(e_\lambda(t)-1) &= \sum_{l=0}^{\infty}(2r)_{l,\lambda}\frac{t^l}{l!}\sum_{m=0}^{\infty}Bel_{m,\lambda}(x)\frac{t^m}{m!} \\
&= \sum_{n=0}^{\infty}\left(\sum_{l=0}^{n}\binom{n}{l}(2r)_{l,\lambda}Bel_{n-l,\lambda}(x)\right)\frac{t^n}{n!}.
\end{aligned} \tag{24}$$

On the other hand, from (8), the right of side of (22) is

$$
\begin{aligned}
\sum_{l=0}^{\infty}\mathbf{B}^{L}_{r,l,\lambda}(x)\frac{(1-e_{\lambda}^{-1}(t))^{l}}{l!} &= \sum_{l=0}^{\infty}\mathbf{B}^{L}_{r,l,\lambda}(x)\frac{(e_{\lambda}(t)-1)^{l}e_{\lambda}^{-l}(t)}{l!} \\
&= \sum_{l=0}^{\infty}\mathbf{B}^{L}_{r,l,\lambda}(x)\sum_{n=l}^{\infty}\left(\sum_{m=l}^{n}\binom{n}{m}S_{2,\lambda}(m,l)(-l)_{n-m,\lambda}\right)\frac{t^{n}}{n!} \\
&= \sum_{n=0}^{\infty}\left(\sum_{l=0}^{n}\sum_{m=l}^{n}\binom{n}{m}(-l)_{n-m,\lambda}S_{2,\lambda}(m,l)\mathbf{B}^{L}_{r,l,\lambda}(x)\right)\frac{t^{n}}{n!}.
\end{aligned}
\tag{25}
$$

Therefore, by comparing coefficients of (24) and (25), we have what we want. □

Theorem 3. *For* $n,r\geq 0$*, we have*

$$
\sum_{l=0}^{mim\{n,2r\}}\binom{n}{l}\binom{2r}{l}(-1)^{l}\,l!\,\mathbf{B}^{L}_{r,n-l,\lambda}(x)=\sum_{k=0}^{n}(-1)^{n+k}S_{1,-\lambda}(n,k)Bel_{k,\lambda}(x).
$$

Proof. Replacing t by $-\log_{-\lambda}(1-t)$ in (13), from the second equation of (14) we observe that

$$
\begin{aligned}
e_{\lambda}^{x}\left(\frac{1}{1-t}-1\right) &= \sum_{k=0}^{\infty}Bel_{k,\lambda}(x)\frac{(-\log_{-\lambda}(1-t))^{k}}{k!} \\
&= \sum_{k=0}^{\infty}Bel_{k,\lambda}(x)(-1)^{k}\sum_{n=k}^{\infty}S_{1,-\lambda}(n,k)\frac{(-t)^{n}}{n!} \\
&= \sum_{n=0}^{\infty}\left(\sum_{k=0}^{n}(-1)^{n+k}S_{1,-\lambda}(n,k)Bel_{k,\lambda}(x)\right)\frac{t^{n}}{n!}.
\end{aligned}
\tag{26}
$$

On the other hand, from (22), we note that

$$
\begin{aligned}
e_{\lambda}^{x}\left(\frac{1}{1-t}-1\right) &= (1-t)^{2r}\sum_{n=0}^{\infty}\mathbf{B}^{L}_{r,n,\lambda}(x)\frac{t^{n}}{n!} \\
&= \sum_{l=0}^{2r}\binom{2r}{l}(-1)^{l}t^{l}\sum_{n=0}^{\infty}\mathbf{B}^{L}_{r,n,\lambda}(x)\frac{t^{n}}{n!} \\
&= \sum_{n=0}^{\infty}\left(\sum_{l=0}^{mim\{n,2r\}}\binom{n}{l}\binom{2r}{l}(-1)^{l}\,l!\,\mathbf{B}^{L}_{r,n-l,\lambda}(x)\right)\frac{t^{n}}{n!}.
\end{aligned}
\tag{27}
$$

Therefore, by comparing coefficients of (26) and (27), we have what we want. □

3. Some identities of degenerate r-Lah-Bell polynomials arising from degenerate Sheffer sequences

From (20) and (22), we observe the λ (degenerate) -Sheffer sequences of $B^{L}_{r,n,\lambda}(x)$ as follows:

$$
\mathbf{B}^{L}_{r,n,\lambda}(x)\ \sim\ \left(\left(\frac{1}{1+t}\right)^{2r},\ \frac{t}{1+t}\right)_{\lambda}.
\tag{28}
$$

