Proceedings of the Jangjeon Mathematical Society
24 (2021), No. 1, pp. 33 - 45

www.jangjeon.or.kr
http://dx.doi.org/10.17777/pjms2021.24.1.33

SOME IDENTITIES OF A NEW TYPE OF DEGENERATE POLY-FROBENIUS-EULER POLYNOMIALS AND NUMBERS

HYE KYUNG KIM[1*] AND WASEEM A. KHAN[2]

ABSTRACT. Many mathematicians have been studying various degenerate versions of special polynomials and numbers not only in some arithmetic and combinatorial aspects but also in applications to differential equations, identities of symmetry and probability theory. Moreover, umbral calculus is one of the important methods for obtaining the symmetric identities for the degenerate version of special numbers and polynomials. Recently, Kim-Kim (J. Math. Anal. Appl.(2020) 23, 124521) introduced the λ-Sheffer sequence and the degenerate Sheffer sequence by replacing the λ- linear functional and λ- differential operator, respectively, instead of the linear functionalal and the differential operator used by Rota. This paper is divided in two parts. In the first part, we introduce a new type of degenerate poly-Frobenius-Euler polynomials and numbers, and derive several combinatorial identities related to those polynomials and numbers without by using degenerate Sheffer sequences. In the second part, we also derive various combinatorial identities related to a new type of degenerate poly-Frobenius-Euler polynomials by using degenerate Sheffer sequences. Some of them include the degenerate and other special polynomials and numbers such as the degenerate falling factorials, the degenerate Bell polynomials, the degenerate Bernoulli polynomials, the degenerate Euler polynomials, the degenerate Daehee polynomials, the Stirling number of the first and second kind, etc.

1. INTRODUCTION

Euler numbers are of fundamental importance in several parts of mathematics and mathematical physics. One of the well-known extensions is given by the Frobenius-Euler numbers and polynomials. These polynomials and numbers arise in many combinatorial and number theory contexts. As for the elementary properties of these sequences, various authors have studied them and obtained many interesting combinational identities (see [1, 7, 24-26]).

Also, umbral calculus, established by Rota in the 1970s, was based on modern concepts such as linear functionals, linear operators, and adjoints. The Sheffer sequences are the key of this theory [5, 6, 12-15, 24, 27-30]. In addition, many mathematicians have been studying various degenerate versions of special polynomials and numbers not only in some arithmetic and combinatorial aspects but also in applications to differential equations, identities of symmetry and probability theory [13-23]. These degenerate versions began when Carlitz introduced the degenerate Bernoulli polynomials and the degenerate Euler polynomials [2]. Kim-Kim [13] recently introduced the λ-Sheffer sequence and the degenerate Sheffer sequence by replacing the λ- linear functional and λ-differential operator, respectively, instead of the linear functional and the differential operator used by Rota. Umbral calculus is one of the important methods for obtaining the symmetric identities for the degenerate version of special numbers and polynomials.

Motivated by the above perspectives, we write this paper in two parts. In the first part, we introduce a new type of degenerate poly-Frobenius-Euler polynomials and numbers, and derive several

2010 *Mathematics Subject Classification.* 11B73; 11B83; 05A19.

Key words and phrases. degenerate poly-Euler polynomials; degenerate poly-Frobenius-Euler polynomials and numbers; degenerate Sheffer sequences; degenerate Bell polynomials; the degenerate Daehee polynomials, degenerate Stirling numbers of the first and second kind.

* is corresponding author.

combinatorial identities related to those polynomials and numbers without by using degenerate Sheffer sequences. In the second part, we also derive various combinatorial identities related to a new type of degenerate poly-Frobenius-Euler polynomials by using degenerate Sheffer sequences. Some of them include the degenerate and other special polynomials and numbers such as the degenerate falling factorials, the degenerate Bell polynomials, the degenerate Bernoulli polynomials, the degenerate Euler polynomials, the degenerate Daehee polynomials, the Stirling number of the first and second kind, etc.

Now, we give some definitions and properties needed in this paper.
For $u \in \mathbb{C}$ with $u \neq 1$, the classical Frobenius-Euler polynomials $h_n(x|u)$ are defined by means of the following generating function

$$\frac{1-u}{e^t-u}e^{xt} = \sum_{n=0}^{\infty} h_n(x|u)\frac{t^n}{n!}, \quad \text{(see [1, 7, 9, 24, 25])}. \tag{1}$$

In the special case when $x=0$, $h_n(u)=h_n(0|u)$ are called n^{th} Frobenius-Euler numbers. Substituting $u=-1$ in (1), $h_n(x:-1)=E_n(x)$, are called the Euler polynomials, (see [2, 4, 9]).

Kim et al. [19] introduced the degenerate Frobenius-Euler polynomials defined by

$$\frac{1-u}{(1+\lambda t)^{\frac{1}{\lambda}}-u}(1+\lambda t)^{\frac{x}{\lambda}} = \sum_{n=0}^{\infty} h_{n,\lambda}(x|u)\frac{t^n}{n!}. \tag{2}$$

When $x=0$, $h_{n,\lambda}(u)=h_{n,\lambda}(0|u)$ are called the degenerate Frobenius-Euler numbers.

For any nonzero $\lambda \in \mathbb{R}$ (or $\mathbb{C}$), the degenerate exponential function is defined by

$$e_\lambda^x(t) = (1+\lambda t)^{\frac{x}{\lambda}}, \quad e_\lambda(t) = (1+\lambda t)^{\frac{1}{\lambda}} = e_\lambda^1(t), \qquad \text{(see [14-23])}. \tag{3}$$

By Taylor expansion, we get

$$e_\lambda^x(t) = \sum_{n=0}^{\infty}(x)_{n,\lambda}\frac{t^n}{n!}, \qquad \text{(see [14-23])}, \tag{4}$$

where $(x)_{0,\lambda}=1,\ (x)_{n,\lambda}=x(x-\lambda)(x-2\lambda)\cdots(x-(n-1)\lambda),\ (n\geq 1)$.
Note that

$$\lim_{\lambda\to 0} e_\lambda^x(t) = \sum_{n=0}^{\infty}\frac{x^n t^n}{n!} = e^{xt}.$$

Carlitz introduced the degenerate Bernoulli polynomials of order r and the degenerate Euler polynomials of order r ($r \in \mathbb{N}$) respectively given by

$$\left(\frac{t}{e_\lambda(t)-1}\right)^r e_\lambda^x(t) = \sum_{n=0}^{\infty} B_{n,\lambda}^{(r)}(x)\frac{t^n}{n!} \qquad \text{(see [2])}, \tag{5}$$

and

$$\left(\frac{2}{e_\lambda(t)+1}\right)^r e_\lambda^x(t) = \sum_{n=0}^{\infty} E_{n,\lambda}^{(r)}(x)\frac{t^n}{n!} \qquad \text{(see [2])}. \tag{6}$$