Lemma 4. *For* $n\geq 0,\ k\in\mathbb{Z}$*, we have*

$$
P(x)=\sum_{k=0}^{n}U_{k}\mathbf{B}^{L}_{r,l,\lambda}(x),\ (n\geq 0),\ \text{ where }\ U_{k}=\frac{1}{k!}\left\langle\left(\frac{1}{1+t}\right)^{2r}\left(\frac{t}{1+t}\right)^{k}\,\middle|\,P(x)\right\rangle_{\lambda},
\tag{29}
$$

where $P(x)\in\mathbb{P}_{n}$.

Proof. For $P(x) \in \mathbb{P}_n$ such that $P(x) = \sum_{l=0}^{n} U_l \, \mathbf{B}^L_{r,l,\lambda}(x)$, from (19), we observe

$$\left\langle \left(\frac{1}{1+t}\right)^{2r}\left(\frac{t}{1+t}\right)^{k} \,\middle|\, P(x)\right\rangle_\lambda = \sum_{l=0}^{n} U_l \left\langle \left(\frac{1}{1+t}\right)^{2r}\left(\frac{t}{1+t}\right)^{k} \,\middle|\, \mathbf{B}^L_{r,l,\lambda}(x)\right\rangle_\lambda = \sum_{l=0}^{n} U_l \, l! \, \delta_{l,k} = k! \, U_k. \tag{30}$$

From (30), we get the desired result. □

Theorem 5. *For $n \geq 0$, we have*

$$\mathbf{B}^L_{r,n,\lambda}(x) = \sum_{k=0}^{n}\left(\binom{n}{k} < 2r >_{n-k}\right)\mathbf{B}^L_{k,\lambda}(x). \tag{31}$$

The inversion formula of (31) *is given by*

$$\mathbf{B}^L_{n,\lambda}(x) = \sum_{k=0}^{n}\left(\frac{1}{k!}\sum_{m=0}^{n}\sum_{l=0}^{k}\binom{k}{l}(-1)^{l+m} < 2r+k >_m L(n,m)\right)\mathbf{B}^L_{r,k,\lambda}(x), \tag{32}$$

where $\mathbf{B}^L_{n,\lambda}(x)$ are the degenerate Lah-Bell polynomials.

Proof. From (7), (20) and (22), we consider the two degenerate Sheffer sequences as follows:

$$\mathbf{B}^L_{r,n,\lambda}(x) \sim \left(\left(\frac{1}{1+t}\right)^{2r}, \frac{t}{1+t}\right)_\lambda \quad \text{and} \quad \mathbf{B}^L_{n,\lambda}(x) \sim \left(1, \frac{t}{1+t}\right)_\lambda. \tag{33}$$

Then from (9), (21) and (33) we have

$$\mathbf{B}^L_{r,n,\lambda}(x) = \sum_{k=0}^{n} c_{n,k}\mathbf{B}^L_{k,\lambda}(x). \tag{34}$$

where

$$c_{n,k} = \frac{1}{k!}\left\langle \left(\frac{1}{1-t}\right)^{2r} t^k \,\middle|\, (x)_{n,\lambda}\right\rangle_\lambda = \frac{1}{k!}\binom{n}{k}k!\left\langle \left(\frac{1}{1-t}\right)^{2r} \,\middle|\, (x)_{n-k,\lambda}\right\rangle_\lambda = \binom{n}{k} < 2r >_{n-k}. \tag{35}$$

Therefore, from (34) and (35) we have the identity (31).

To find the inversion formula of (31), put $P(x) = \mathbf{B}^L_{k,\lambda}(x) \in \mathbb{P}_{,n}$.

In addition, from (2), (7) and (8), we note that $\mathbf{B}^L_{n,\lambda}(x) = \sum_{m=0}^{n} L(n,m)(x)_{m,\lambda}$.

By using (9) and (29), we have

$$\mathbf{B}^L_{k,\lambda}(x) = \sum_{k=0}^{n} U_k \mathbf{B}^L_{r,n,\lambda}(x), \tag{36}$$

where

$$
\begin{aligned}
U_k &= \frac{1}{k!}\left\langle \left(\frac{1}{1+t}\right)^{2r}\left(\frac{t}{1+t}\right)^{k} \;\middle|\; \mathbf{B}^L_{n,\lambda}(x)\right\rangle_\lambda \\
&= \frac{1}{k!}\left\langle \left(\frac{1}{1+t}\right)^{2r}\left(\frac{t}{1+t}\right)^{k} \;\middle|\; \sum_{m=0}^{n} L(n,m)(x)_{m,\lambda}\right\rangle_\lambda \\
&= \frac{1}{k!}\sum_{m=0}^{n} L(n,m)\sum_{l=0}^{k}\binom{k}{l}(-1)^l\left\langle \left(\frac{1}{1+t}\right)^{2r+l} \;\middle|\; (x)_{m,\lambda}\right\rangle_\lambda \\
&= \frac{1}{k!}\sum_{m=0}^{n} L(n,m)\sum_{l=0}^{k}\binom{k}{l}(-1)^{l+m} < 2r+k >_m .
\end{aligned} \tag{37}
$$

Therefore, from (36) and (37) we get the identity (32). □

Theorem 6. *For $n \geq 0$, we have*

$$
\mathbf{B}^L_{r,n,\lambda}(x) = \sum_{k=0}^{n}\Big(L_r(n,k)\Big)(x)_{k,\lambda} = \sum_{k=0}^{n}\left(\sum_{l=0}^{n}\binom{n}{l}L(l,k) < 2r >_{n-l}\right)(x)_{k,\lambda}. \tag{38}
$$

The inversion formula of (38) *is given by*

$$
(x)_{n,\lambda} = \sum_{k=0}^{n}\left(\binom{n}{k}(-1)^{n-k} < 2r+k >_{n-k}\right)\mathbf{B}^L_{r,k,\lambda}(x). \tag{39}
$$

Proof. From (8), (20) and (22), we consider the following two Sheffer sequence as follows:

$$
\mathbf{B}^L_{r,n,\lambda}(x) \sim \left(\left(\frac{1}{1+t}\right)^{2r},\ \frac{t}{1+t}\right)_\lambda \quad \text{and} \quad (x)_{n,\lambda} \sim \big(1,\ t\big)_\lambda . \tag{40}
$$

From (2), (21) and (40), we have

$$
\mathbf{B}^L_{r,n,\lambda}(x) = \sum_{k=0}^{n} c_{n,k}(x)_{k,\lambda}, \tag{41}
$$

where

$$
\begin{aligned}
c_{n,k} &= \frac{1}{k!}\left\langle (1-t)^{-2r}\left(\frac{t}{1-t}\right)^{k} \;\middle|\; (x)_{n,\lambda}\right\rangle_\lambda = L_r(n,k). \\
\text{or}\quad c_{n,k} &= \left\langle (1-t)^{-2r} \;\middle|\; \left(\frac{1}{k!}\left(\frac{t}{1-t}\right)^{k}\right)_\lambda (x)_{n,\lambda}\right\rangle_\lambda = \sum_{l=0}^{n}\binom{n}{l}L(l,k) < 2r >_{n-l} .
\end{aligned} \tag{42}
$$

Therefore, From (41) and (42) we have the identity (38).
To find the inversion formula of (38), put $P(x) = (x)_{n,\lambda} \in \mathbb{P}_n$.
By using (9) and (29), we have

$$
(x)_{n,\lambda} = \sum_{k=0}^{n} U_k \mathbf{B}^L_{k,\lambda}(x), \tag{43}
$$

where

$$
\begin{aligned}
U_k &= \frac{1}{k!}\left\langle \left(\frac{1}{1+t}\right)^{2r}\left(\frac{t}{1+t}\right)^{k} \;\middle|\; (x)_{n,\lambda}\right\rangle_\lambda \\
&= \binom{n}{k}\left\langle \left(\frac{1}{1+t}\right)^{2r+k} \;\middle|\; (x)_{n-k,\lambda}\right\rangle_\lambda = \binom{n}{k}(-1)^{n-k} < 2r+k >_{n-k} .
\end{aligned} \tag{44}
$$

Therefore, from (43) and (44), we have the identity (39). □

Theorem 7. *For $n \geq 0$, $k, r \in \mathbb{N}$ is given by*

$$\mathbf{B}^L_{r,n,\lambda}(x) = \sum_{k=0}^{n}\left(\sum_{l=k}^{n}\sum_{m=0}^{n-l}\binom{n}{l}\frac{(1)_{m+1,\lambda}}{(m+1)}L_r(l,k)L(n-l,m)\right)\beta_{k,\lambda}(x). \tag{45}$$

The inversion formula of (51) *is given by*

$$\beta_{n,\lambda}(x) = \sum_{k=0}^{n}\left(\frac{1}{k!}\sum_{m=0}^{n}\sum_{l=0}^{k}\binom{n}{m}\binom{k}{l}(-1)^{l+n-m} < 2r+l >_{n-m} \beta_{m,\lambda}\right)\mathbf{B}^L_{r,k,\lambda}(x). \tag{46}$$

where $\beta_{n,\lambda}(x)$ are the degenerate Bernoulli polynomials.