When $r=1$ and $x=0$, $B_{n,\lambda}=B_{n,\lambda}^{(1)}(0)$ are called the degenerate Bernoulli numbers, and $E_{n,\lambda} = EB_{n,\lambda}^{(1)}(0)$ are called the degenerate Euler numbers.
Note that $\lim_{\lambda\to 0} B_{n,\lambda}^{(r)}(x) = B_n^{(r)}(x), (n\geq 0)$ and $\lim_{\lambda\to 0} E_{n,\lambda}^{(r)}(x) = E_n^{(r)}(x), (n\geq 0)$, where

$$\left(\frac{t}{e^t-1}\right)^r e^{xt} = \sum_{n=0}^{\infty} B_n^{(r)}(x)\frac{t^n}{n!} \ \text{ and } \ \left(\frac{2}{e^t+1}\right)^r e^{xt} = \sum_{n=0}^{\infty} E_n^{(r)}(x)\frac{t^n}{n!} \quad \text{(see [2])},$$

The degenerate Daehee polynomials are defined by

$$\frac{\log_\lambda(1+t)}{t}(1+t)^x=\sum_{n=0}^{\infty}D_{n,\lambda}(x)\frac{t^n}{n!}\qquad(\text{see [20]}),\tag{7}$$

where $\log_\lambda(t)=\frac{1}{\lambda}(t^\lambda-1)$. When $x=0$, $D_{n,\lambda}=D_{n,\lambda}(0)$ are called the degenerate Daehee numbers.

The Bell polynomials are defined by the generating function

$$e^{x(e^t-1)}=\sum_{n=0}^{\infty}Bel_n(x)\frac{t^n}{n!},\qquad(\text{see [11, 21, 22]}).\tag{8}$$

Kim-Kim introduced the degenerate Bell polynomials given by

$$e_\lambda^x(e_\lambda(t)-1)=\sum_{l=0}^{\infty}Bel_{l,\lambda}(x)\frac{t^l}{l!},\qquad(\text{see [21]}).\tag{9}$$

Recently, Kim-Kim [10] introduced the degenerate polylogarithm function defined by

$$l_{k,\lambda}(x)=\sum_{n=1}^{\infty}\frac{(-\lambda)^{n-1}(1)_{n,1/\lambda}}{(n-1)!n^k}x^n,\ \ (k\in\mathbb{Z},\quad |x|<1),\qquad(\text{see [10]}).\tag{10}$$

We note that $\lim_{\lambda\to 0}l_{k,\lambda}(x)=\sum_{n=1}^{\infty}\frac{x^n}{n^k}=\mathrm{Li}_k(x)$.

From (10), we see that

$$l_{1,\lambda}(x)=\sum_{n=1}^{\infty}\frac{(-\lambda)^{n-1}(1)_{n,1/\lambda}}{n!}x^n=-\log_\lambda(1-x).\tag{11}$$

They also studied a new type of degenerate poly-Bernoulli polynomials and numbers, by using the degenerate polylogarithm function as follows:

$$\frac{l_{k,\lambda}(1-e_\lambda(-t))}{1-e_\lambda(-t)}e_\lambda^x(-t)=\sum_{n=0}^{\infty}\beta_{n,\lambda}^{(k)}(x)\frac{t^n}{n!},\qquad(\text{see [10]}).$$

When $x=0$, $\beta_{n,\lambda}^{(k)}=\beta_{n,\lambda}^{(k)}(0)$ are called the degenerate poly-Bernoulli numbers.
Moreover, they observed that

$$\sum_{n=0}^{\infty}\beta_{n,\lambda}^{(1)}\frac{t^n}{n!}=\frac{1}{1-e_\lambda(-t)}l_{1,\lambda}(1-e_\lambda(-t))=\frac{-t}{e_\lambda(-t)-1}=\sum_{n=0}^{\infty}(-1)^nB_{n,\lambda}\frac{t^n}{n!}\qquad(\text{see [10]}).$$

Kim, H. K. [16] considered the degenerate type 2 poly-Euler polynomials which are given by the generating function to be

$$\frac{l_k(1-e_\lambda(2t)))}{t(e_\lambda(t)+e_\lambda^{-1}(t))}e_\lambda^x(t)=\sum_{n=0}^{\infty}\mathscr{E}_{n,\lambda}^{(k)}(x)\frac{t^n}{n!},\qquad(\text{see [16]}).\tag{12}$$

When $x=0, \mathscr{E}_n^{(k)}=\mathscr{E}_n^{(k)}(0)$ are called type 2 poly-Euler numbers.

Since $l_{1,\lambda}(1-e_\lambda(2t))=2t$, we see that $\mathscr{E}_{n,\lambda}^{(1)}(x)=E_{n,\lambda}^*(x)$ $(n\geq 0)$ are the type 2 degenerate Euler polynomials, where

$$\frac{2}{e_\lambda(t)+e_\lambda^{-1}(t))}e_\lambda^x(t)=\sum_{n=0}^{\infty}E_{n,\lambda}^*(x)\frac{t^n}{n!}\qquad(\text{see [16]}),$$

Let $\mathbb{C}$ be the complex number field and let $\mathscr{F}$ be the set of all power series in the variable t over $\mathbb{C}$ with

$$\mathscr{F} = \left\{ f(t) = \sum_{k=0}^{\infty} a_k \frac{t^k}{k!} \,\middle|\, a_k \in \mathbb{C} \right\}. \tag{13}$$

Let $\mathbb{P} = \mathbb{C}[x]$ and $\mathbb{P}^*$ be the vector space all linear functional on $\mathbb{P}$.

$$\mathbb{P}_n = \{\, P(x) \in \mathbb{C}[x] \mid degP(x) \leq n\}, \;\; (n \geq 0). \tag{14}$$

Then $\mathbb{P}_n$ is an $(n+1)$-dimensional vector space over $\mathbb{C}$.

In particular, $P(x)_\lambda = b_0(x)_{0,\lambda} + b_1(x)_{1,\lambda} + \cdots + b_n(x)_{n,\lambda} \in \mathbb{P}_n$.

For $f(t) = \sum_{k=0}^{\infty} a_k \frac{t^k}{k!} \in \mathscr{F}$ and a fixed nonzero real number λ, Kim-Kim introduced recently the linear functional $\langle f(t) \mid \cdot \rangle_\lambda$ on $\mathbb{P}$, called λ-linear functional given by $f(t)$, defined by

$$\langle f(t) \mid (x)_{n,\lambda} \rangle_\lambda = a_n, \quad \text{for all } n \geq 0 \qquad \text{(see [13]).} \tag{15}$$

and by (15), we note that

$$\langle t^k \mid (x)_{n,\lambda} \rangle_\lambda = n!\delta_{n,k}, \quad (n,\ k \geq 0), \tag{16}$$

where $\delta_{n,k}$ is the Kronecker's symbol.