Proof. From (10), (20) and (22), we consider the following two degenerate Sheffer sequences.

$$\mathbf{B}^L_{r,n,\lambda}(x) \sim \left(\left(\frac{1}{1+t}\right)^{2r}, \frac{t}{1+t}\right)_\lambda \quad \text{and} \quad \beta_{n,\lambda}(x) \sim \left(\frac{e_\lambda(t)-1}{t}, t\right)_\lambda. \tag{47}$$

From (2), (4), (21) and (47), we have

$$\mathbf{B}^L_{r,n,\lambda}(x) = \sum_{k=0}^{n} c_{n,k}\beta_{k,\lambda}(x), \tag{48}$$

where

$$\begin{aligned} c_{n,k} &= \frac{1}{k!}\left\langle \frac{e_\lambda(\frac{t}{1-t})-1}{\frac{t}{1-t}}\left(\frac{1}{1-t}\right)^{2r}\left(\frac{t}{1-t}\right)^{k} \,\middle|\, (x)_{n,\lambda}\right\rangle_\lambda \\ &= \left\langle \frac{e_\lambda(\frac{t}{1-t})-1}{\frac{t}{1-t}} \,\middle|\, \left(\frac{1}{k!}\left(\frac{1}{1-t}\right)^{2r}\left(\frac{t}{1-t}\right)^{k}\right)_\lambda (x)_{n,\lambda}\right\rangle_\lambda \\ &= \sum_{l=k}^{n}\binom{n}{l}L_r(l,k)\left\langle \sum_{m=1}^{\infty}(1)_{m,\lambda}\frac{1}{m!}\left(\frac{t}{1-t}\right)^{m-1} \,\middle|\, (x)_{n-l,\lambda}\right\rangle_\lambda \\ &= \sum_{l=k}^{n}\binom{n}{l}L_r(l,k)\sum_{m=0}^{n-l}(1)_{m+1,\lambda}\frac{1}{(m+1)}\left\langle \frac{1}{m!}\left(\frac{t}{1-t}\right)^{m} \,\middle|\, (x)_{n-l,\lambda}\right\rangle_\lambda \\ &= \sum_{l=k}^{n}\binom{n}{l}L_r(l,k)\sum_{m=0}^{n-l}(1)_{m+1,\lambda}\frac{1}{(m+1)}L(n-l,m). \end{aligned} \tag{49}$$

Therefore from (48) and (49), we get the identity (45).

To find the inversion formula of (45), put $P(x) = \beta_{n,\lambda}(x) \in \mathbb{P}_n$.

In addition, from (8) and (10), we note that $\beta_{n,\lambda}(x) = \sum_{m=0}^{n}\binom{n}{m}\beta_{m,\lambda}(x)_{n-m,\lambda}$.

By using (9) and (29), we have

$$\beta_{n,\lambda}(x) = \sum_{k=0}^{n} U_k \mathbf{B}^L_{r,k,\lambda}(x), \ (n \geq 0). \tag{50}$$

where

$$(51)\quad \begin{aligned} U_k &= \frac{1}{k!}\left\langle \left(\frac{1}{1+t}\right)^{2r}\left(\frac{t}{1+t}\right)^{k} \,\middle|\, \beta_{n,\lambda}(x)\right\rangle_{\lambda} \\ &= \frac{1}{k!}\left\langle \left(\frac{1}{1+t}\right)^{2r}\left(\frac{t}{1+t}\right)^{k} \,\middle|\, \sum_{m=0}^{n}\binom{n}{m}\beta_{m,\lambda}(x)_{n-m,\lambda}\right\rangle_{\lambda} \\ &= \frac{1}{k!}\sum_{m=0}^{n}\binom{n}{m}\beta_{m,\lambda}\sum_{l=0}^{k}\binom{k}{l}(-1)^{l}\left\langle \left(\frac{1}{1+t}\right)^{2r+l} \,\middle|\, (x)_{n-m,\lambda}\right\rangle_{\lambda} \\ &= \frac{1}{k!}\sum_{m=0}^{n}\binom{n}{m}\beta_{m,\lambda}\sum_{l=0}^{k}\binom{k}{l}(-1)^{l+n-m} < 2r+l >_{n-m}. \end{aligned}$$

Therefore, from (50) and (51) we have the identity (46). □

Theorem 8. *For $n \geq 0$, $k, r \in \mathbb{N}$, we have*

$$(52)\quad \mathbf{B}^{L}_{r,n,\lambda}(x) = \frac{1}{2}\sum_{k=0}^{n}\left(\sum_{m=0}^{n}\sum_{l=0}^{n-m}\binom{n}{m}(1)_{l,\lambda}L_r(m,k)L(n-m,l) + L_r(n,k)\right)E_{k,\lambda}(x).$$

The inversion formula of (52) *is given by*

$$(53)\quad E_{n,\lambda}(x) = \sum_{k=0}^{n}\left(\frac{1}{k!}\sum_{m=0}^{n}\sum_{l=0}^{k}\binom{n}{m}\binom{k}{l}(-1)^{l+n-m} < 2r+l >_{n-m} E_{m,\lambda}\right)\mathbf{B}^{L}_{r,k,\lambda}(x).$$

where $E_{n,\lambda}(x)$ are the degenerate Euler polynomials.

Proof. From (11), (20) and (22), we consider the following two degenerate Sheffer sequences as follows:

$$(54)\quad \mathbf{B}^{L}_{r,n,\lambda}(x) \sim \left(\left(\frac{1}{1+t}\right)^{2r}, \frac{t}{1+t}\right)_{\lambda} \quad \text{and} \quad E_{n,\lambda}(x) \sim \left(\frac{e_{\lambda}(t)+1}{2}, t\right)_{\lambda}.$$

From (2), (4), (21) and (54), we have

$$(55)\quad \mathbf{B}^{L}_{r,n\lambda}(x) = \sum_{k=0}^{n} c_{n,k}E_{k,\lambda}(x),$$

where

$$(56)\quad \begin{aligned} c_{n,k} &= \frac{1}{k!}\left\langle \frac{e_{\lambda}(\frac{t}{1-t})+1}{2}\left(\frac{1}{1-t}\right)^{2r}\left(\frac{t}{1-t}\right)^{k} \,\middle|\, (x)_{n,\lambda}\right\rangle_{\lambda} \\ &= \frac{1}{2}\sum_{m=0}^{n}\binom{n}{m}L_r(m,k)\left\langle e_{\lambda}\left(\frac{t}{1-t}\right)+1 \,\middle|\, (x)_{n-m,\lambda}\right\rangle_{\lambda} \\ &= \frac{1}{2}\sum_{m=0}^{n}\binom{n}{m}L_r(m,k)\left(\sum_{l=0}^{n-m}(1)_{l,\lambda}\left\langle \frac{1}{l!}\left(\frac{t}{1-t}\right)^{l} \,\middle|\, (x)_{n-m,\lambda}\right\rangle_{\lambda} + \left\langle 1 \,\middle|\, (x)_{n-m,\lambda}\right\rangle_{\lambda}\right) \\ &= \frac{1}{2}\left(\sum_{m=0}^{n}\binom{n}{m}L_r(m,k)\sum_{l=0}^{n-m}(1)_{l,\lambda}L(n-m,l) + L_r(n,k)\right). \end{aligned}$$

Therefore, from (55) and (56), we get the identity (52).
To find the inversion formula of (52), put $P(x) = E_{n,\lambda}(x) \in \mathbb{P}_n$.
From (9) and (29), we have

$$(57)\quad E_{n,\lambda}(x) = \sum_{k=0}^{n} U_k\mathbf{B}^{L}_{r,k,\lambda}(x), \ (n \geq 0).$$

where

$$
\begin{aligned}
U_k &= \frac{1}{k!}\left\langle \left(\frac{1}{1+t}\right)^{2r}\left(\frac{t}{1+t}\right)^{k} \Big| E_{n,\lambda}(x)\right\rangle_{\lambda} \\
&= \frac{1}{k!}\left\langle \left(\frac{1}{1+t}\right)^{2r}\left(1-\frac{1}{1+t}\right)^{k} \Big| \sum_{m=0}^{n}\binom{n}{m}E_{m,\lambda}(x)_{n-m,\lambda}\right\rangle_{\lambda} \\
&= \frac{1}{k!}\sum_{m=0}^{n}\binom{n}{m}E_{m,\lambda}\sum_{l=0}^{k}\binom{k}{l}(-1)^{l}\left\langle \left(\frac{1}{1+t}\right)^{2r+l} \Big| (x)_{n-m,\lambda}\right\rangle_{\lambda} \\
&= \frac{1}{k!}\sum_{m=0}^{n}\binom{n}{m}E_{m,\lambda}\sum_{l=0}^{k}\binom{k}{l}(-1)^{l+n-m} < 2r+l >_{n-m}.
\end{aligned} \tag{58}
$$