For each $\lambda \in \mathbb{R}$, and each nonnegative integer k, they also defined the differential operator on $\mathbb{P}$ by

$$(t^k)_\lambda (x)_{n,\lambda} = \begin{cases} (n)_k (x)_{n-k,\lambda}, & \text{if } \; k \leq n, \\ 0 & \text{if } \; k \geq n, \end{cases} \quad \text{(see [13]),} \tag{17}$$

and any power series $f(t) = \sum_{k=0}^{\infty} a_k \frac{t^k}{k!} \in \mathscr{F}$ yields the differential operator on $\mathbb{P}$, called λ-differential operator given by $f(t)$, which is defined by

$$(f(t))_\lambda (x)_{n,\lambda} = \sum_{k=0}^{n} \binom{n}{k} a_k (x)_{n-k,\lambda}, \quad (n \geq 0). \tag{18}$$

Note that we should note that different λ's give rise to different linear functionals on $\mathbb{P}$ (see [13]). We also observe that, for $\lambda = 0$, the linear functional $\langle f(t) \mid \cdot \rangle$ agrees with the one in $\langle f(t) \mid x^n \rangle = a_k$, $(k \geq 0)$.

The order $o(f(t))$ of a power series $f(t)(\neq 0)$ is the smallest integer k for which the coefficient of t^k does not vanish.

If $f(t)$ is a series with $o(f(t)) = 1$, then $f(t)$ is called a delta series.

If $f(t)$ is a series with $o(f(t)) = 0$, then $f(t)$ is called an invertible series. When $f(t)$ is a delta series, there exists the compositional inverse $\overline{f}(t)$ of $f(t)$ with $\overline{f}(f(t)) = f(\overline{f}(t)) = t$.

The sequence $s_{n,\lambda}(x)$ is a λ-Sheffer for $(g(t), f(t))$ if and only if

$$\frac{1}{g\big(\overline{f}(t)\big)} e_\lambda^x \big(\overline{f}(t)\big) \;=\; \sum_{k=0}^{\infty} \frac{s_{k,\lambda}(x)}{k!} t^k \qquad \text{(see [13]),} \tag{19}$$

for all $y \in \mathbb{C}$.

Assume that for each $\lambda \in \mathbb{R}^*$ of the set of nonzero real numbers, $s_{n,\lambda}(x)$ is λ-Sheffer for $(g_\lambda(t), f_\lambda(t))$. Assume also that $\lim_{\lambda\to 0} f_\lambda(t) = f(t)$ and $\lim_{\lambda\to 0} g_\lambda(t) = g(t)$, for some delta series $f(t)$ and an invertible series $g(t)$. Then $\lim_{\lambda\to 0}\overline{f}_\lambda(t) = \overline{f}(t)$, where $\overline{f}(t)$ is the compositional inverse of $f(t)$ with $\overline{f}(f(t)) = f(\overline{f}(t)) = t$. We note that $\lim_{\lambda\to 0} s_{k,\lambda}(x) = s_k(x)$.

In this case, Kim-Kim called that the family $\{s_{n,\lambda}(x)\}_{\lambda\in\mathscr{R}-\{0\}}$ of λ-Sheffer sequences $s_{n,\lambda}$ are the degenerate (Sheffer) sequences for the Sheffer polynomial $s_n(x)$.

For $s_{n,\lambda}(x) \sim (g(t), f(t))_\lambda$ and $r_{n,\lambda}(x) \sim (h(t), g(t))_\lambda$, $(n \geq 0)$, we have

(20)

$$s_{n,\lambda}(x) = \sum_{k=0}^{n} c_{n,k} r_{k,\lambda}(x), \quad (n \geq 0), \quad \text{where} \quad c_{n,k} = \frac{1}{k!}\left\langle \frac{h(\overline{f}(t))}{g(\overline{f}(t))}\left(l(\overline{f}(t))\right)^k \mid (x)_{n,\lambda} \right\rangle_\lambda, \qquad \text{(see [13]).}$$

For $n \geq 0$, it is well known that the degenerate Stirling numbers of the first and second kind respectively are given by

$$(x)_n = \sum_{l=0}^{n} S_{1,\lambda}(n,l)(x)_{l,\lambda}, \qquad \text{(see [10, 18]),} \tag{21}$$

and

$$(x)_{n,\lambda} = \sum_{l=0}^{n} S_{2,\lambda}(n,l)(x)_l, \qquad \text{(see [10, 18]).} \tag{22}$$

where $(x)_0 = 1,\ (x)_n = x(x-1)\ldots(x-n+1),\ (n \geq 1)$.

On the other hand, the degenerate Stirling numbers of the first and second kind are also respectively given by

$$\frac{1}{k!}\left(\log_\lambda(1+t)\right)^k = \sum_{n=k}^{\infty} S_{1,\lambda}(n,k)\frac{t^n}{n!} \quad (k \geq 0), \qquad \text{(see [10, 18]),} \tag{23}$$

and

$$\frac{1}{k!}\left(e_\lambda(t)-1\right)^k = \sum_{n=k}^{\infty} S_{2,\lambda}(n,k)\frac{t^n}{n!} \quad (k \geq 0), \qquad \text{(see [10, 18]).} \tag{24}$$

2. New type of degenerate poly-Frobenius-Euler polynomials and numbers

In this section, we introduce the new type of degenerate poly-Frobenius-Euler polynomials and derive several identities related to these polynomials and numbers without using degenerate Sheffer sequences.

Now, we give the new type of degenerate poly-Frobenius-Euler polynomials by

$$\frac{l_{k,\lambda}(1-e_\lambda(-(1-u)t))}{t(e_\lambda(t)-u)} e_\lambda^x(t) = \sum_{n=0}^{\infty} h_{n,\lambda}^{(k)}(x|u)\frac{t^n}{n!}, \tag{25}$$

when $x = 0$, $h_{n,\lambda}^{(k)}(u) := h_{n,\lambda}^{(k)}(0|u)$ are called the degenerate poly-Frobenius-Euler numbers.

When $k = 1$, $\dfrac{(1-u)t}{t(e_\lambda(t)-u)} e_\lambda^x(t) = \sum_{n=0}^{\infty} h_{n,\lambda}(x|u)\dfrac{t^n}{n!}$ are the degenerate Frobenius-Euler polynomials.

It is easy to show the following theorem from (4) and (25).

Theorem 1. *For $n \geq 0$, we have*

$$h_{n,\lambda}^{(k)}(x|u) = \sum_{l=0}^{n} \binom{n}{l} h_{l,\lambda}^{(k)}(u)(x)_{n-l,\lambda} \quad and \quad h_{n,\lambda}^{(k)}(-x|u) = \sum_{l=0}^{n} \binom{n}{l} (-1)^{n-l} h_{l,\lambda}^{(k)}(u) < x >_{n-l,\lambda},$$

where $< x >_{0,\lambda} = 1$ *and* $< x >_{n,\lambda} = x(x+\lambda)(x+2\lambda)\cdots(x+(n-1)\lambda)$, $(n \geq 1)$.

Lemma 2. *For $n \geq 0$, we have*

$$l_{k,\lambda}(1-e_\lambda(-(1-u)t)) = \sum_{n=1}^{\infty} \left(\sum_{m=1}^{n} \frac{(1)_{m,\frac{1}{\lambda}}(-1)^{n-1}\lambda^{m-1}(1-u)^n}{m^{k-1}} S_{2,\lambda}(n,m) \right) \frac{t^n}{n!}.$$

Proof. From (10) and (24), we observe

$$\begin{aligned} l_{k,\lambda}(1-e_\lambda(-(1-u)t)) &= \sum_{m=1}^{\infty} \frac{(1)_{m,\frac{1}{\lambda}}(-\lambda)^{m-1}}{(m-1)!m^k} (1-e_\lambda(-(1-u)t))^m \\ &= \sum_{m=1}^{\infty} \frac{(1)_{m,\frac{1}{\lambda}}(-\lambda)^{m-1}}{m^{k-1}} (-1)^m \frac{(e_\lambda(-(1-u)t)-1)^m}{m!} \\ &= \sum_{n=1}^{\infty} \left(\sum_{m=1}^{n} \frac{(1)_{m,\frac{1}{\lambda}}(-1)^{n-1}\lambda^{m-1}(1-u)^n}{m^{k-1}} S_{2,\lambda}(n,m) \right) \frac{t^n}{n!}. \end{aligned}$$

Thus, we have the desired result. □

Theorem 3. *For $n \geq 0$, $k \in \mathbb{Z}$ we have*

$$h_{n,\lambda}^{(k)}(1|u) - uh_{n,\lambda}^{(k)}(u) = \sum_{m=1}^{n+1} \frac{(1)_{m,\frac{1}{\lambda}}(-1)^n\lambda^{m-1}(1-u)^{n+1}}{m^{k-1}(n+1)} S_{2,\lambda}(n+1,m).$$

Proof. By using the first identity of Theorem 1 with $x = 1$ and (25) with $x = 0$, we have

$$\begin{aligned} \frac{1}{t} l_{k,\lambda}(1-e_\lambda(-(1-u)t)) &= (e_\lambda(t)-u) \sum_{l=0}^{\infty} h_{l,\lambda}^{(k)}(u) \frac{t^l}{l!} \\ &= \left(\sum_{m=0}^{\infty} (1)_{m,\lambda} \frac{t^m}{m!} - u \right) \sum_{l=0}^{\infty} h_{l,\lambda}^{(k)}(u) \frac{t^l}{l!} \\ &= \sum_{n=0}^{\infty} \left(\sum_{l=0}^{n} \binom{n}{l} (1)_{n-l,\lambda} h_{l,\lambda}^{(k)}(u) - uh_{n,\lambda}^{(k)}(u) \right) \frac{t^n}{n!} \\ &= \sum_{n=0}^{\infty} \left(h_{n,\lambda}^{(k)}(1|u) - uh_{n,\lambda}^{(k)}(u) \right) \frac{t^n}{n!}. \end{aligned} \tag{26}$$

On the other hand, from Lemma 2, we get

$$\begin{aligned} \frac{1}{t} l_{k,\lambda}(1-e_\lambda(-(1-u)t)) &= \frac{1}{t} \sum_{n=1}^{\infty} \left(\sum_{m=1}^{n} \frac{(1)_{m,\frac{1}{\lambda}}(-1)^{n-1}\lambda^{m-1}(1-u)^n}{m^{k-1}} S_{2,\lambda}(n,m) \right) \frac{t^n}{n!} \\ &= \sum_{n=0}^{\infty} \left(\sum_{m=1}^{n+1} \frac{(1)_{m,\frac{1}{\lambda}}(-1)^n\lambda^{m-1}(1-u)^{n+1}}{m^{k-1}(n+1)} S_{2,\lambda}(n+1,m) \right) \frac{t^n}{n!}. \end{aligned} \tag{27}$$

Therefore, by comparing the coefficients of (26) and (27), we have the desired result. □

Corollary 4. *For $n \geq 0$, we have*

$$\sum_{m=1}^{n+1} \frac{(1)_{m,\frac{1}{\lambda}}(-1)^n\lambda^{m-1}(1-u)^{n+1}}{(n+1)} S_{2,\lambda}(n+1,m) = \begin{cases} 1-u, & \text{if } n=0, \\ 0, & \text{otherwise.} \end{cases}$$

Proof. When $k=1$, we observe from (27) that

$$\begin{aligned} \frac{1}{t} l_{1,\lambda}(1-e_\lambda(-(1-u)t)) &= 1-u \\ &= \sum_{n=0}^{\infty}\left(\sum_{m=1}^{n+1} \frac{(1)_{m,\frac{1}{\lambda}}(-1)^n\lambda^{m-1}(1-u)^{n+1}}{(n+1)} S_{2,\lambda}(n+1,m)\right)\frac{t^n}{n!} \end{aligned} \tag{28}$$

Therefore, by comparing the coefficients on both side of (28), we have what we want. □

Now, we note that

$$\frac{d}{dt}e_\lambda(t) = e_\lambda^{1-\lambda}(t) \quad \text{and} \quad \frac{d}{dt} l_{k,\lambda}(t) = \frac{1}{t} l_{k-1,\lambda}(t) \tag{29}$$

Moreover, we have

$$l_{k,\lambda}(x) = \int_0^x \frac{1}{t} \underbrace{\int_0^t \frac{1}{t} \cdots \int_0^t \frac{1}{t}}_{(k-2)-times} l_{1,\lambda}(x)\, dt\, dt \cdots dt. \tag{30}$$

From (29) and (30), for $n \geq 2$ and $k \in \mathbb{Z}$, we have

$$\begin{aligned} l_{k,\lambda}(1-e_\lambda(-(1-u)t)) &= (1-u)\int_0^x \frac{-(1-u)e_\lambda^{1-\lambda}(-(1-u)t)}{e_\lambda(-(1-u)t)-1} \\ &\times \underbrace{\int_0^x \frac{-(1-u)e_\lambda^{1-\lambda}(-(1-u)t)}{e_\lambda(-(1-u)t)-1} \cdots \int_0^x \frac{-(1-u)e_\lambda^{1-\lambda}(-(1-u)t)}{e_\lambda(-(1-u)t)-1}}_{(k-2)-times} t\; dt\, dt \cdots dt. \end{aligned} \tag{31}$$