Therefore, from (57) and (58) we have the identity (53). □

Theorem 9. *For $n \geq 0$, we have*

$$
\mathbf{B}^{L}_{r,n,\lambda}(x) = \sum_{k=0}^{n}\left(\sum_{j=k}^{n}\sum_{l=j}^{n}\sum_{m=0}^{n-l}\binom{n}{l}\frac{(1)_{m+1,\lambda}}{m+1}S_{2,\lambda}(j,k)L_r(l,j)L(n-l,m)\right)D_{k,\lambda}(x). \tag{59}
$$

The inversion formula of (59)*is given by*

$$
D_{n,\lambda}(x) = \sum_{k=0}^{n}\left(\frac{1}{k!}\sum_{m=0}^{n}\sum_{j=0}^{n-m}\sum_{l=0}^{k}\binom{n}{m}\binom{k}{l}(-1)^{l+j} < 2r+l >_{j} S_{1,\lambda}(n-m,j)D_{m,\lambda}\right)\mathbf{B}^{L}_{r,k,\lambda}(x). \tag{60}
$$

where $D_{n,\lambda}(x)$ are the degenerate Daehee polynomials.

Proof. From (12), (20) and (22), we consider the following two degenerate Sheffer sequences.

$$
\mathbf{B}^{L}_{r,n,\lambda}(x) \sim \left(\left(\frac{1}{1+t}\right)^{2r}, \frac{t}{1+t}\right)_{\lambda} \quad \text{and} \quad D_{n,\lambda}(x) \sim \left(\frac{e_{\lambda}(t)-1}{t},\ e_{\lambda}(t)-1\right)_{\lambda}. \tag{61}
$$

From (2), (4), (21)and (61), we have

$$
\mathbf{B}^{L}_{r,n,\lambda}(x) = \sum_{k=0}^{n}c_{n,k}D_{k,\lambda}(x), \tag{62}
$$

where

$$
\begin{aligned}
c_{n,k} &= \frac{1}{k!}\left\langle \frac{e_{\lambda}(\frac{t}{1-t})-1}{\frac{t}{1-t}}\left(\frac{1}{1-t}\right)^{2r}\left(e_{\lambda}\left(\frac{t}{1-t}\right)-1\right)^{k} \Big| (x)_{n,\lambda}\right\rangle_{\lambda} \\
&= \left\langle \frac{e_{\lambda}(\frac{t}{1-t})-1}{\frac{t}{1-t}}\left(\frac{1}{1-t}\right)^{2r} \Big| \left(\frac{1}{k!}\left(e_{\lambda}\left(\frac{t}{1-t}\right)-1\right)^{k}\right)_{\lambda}(x)_{n,\lambda}\right\rangle_{\lambda} \\
&= \sum_{j=k}^{n}S_{2,\lambda}(j,k)\left\langle \frac{e_{\lambda}(\frac{t}{1-t})-1}{\frac{t}{1-t}} \Big| \left(\frac{1}{j!}\left(\frac{1}{1-t}\right)^{2r}\left(\frac{t}{1-t}\right)^{j}\right)_{\lambda}(x)_{n,\lambda}\right\rangle_{\lambda} \\
&= \sum_{j=k}^{n}S_{2,\lambda}(j,k)\sum_{l=j}^{n}\binom{n}{l}L_r(l,j)\sum_{m=0}^{n-l}(1)_{m+1,\lambda}\frac{1}{m+1}\left\langle \frac{1}{m!}\left(\frac{t}{1-t}\right)^{m} \Big| (x)_{n-l,\lambda}\right\rangle_{\lambda} \\
&= \sum_{j=k}^{n}S_{2,\lambda}(j,k)\sum_{l=j}^{n}\binom{n}{l}L_r(l,j)\sum_{m=0}^{n-l}(1)_{m+1,\lambda}\frac{1}{m+1}L(n-l,m).
\end{aligned} \tag{63}
$$

Therefore, from (62) and (63), we get the identity (59).

To find the inversion formula of (59), put $P(x) = D_{n,\lambda}(x) \in \mathbb{P}_n$.