Theorem 5. *For $n \geq 1$ and $k=2$, we have*

$$h_{n,\lambda}^{(2)}(u) = \sum_{l=0}^{n}\binom{n}{l}\frac{(-1)^l(1-u)^l}{l+1} B_{l,\lambda}(1-\lambda) h_{n-l,\lambda}(u).$$

Proof. By using (5) and (31), we have

$$\begin{aligned} \sum_{n=0}^{\infty} h_{n,\lambda}^{(2)}(u)\frac{x^n}{n!} &= \frac{1-u}{x(e_\lambda(x)-u)}\int_0^x \frac{-(1-u)t}{e_\lambda(-(1-u)t)-1} e_\lambda^{1-\lambda}(-(1-u)t)dt \\ &= \frac{1-u}{x(e_\lambda(x)-u)}\int_0^x \sum_{l=0}^{\infty} B_{l,\lambda}(1-\lambda)\frac{(-(1-u)t)^l}{l!}dt \\ &= \frac{1-u}{e_\lambda(x)-u}\sum_{l=0}^{\infty}\frac{(-1)^l(1-u)^l B_{l,\lambda}(1-\lambda)}{l+1}\frac{x^l}{l!} \\ &= \sum_{n=0}^{\infty}\left(\sum_{l=0}^{n}\binom{n}{l}\frac{(-1)^l(1-u)^l}{l+1}B_{l,\lambda}(1-\lambda)h_{n-l,\lambda}(u)\right)\frac{x^n}{n!}\Bigg) \end{aligned} \tag{32}$$

Therefore, by comparing coefficients of both side of (32). □

We have the following theorem by using the similar method in Theorem 5.

Theorem 6. *For $n \geq 0$, $k \in \mathbb{Z}$ we have*

$$h_{n,\lambda}^{(k)}(u) = \sum_{m=0}^{n} \binom{n}{m} \sum_{m_1+m_2+\cdots+m_{k-1}=m} (-1)^m (1-u)^m \binom{m}{m_1\ m_2 \cdots m_{k-1}} \times \frac{B_{m_1,\lambda}(1-\lambda)}{m_1+1} \cdots \frac{B_{m_{k-1},\lambda}(1-\lambda)}{m_1+m_2+\cdots m_{k-1}+1} h_{n-m,\lambda}(u).$$

Theorem 7. *For $n \geq 0$, $k \in \mathbb{Z}$, we have*

$$h_{n,\lambda}^{(k)}(x|u) = \sum_{l=0}^{n} \sum_{m=0}^{l} \binom{n}{l} (x)_m S_{2,\lambda}(l,m) h_{n-l,\lambda}^{(k)}(u).$$

Proof. From (24) and (25), we have

$$\begin{aligned} \sum_{n=0}^{\infty} h_{n,\lambda}^{(k)}(x|u)\frac{t^n}{n!} &= \frac{l_{k,\lambda}\left(1-e_\lambda(-(1-u)t)\right)}{t(e_\lambda(t)-1)} \left(e_\lambda(t)-1+1\right)^x \\ &= \sum_{j=0}^{\infty} h_{j,\lambda}^{(k)}(u)\frac{t^j}{j!} \sum_{m=0}^{\infty} (x)_m \frac{(e_\lambda(t)-1)^m}{m!} \\ &= \sum_{j=0}^{\infty} h_{j,\lambda}^{(k)}(u)\frac{t^j}{j!} \sum_{l=0}^{\infty} \sum_{m=0}^{l} (x)_m S_{2,\lambda}(l,m)\frac{t^l}{l!} \\ &= \sum_{n=0}^{\infty} \left(\sum_{l=0}^{n} \sum_{m=0}^{l} \binom{n}{l} (x)_m S_{2,\lambda}(l,m) h_{n-l,\lambda}^{(k)}(u) \right) \frac{t^n}{n!}. \end{aligned} \tag{33}$$

Therefore, by comparing the coefficients of (33), we get the desired result. □

3. Some identities of degenerate poly-Frobenius-Euler polynomials arising from degenerate Sheffer sequences

In this section, we study combinatorial identities related to the new type of the degenerate poly-Frobenius-Euler polynomials and numbers by using the degenerate Sheffer sequences.

From (19) and (25), we observe that $h_{n,\lambda}^{(k)}(x|u)$ is the degenerate λ-Sheffer sequences for

$$\left(\frac{t(e_\lambda(t)-u)}{l_{k,\lambda}(1-e_\lambda(-(1-u)t))},\ t \right). \tag{34}$$

Theorem 8. *For $n \geq 0$, we have*

$$h_{n,\lambda}^{(k)}(x|u) = \sum_{s=0}^{n} \left(\binom{n}{s} h_{n-s,\lambda}^{(k)}(u) \right) (x)_{s,\lambda}.$$

Proof. From (4), (19) and (34) we consider the following two Sheffer sequence as follows:

$$h_{n,\lambda}^{(k)}(x|u) \sim \left(\frac{t(e_\lambda(t)-u)}{l_{k,\lambda}(1-e_\lambda(-(1-u)t))},\ t \right)_\lambda \quad \text{and} \quad (x)_{n,\lambda} \sim \left(1,\ t\right)_\lambda. \tag{35}$$

From (20) and (35), we have

$$h_{n,\lambda}^{(k)}(x|u) = \sum_{s=0}^{n} c_{n,s} (x)_{s,\lambda}. \tag{36}$$

where

$$
\begin{aligned}
c_{n,s} &= \frac{1}{s!}\left\langle \left(\frac{l_{k,\lambda}(1-e_\lambda(-(1-u)t))}{t(e_\lambda(t)-u)} \right) t^s \,\middle|\, (x)_{n,\lambda} \right\rangle_\lambda \\
&= \frac{1}{s!}\left\langle \left(\frac{l_{k,\lambda}(1-e_\lambda(-(1-u)t))}{t(e_\lambda(t)-u)} \right) \,\middle|\, (t^s)_\lambda (x)_{n,\lambda} \right\rangle_\lambda \\
&= \frac{1}{s!}\binom{n}{s} s! \left\langle \frac{l_{k,\lambda}(1-e_\lambda(-(1-u)t))}{t(e_\lambda(t)-u)} \,\middle|\, (x)_{n-s,\lambda} \right\rangle_\lambda = \binom{n}{s} h^{(k)}_{n-s,\lambda}(u).
\end{aligned} \tag{37}
$$

From (36) and (37), we have the desired result.

□

Theorem 9. *For $n \geq 0$, we have*

$$
h^{(k)}_{n,\lambda}(x|u) = \sum_{s=0}^{n} \left(\sum_{l=s}^{n} \binom{n}{l} S_{1,\lambda}(l,s) h^{(k)}_{n-l,\lambda}(u) \right) Bel_{s,\lambda}(x),
$$

where $Bel_{n,\lambda}(x)$ are the degenerate Bell polynomials.

Proof. From (9), (19) and (34), we consider two degenerate Sheffer sequences as follows:

$$
h^{(k)}_{n,\lambda}(x|u) \sim \left(\frac{t(e_\lambda(t)-u)}{l_{k,\lambda}(1-e_\lambda(-(1-u)t))},\ t \right)_\lambda \quad \text{and} \quad Bel_{n,\lambda} \sim (1,\ \log_\lambda(1+t))_\lambda. \tag{38}
$$

From (20), (23) and (38), we have

$$
h^{(k)}_{n,\lambda}(x|u) = \sum_{s=0}^{n} c_{n,s} Bel_{s,\lambda}(x), \tag{39}
$$

where

$$
\begin{aligned}
c_{n,s} &= \frac{1}{s!}\left\langle \left(\frac{t(e_\lambda(t)-u)}{l_{k,\lambda}(1-e_\lambda(-(1-u)t))} \right)^{-1} \left(\log_\lambda(1+t)\right)^s \,\middle|\, (x)_{n,\lambda} \right\rangle_\lambda \\
&= \left\langle \frac{l_{k,\lambda}(1-e_\lambda(-(1-u)t))}{t(e_\lambda(t)-u)} \,\middle|\, \left(\frac{(\log_\lambda(1+t))^s}{s!} \right)_\lambda (x)_{n,\lambda} \right\rangle_\lambda \\
&= \sum_{l=s}^{n} \binom{n}{l} S_{1,\lambda}(l,s) h^{(k)}_{n-l,\lambda}(u).
\end{aligned} \tag{40}
$$

Therefore, from (39) and (40), we get what we want.

□

Theorem 10. *For $n \geq 0$, we have*

$$
h^{(k)}_{n,\lambda}(x|u) = \sum_{s=0}^{n} \left(\sum_{l=s}^{n} \binom{n}{l} S_{2,\lambda}(l,s) h^{(k)}_{n-l,\lambda}(u) \right) (x)_s.
$$

Proof. Since

$$
e^x_\lambda(\log(1+t)) = (1+t)^x = \sum_{n=0}^{\infty} (x)_n \frac{t^n}{n!}, \tag{41}
$$

we have $(x)_n \sim (1, e_\lambda(t)-1)_\lambda$ by using (19).

From (34) and (41), we consider the two degenerate Sheffer sequences as follows:

$$h_{n,\lambda}^{(k)}(x|u) \sim \left(\frac{t(e_\lambda(t)-u)}{l_{k,\lambda}(1-e_\lambda(-(1-u)t))},\ t\right)_\lambda \quad \text{and} \quad (x)_n \sim (1,\ e_\lambda(t)-1)_\lambda. \tag{42}$$

Thus from (20), (24) and (42), we have

$$h_{n,\lambda}^{(k)}(x|u) = \sum_{s=0}^{n} c_{n,s}(x)_s,\ (n \geq 0), \tag{43}$$

where

$$\begin{aligned} c_{n,s} &= \frac{1}{s!}\left\langle \left(\frac{t(e_\lambda(t)-u)}{l_{k,\lambda}(1-e_\lambda(-(1-u)t))}\right)^{-1} (e_\lambda(t)-1)^s \,\middle|\, (x)_{n,\lambda} \right\rangle_\lambda \\ &= \left\langle \frac{l_{k,\lambda}(1-e_\lambda(-(1-u)t))}{t(e_\lambda(t)-u)} \,\middle|\, \left(\frac{(e_\lambda(t)-1)^s}{s!}\right)_\lambda (x)_{n,\lambda} \right\rangle_\lambda \\ &= \sum_{l=s}^{n} \binom{n}{l} S_{2,\lambda}(l,s) h_{n-l,\lambda}^{(k)}(u). \end{aligned} \tag{44}$$

Therefore, from (43) and (44), we have what we want. □

Theorem 11. *For $n \geq 0$, $k,\ r \in \mathbb{N}$, we have*

$$h_{n,\lambda}^{(k)}(x|u) = \sum_{s=0}^{n} \left(\sum_{m=0}^{n-s} \binom{n}{s} \binom{n-s}{m} \frac{(1)_{m+1,\lambda}}{m+1} h_{n-s-m,\lambda}^{(k)}(u) \right) B_{s,\lambda}(x),$$

where $B_{s,\lambda}(x)$ are the degenerate Bernoulli polynomials.

Proof. From (5), (19) and (34), we have two degenerate Sheffer sequences

$$h_{n,\lambda}^{(k)}(x|u) \sim \left(\frac{t(e_\lambda(t)-u)}{l_{k,\lambda}(1-e_\lambda(-(1-u)-t))},\ t\right)_\lambda \quad \textit{and} \quad B_{n,\lambda}(x) \sim \left(\frac{e_\lambda(t)-1}{t},\ t\right)_\lambda. \tag{45}$$

From (20), (22) and (45), we have

$$h_{n,\lambda}^{(k)}(x|u) = \sum_{s=0}^{n} c_{n,s} B_{s,\lambda}(x), \tag{46}$$

where

$$\begin{aligned} c_{n,s} &= \frac{1}{s!}\left\langle \left(\frac{e_\lambda(t)-1}{t}\right)\left(\frac{l_{k,\lambda}(1-e_\lambda(-(1-u)t)}{t(e_\lambda(t)-u)}\right) t^s \,\middle|\, (x)_{n,\lambda} \right\rangle_\lambda \\ &= \frac{1}{s!}\left\langle \left(\frac{e_\lambda(t)-1}{t}\right)\left(\frac{l_{k,\lambda}(1-e_\lambda(-(1-u)t)}{t(e_\lambda(t)-u)}\right) \,\middle|\, (t^s)_\lambda (x)_{n,\lambda} \right\rangle_\lambda \\ &= \binom{n}{s} \sum_{m=0}^{n-s} (1)_{m+1,\lambda} \frac{1}{(m+1)!} \left\langle \frac{l_{k,\lambda}(1-e_\lambda(-(1-u)t)}{t(e_\lambda(t)-u)} \,\middle|\, (t^m)_\lambda (x)_{n-s,\lambda} \right\rangle_\lambda \\ &= \binom{n}{s} \sum_{m=0}^{n-s} (1)_{m+1,\lambda} \frac{1}{(m+1)!} \binom{n-s}{m} m! \left\langle \frac{l_{k,\lambda}(1-e_\lambda(-(1-u)t)}{t(e_\lambda(t)-u)} \,\middle|\, (x)_{n-s-m,\lambda} \right\rangle_\lambda \\ &= \binom{n}{s} \sum_{m=0}^{n-s} (1)_{m+1,\lambda} \frac{1}{m+1} \binom{n-s}{m} h_{n-s-m,\lambda}^{(k)}(u). \end{aligned} \tag{47}$$