In addition, from (12) and $(1+x)^x=\sum_{n=0}^{\infty}(x)_n\frac{t^n}{n!}$, we note that $D_{n,\lambda}(x)=\sum_{m=0}^{n}\binom{n}{m}D_{m,\lambda}(x)_{n-m}$. From the first equation of (14) and (29), we have

$$D_{n,\lambda}(x)=\sum_{k=0}^{n}U_k\mathbf{B}^{L}_{r,k,\lambda}(x), \tag{64}$$

where

$$\begin{aligned}U_k&=\frac{1}{k!}\left\langle\left(\frac{1}{1+t}\right)^{2r}\left(\frac{t}{1+t}\right)^{k}\,\middle|\,D_{n,\lambda}(x)\right\rangle_{\lambda}\\&=\frac{1}{k!}\left\langle\left(\frac{1}{1+t}\right)^{2r}\left(\frac{t}{1+t}\right)^{k}\,\middle|\,\sum_{m=0}^{n}\binom{n}{m}D_{m,\lambda}(x)_{n-m}\right\rangle_{\lambda}\\&=\frac{1}{k!}\sum_{m=0}^{n}\binom{n}{m}D_{m,\lambda}\left\langle\left(\frac{1}{1+t}\right)^{2r}\left(1-\frac{1}{1+t}\right)^{k}\,\middle|\,\sum_{j=0}^{n-m}S_{1,\lambda}(n-m,j)(x)_{j,\lambda}\right\rangle_{\lambda}\\&=\frac{1}{k!}\sum_{m=0}^{n}\binom{n}{m}D_{m,\lambda}\sum_{j=0}^{n-m}S_{1,\lambda}(n-m,j)\sum_{l=0}^{k}\binom{k}{l}(-1)^{l}\left\langle\left(\frac{1}{1+t}\right)^{2r+l}\,\middle|\,(x)_{j,\lambda}\right\rangle_{\lambda}\\&=\frac{1}{k!}\sum_{m=0}^{n}\binom{n}{m}D_{m,\lambda}\sum_{j=0}^{n-m}S_{1,\lambda}(n-m,j)\sum_{l=0}^{k}\binom{k}{l}(-1)^{l+j}<2r+l>_{j}\,.\end{aligned}\tag{65}$$

Therefore, from (64) and (65), we get the identity (60). ☐

Theorem 10. *For $n\ge 0$, we have*

$$\mathbf{B}^{L}_{r,n,\lambda}(x)=\sum_{k=0}^{n}\left(\sum_{l=k}^{n}\binom{n}{l}<2r+l>_{n-l}S_{1,\lambda}(l,k)\right)Bel_{k,\lambda}(x). \tag{66}$$

The inversion formula of (66) *is given by*

$$Bel_{n,\lambda}(x)=\sum_{k=0}^{n}\left(\frac{1}{k!}\sum_{m=0}^{n}\sum_{l=0}^{k}\binom{k}{l}(-1)^{l+m}<2r+l>_{m}S_{2,\lambda}(n,m)\right)\mathbf{B}^{L}_{r,k,\lambda}(x). \tag{67}$$

where $Bel_{n,\lambda}(x)$ are the degenerate Bell polynomials.

Proof. From (13), (20) and (22), we consider two degenerate Sheffer sequences as follows:

$$\mathbf{B}^{L}_{r,n,\lambda}(x)\sim\left(\left(\frac{1}{1+t}\right)^{2r},\ \frac{t}{1+t}\right)_{\lambda}\quad\text{and}\quad Bel_{n,\lambda}\sim(1,\ \log_{\lambda}(1+t))_{\lambda}. \tag{68}$$

From the second equation of (14), (21) and (68), we have

$$\mathbf{B}^{L}_{r,n,\lambda}(x)=\sum_{k=0}^{n}c_{n,k}Bel_{k,\lambda}(x), \tag{69}$$