Therefore, from (46) and (47), we have what we want. □

Theorem 12. *For $n \geq 0$, $k,\ r \in \mathbb{N}$, we have*

$$h_{n,\lambda}^{(k)}(x) = \sum_{s=0}^{n} \left(\frac{1}{2^r} \binom{n}{s} \sum_{m=0}^{n-s} \sum_{j=0}^{r} \binom{n-s}{m} \binom{r}{j} (j)_{n-s-m,\lambda} h_{m,\lambda}^{(k)}(u) \right) E_{s,\lambda}^{(r)}(x),$$

where $E_{s,\lambda}^{(r)}(x)$ are the degenerate Euler polynomials of order r.

Proof. From (5), (19) and (34), we have two degenerate Sheffer sequences as follows:

$$h_{n,\lambda}^{(k)}(x|u) \sim \left(\frac{t(e_\lambda(t)-u)}{l_{k,\lambda}(1-e_\lambda(-(1-u)t))},\ t \right)_\lambda \quad \text{and} \quad E_{n,\lambda}^{(r)}(x) \sim \left(\left(\frac{e_\lambda(t)+1}{2} \right)^r,\ t \right)_\lambda . \tag{48}$$

From (4), (20) and (48), we have

$$h_{n,\lambda}^{(k)}(x|u) = \sum_{s=0}^{n} c_{n,s} E_{s,\lambda}^{(r)}(x), \tag{49}$$

where

$$\begin{aligned} c_{n,s} &= \frac{1}{s!} \left\langle \left(\frac{e_\lambda(t)+1}{2} \right)^r \left(\frac{l_{k,\lambda}(1-e_\lambda(-(1-u)t))}{t(e_\lambda(t)-u)} \right) t^s \;\middle|\; (x)_{n,\lambda} \right\rangle_\lambda \\ &= \frac{1}{2^r} \binom{n}{s} \left\langle (e_\lambda(t)+1)^r \left(\frac{l_{k,\lambda}(1-e_\lambda(-(1-u)t))}{t(e_\lambda(t)-u)} \right) \middle| \; (x)_{n-s,\lambda} \right\rangle_\lambda \\ &= \frac{1}{2^r} \binom{n}{s} \sum_{m=0}^{n-s} \binom{n-s}{m} h_{m,\lambda}^{(k)}(u) \langle (e_\lambda(t)+1)^r \mid (x)_{n-s-m,\lambda} \rangle_\lambda \\ &= \frac{1}{2^r} \binom{n}{s} \sum_{m=0}^{n-s} \binom{n-s}{m} h_{m,\lambda}^{(k)}(u) \sum_{j=0}^{r} \binom{r}{j} (j)_{n-s-m,\lambda} . \end{aligned} \tag{50}$$

Therefore, from (49) and (50) we have the desired result. □

Theorem 13. *For $n \geq 0$, $k, r \in \mathbb{Z}$, we have*

$$h_{n,\lambda}^{(k)}(x|u) = \sum_{s=0}^{n} \left(\sum_{l=s}^{n} \sum_{m=0}^{n-l} \binom{n}{l} \binom{n-l}{m} \frac{(1)_{n-l-m+1,\lambda}}{n-l-m+1} S_{2,\lambda}(l,s) h_{m,\lambda}^{(k)}(u) \right) D_{s,\lambda}(x),$$

where $D_{n,\lambda}(x)$ are the degenerate Daehee polynomials.

Proof. From (7), (19) and (34),, we consider the following two degenerate Sheffer sequences.

$$h_{n,\lambda}^{(k)}(x|u) \sim \left(\frac{t(e_\lambda(t)-u)}{l_{k,\lambda}(1-e_\lambda(-(1-u)t))},\ t \right)_\lambda \quad \text{and} \quad D_{n,\lambda}(x) \sim \left(\frac{e_\lambda(t)-1}{t},\ e_\lambda(t)-1 \right)_\lambda . \tag{51}$$

From (20), (24) and (51), we have

$$h_{n,\lambda}(x|u) = \sum_{s=0}^{n} c_{n,s} D_{s,\lambda}(x), \tag{52}$$

where

$$
\begin{aligned}
c_{n,s} &= \frac{1}{s!}\left\langle \left(\frac{e_\lambda(t)-1}{t}\right)\left(\frac{l_{k,\lambda}(1-e_\lambda(-(1-u)t))}{t(e_\lambda(t)-u)}\right)(e_\lambda(t)-1)^s \middle| (x)_{n,\lambda}\right\rangle_\lambda \\
&= \left\langle \left(\frac{e_\lambda(t)-1}{t}\right)\left(\frac{l_{k,\lambda}(1-e_\lambda(-(1-u)t))}{t(e_\lambda(t)-u)}\right) \middle| \left(\frac{(e_\lambda(t)-1)^s}{s!}\right)_\lambda (x)_{n,\lambda}\right\rangle_\lambda \\
&= \sum_{l=s}^{n}\binom{n}{l}S_{2,\lambda}(l,s)\left\langle \left(\frac{e_\lambda(t)-1}{t}\right)\left(\frac{l_{k,\lambda}(1-e_\lambda(-(1-u)t))}{t(e_\lambda(t)-u)}\right) \middle| (x)_{n-l,\lambda}\right\rangle_\lambda \\
&= \sum_{l=s}^{n}\binom{n}{l}S_{2,\lambda}(l,s)\sum_{m=0}^{n-l}\binom{n-l}{m}h_{m,\lambda}^{(k)}(u)\left\langle \sum_{j=0}^{\infty}\frac{(1)_{j+1,\lambda}}{j+1}\frac{t^j}{j!} \middle| (x)_{n-l-m,\lambda}\right\rangle_\lambda \\
&= \sum_{l=s}^{n}\binom{n}{l}S_{2,\lambda}(l,s)\sum_{m=0}^{n-l}\binom{n-l}{m}h_{m,\lambda}^{(k)}(u)\,\frac{(1)_{n-l-m+1,\lambda}}{n-l-m+1}.
\end{aligned} \tag{53}
$$

Therefore, from (52), (53), we have what we want.

□

Acknowledgments

The authors thank Jangjeon Institute for Mathematical Science for the support of this research.