where

$$\begin{aligned}
c_{n,k} &= \frac{1}{k!}\left\langle (1-t)^{-2r}\left(\log_\lambda\left(1+\frac{t}{1-t}\right)\right)^k \,\middle|\, (x)_{n,\lambda}\right\rangle_\lambda \\
&= \left\langle (1-t)^{-2r} \,\middle|\, \left(\frac{1}{k!}\left(\log_\lambda\left(1+\frac{t}{1-t}\right)\right)^k\right)_\lambda (x)_{n,\lambda}\right\rangle_\lambda \\
&= \sum_{l=k}^{n} S_{1,\lambda}(l,k)\frac{1}{l!}\left\langle (1-t)^{-2r-l} \,\middle|\, (t^l)_\lambda (x)_{n,\lambda}\right\rangle_\lambda \qquad (70)\\
&= \sum_{l=k}^{n}\binom{n}{l} S_{1,\lambda}(l,k)\left\langle \sum_{j=0}^{\infty} < 2r+l >_j \frac{t^j}{j!} \,\middle|\, (x)_{n-l,\lambda}\right\rangle_\lambda \\
&= \sum_{l=k}^{n}\binom{n}{l} S_{1,\lambda}(l,k) < 2r+l >_{n-l}.
\end{aligned}$$

Therefore, from (69) and (70), we get the identity (66).

To find the inversion formula of (66), put $P(x) = Bel_{n,\lambda}(x) \in \mathbb{P}_n$.

In addition, from (8), (13) and (15), we note that $Bel_{n,\lambda}(x) = \sum_{m=0}^{n} S_{2,\lambda}(n,m)(x)_{m,\lambda}$.

From (9) and (29), we have

$$Bel_{n,\lambda}(x) = \sum_{k=0}^{n} U_k \mathbf{B}^L_{r,k,\lambda}(x), \qquad (71)$$

where

$$\begin{aligned}
U_k &= \frac{1}{k!}\left\langle \left(\frac{1}{1+t}\right)^{2r}\left(\frac{t}{1+t}\right)^k \,\middle|\, Bel_{n,\lambda}(x)\right\rangle_\lambda \\
&= \frac{1}{k!}\left\langle \left(\frac{1}{1+t}\right)^{2r}\left(1-\frac{1}{1+t}\right)^k \,\middle|\, \sum_{m=0}^{n} S_{2,\lambda}(n,m)(x)_{m,\lambda}\right\rangle_\lambda \qquad (72)\\
&= \frac{1}{k!}\sum_{m=0}^{n} S_{2,\lambda}(n,m)\sum_{l=0}^{k}\binom{k}{l}(-1)^l\left\langle \left(\frac{1}{1+t}\right)^{2r+l} \,\middle|\, (x)_{m,\lambda}\right\rangle_\lambda \\
&= \frac{1}{k!}\sum_{m=0}^{n} S_{2,\lambda}(n,m)\sum_{l=0}^{k}\binom{k}{l}(-1)^{l+m} < 2r+k >_m.
\end{aligned}$$

Therefore, from (71) and (72), we get the identity (67).

□

Theorem 11. *For $n \geq 0$, we have*

$$\mathbf{B}^L_{r,n,\lambda}(x) = \sum_{k=0}^{n}\left(\sum_{l=k}^{n} S_{2,\lambda}(l,k)L_r(n,l)\right)(x)_k, \ (n \geq 0). \qquad (73)$$

Proof. Since

$$e^x_\lambda(\log(1+t)) = (1+t)^x = \sum_{n=0}^{\infty}(x)_n\frac{t^n}{n!}, \qquad (74)$$

we have $(x)_n \sim (1, e_\lambda(t)-1)_\lambda$.

From (20), (22) and (74), we consider the two degenerate Sheffer sequences as follows:

$$\mathbf{B}^{L}_{r,n,\lambda}(x) \sim \left(\left(\frac{1}{1+t}\right)^{2r},\ \frac{t}{1+t}\right)_{\lambda} \quad \text{and} \quad (x)_n \sim (1,\ e_{\lambda}(t)-1)_{\lambda}. \tag{75}$$

Thus from (4), (21) and (75), we have

$$\mathbf{B}^{L}_{r,n,\lambda}(x) = \sum_{k=0}^{n} c_{n,k}(x)_k,\ (n \geq 0), \tag{76}$$

where

$$\begin{aligned} c_{n,k} &= \frac{1}{k!}\left\langle (1-t)^{-2r}\left(e_{\lambda}\left(\frac{t}{1-t}\right)-1\right)^{k} \middle| (x)_{n,\lambda}\right\rangle_{\lambda} \\ &= \sum_{l=k}^{n} S_{2,\lambda}(l,k)\left\langle (1-t)^{-2r}\frac{1}{l!}\left(\frac{t}{1-t}\right)^{l} \middle| (x)_{n,\lambda}\right\rangle_{\lambda} = \sum_{l=k}^{n} S_{2,\lambda}(l,k)L_r(n,l). \end{aligned} \tag{77}$$

Therefore, from (76) and (77), we have the desired result. □

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