Funding

This work was supported by the Basic Science Research Program, the National Research Foundation of Korea, the Ministry of Education, (NRF-2018R1D1A1B07049584).

References

[1] Araci, S.; Acikgoz, M. *A note on the Frobenius-Euler numbers and polynomials associated with Bernstein polynomials,* Adv. Stud. Contemp. Math. (Kyungshang) (2012), **22(3)**, 399-406.

[2] Carlitz, L. *Degenerate Stirling, Bernoulli and Eulerian numbers,* Utilitas Math. (1979), **15**, 51-88.

[3] Carlitz, L. *Weighted Stirling numbers of the first and second kind,* Fibonacci Quart. (1980), **18**, 147-162.

[4] Comtet, L. *Advanced combinatorics. The art of finite and infinite expansions,* Revised and enlarged edition. D. Reidel Publishing Co., Dordrecht, 1974. xi+343 pp. ISBN: 90-277-0441-4 05-02.

[5] Dere, R.; Simsek, Y. *Applications of umbral algebra to some special polynomials.* Adv. Stud. Contemp. Math. (Kyungshang) (2012), **22**, no. 3, 433-438.

[6] Ernst, T. *Examples of a q-umbral calculus.* Adv. Stud. Contemp. Math. (Kyungshang) (2008), **16** , no. 1, 1-22.

[7] Khan, W. A. *A new class of degenerate Frobenius-Euler Hermite polynomials,* Adv. Stud. Contemp. Math. (2018), **30(4)**, 567-576.

[8] Kim, D. S.; Kim, T.; Kwon, H.-I.; Mansour T. *Nonlinear differential equation for Korobov numbers,* Adv. Stud. Contemp. Math. (Kyungshang) (2016), **26(4)**, 733-740.

[9] Kim, D. S.; Kim, T.; Seo, J. J. *Higher-order Bernoulli, Frobenius-Euler and Euler polynomials,* J. Comput. Anal. Appl. (2014), **17(1)**, 147-155.

[10] Kim, D. S.; Kim, T. *A note on a new type of degenerate Bernoulli numbers,* Russ. J. Math. Phys. (2020), **27(2)**, 227-235.

[11] Kim, D. S.; Kim, T. *Some identities of Bell polynomials,* Sci. China Math. **58** (2015), **no. 10**, 2095-2104.

[12] Kim, D. S.; Kim, T. *Some identities of Bernoulli and Euler polynomials arising from umbral calculus,* Adv. Stud. Contemp. Math. (Kyungshang) (2013), **23(1)**, 159-171.

[13] Kim, D. S.; Kim, T. *Degenerate Sheffer sequence and λ-Sheffer sequence,* J. Math. Anal. Appl.(2020), **23**, 124521.

[14] Kim, D. S.; Kim, T.; Kwon, H. I.; Mansour, T. *Degenerate poly-Bernoulli polynomials with umbral calculus viewpoit.* Jour. of Inequalities and Applications (2015), **2015**, 228.

[15] Kim, T.; Kim, D. S. *Some identities of extended degenerate r-central Bell polynomials arising from umbral calculus.* Rev. R. Acad. Cienc. Exactas. Nat. Ser. A Mat. RACSAM (2020), **114(1)**, 19pp.

[16] Kim, H. K. *Note on a new type of degenerate poly-type2 Euler polynomials and numbers*, submitted.

[17] Kim, T.; Kim, D. S.; Kwon, J.; Lee, H. *Degenerate polyexponential functions and type 2 degenerate poly-Bernoulli numbers and polynomials.* Adv. Difference Equ. (2020), Paper No. **168**, 12pp.

[18] Kim, T. *A note on degenerate Stirling polynomials of the second kind,* Proc. Jangjeon Math. Soc. (2017), **20(3)**, 319-331.

[19] Kim, T.; Kim, D. S. *An identity of symmetry for the degenerate Frobenius-Euler polynomials,* Math. Slovaca. (2018), **68(1)**, 239-243.

[20] Kim, T.; Kim, D. S.; Kim, H. Y.; Kwon, J. *Some results on degenerate Daehee and Bernoulli numbers and polynomials,* Adv. Difference Equ. (2020), **2020:311**, 13pp.

[21] Kim, T.; Kim, D. S.; Kim, H. Y.; Kwon, J. *Some identities of degenerate Bell polynomials,* Mathematics (2020), **8(40)** , 8pp.

[22] Kim, T.; Kim, D. S.; Lee, H.; Kwon, J. *A note no some identities of new type degenerate Bell polynomials*, Mathematics (2019) **7**, 1086, 12pp.

[23] Kim, T.; Kim, D. S. *Note on the Degenerate Gamma Function,* Russ. J. Math. Phys. (2020) **27(3)**, 352-358.

[24] Kim, T.; Mansour, T. *Umbral calculus associated with Frobenius-type Eulerian polynomials.* Russ. J. Math. Phys. (2014), **21(4)**, 484-493.

[25] Kurt, B.; Simsek, Y. *On the generalized Apostol type Frobenius Euler polynomials. Advances in Differences equations,* (2013), 1-9.

[26] Kurt, B.; Simsek, Y. *Frobenius Euler type polynomials related to Hermite-Bernoulli polynomials. Proceedings of the International Conferences on Numerical Analysis and Applied Mathematics,* Amer, Inst. Phys. Conf. Proc. (2011), 1389, 385-388.

[27] Kwaśniewski, A. K. *On ψ-umbral extensions of Stirling numbers and Dobinski-like formulas.* Adv. Stud. Contemp. Math. (Kyungshang) (2006), **12(1)**, 73-100.

[28] Ma. Y.; Kim, D. S.; Kim, T.; Kim, H. Y.; Lee, H. S. *Some identities of Lah-Bell polynomials*, Adv. Difference Equ., Paper No. **2020:510** (2020), 10pp.

[29] Roman, S. *More on the umbral calculus, with emphasis on the q-umbral calculus.* J. Math. Anal. Appl. (1985), **107(1)**, 222-254.

[30] Roman, S. *The umbral calculus.* Pure and Applied Mathematics, 111. Academic Press, Inc. [Harcourt Brace Jovanovich, Publishers], New York, 1984. x+193 pp.

DEPARTMENT OF MATHEMATICS EDUCATION, DAEGU CATHOLIC UNIVERSITY, GYEONGSAN 38430, REPUBLIC OF KOREA

E-mail address: hkkim@cu.ac.kr

DEPARTMENT OF MATHEMATICS AND NATURAL SCIENCES, PRINCE MOHAMMAD BIN FAHD UNIVERSITY, P.O BOX 1664, AL KHOBAR 31952, SAUDI ARABIA

E-mail address: wkhan1@pmu.edu.sa