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SOME IDENTITIES OF CARLITZ'S TYPE DEGENERATE q-CHANGHEE POLYNOMIALS AND NUMBERS

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ABSTRACT. Recently several authors have studied the degenerate q-special polynomials related to Carlitz's degenerate polynomials. In this paper, we introduce the Carlitz's type degenerate q-Changhee numbers and polynomials. And, we study some explicit identities and properties for the Carlitz's type degenerate q-Changhee numbers and polynomials arising from p-adic q-integral on $\mathbb{Z}_p$.

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1. INTRODUCTION

Throughout this paper p is a fixed odd prime number. We use the notations $\mathbb{Z}_p$ to express the ring of p-adic integers , $\mathbb{Q}_p$ the field of p-adic rational numbers and $\mathbb{C}_p$ the completion of algebraic closure of $\mathbb{Q}_p$. The p-adic norm $|\cdot|_p$ is normalized as $|p|_p = \frac{1}{p}$. For $q, x \in \mathbb{C}_p$ with $|q-1|_p < p^{-\frac{1}{p-1}}$. We define the *$q$-analogue of a number* x to be $[x]_q = \frac{1-q^x}{1-q}$. Note that $\lim_{q\to 1}[x]_q = x$.

Let $C(\mathbb{Z}_p)$ be the set of all $\mathbb{C}_p$-valued continuous functions on $\mathbb{Z}_p$. For a $f \in C(\mathbb{Z}_p)$, Kim introduced the fermionic p-adic q-integral $I_{-q}(f)$ on $\mathbb{Z}_p$ (see [17, 18, 20]).

$$I_{-q}(f) = \int_{\mathbb{Z}_p} f(x)d\mu_{-q}(x) = \lim_{N\to\infty} \frac{1}{[p^N]_{-q}} \sum_{x=0}^{p^N-1} f(x)(-q)^x. \tag{1}$$

By (1), the following distribution relation is well-known in [18].

$$\begin{aligned} \int_{\mathbb{Z}_p} f(x)d\mu_{-q}(y) &= \lim_{N\to\infty} \frac{1}{[p^N]_{-q}} \sum_{x=0}^{p^N-1} f(x)(-q)^x \\ &= \lim_{N\to\infty} \frac{1}{[dp^N]_{-q}} \sum_{a=0}^{d-1} \sum_{x=0}^{p^N-1} f(a+dx)(-q)^{a+dx}, \end{aligned} \tag{2}$$

where $d \in \mathbb{N}$ with $d \equiv 1$ $(mod\ 2)$.

The *q-Euler polynomials* $E_{n,q}(x)$ are defined by the fermionic p-adic q-integral on $\mathbb{Z}_p$,

$$\int_{\mathbb{Z}_p} e^{[x+y]_q t} d\mu_{-q}(y) = \sum_{n=0}^{\infty} E_{n,q}(x)\frac{t^n}{n!}, \quad \text{(see [1-4, 9-15, 17, 22, 24])}. \tag{3}$$

In particular, if $x = 0$, then $E_{n,q} = E_{n,q}(0)$ are called the *q-Euler numbers.* By (3), we know that

$$\int_{\mathbb{Z}_p} [x+y]_q^n \, d\mu_{-q}(x), \ n \in \mathbb{N} \cup \{0\}. \tag{4}$$

For $r \in \mathbb{N}$, *higher-order q-Euler polynomials* $E^r_{n,q}(x)$ are defined by

$$\int_{\mathbb{Z}_p} \cdots \int_{\mathbb{Z}_p} e^{[x_1+\cdots+x_r+x]_q} d\mu_{-q}(x_1)\cdots d\mu_{-q}(x_r) = \sum_{n=0}^{\infty} E^{(r)}_{n,q}(x)\frac{t^n}{n!}, \ \text{(see [1,9,12,15,16])}. \tag{5}$$

The *Bell polynomials* are given by the generating function

$$\sum_{n=0}^{\infty} Bel_n(x)\frac{t^n}{n!} = e^{x(e^t-1)} \ \text{(see [7, 10])}.$$

When $x = 1$, $Bel_n = Bel_n(1)$ are called the *Bell numbers.*

The Stirling number of the first kind is defined by

$$(x)_m = \sum_{n=m}^{\infty} S_1(n,m)x^n, \ \text{(see [5, 7])}, \tag{6}$$

where $(x)_0 = 1, (x)_m = x(x-1)\cdots(x-m+1), (m \geq 1)$.

By (4), we easily obtain

$$\frac{1}{m!}(\log(1+t))^m = \sum_{n=m}^{\infty} S_1(n,m)\frac{t^n}{n!}, \ \text{(see [5, 7])}. \tag{7}$$

In the inverse expression to (4), the Stirling number of the second kind in the following:

$$x^m = \sum_{n=m}^{\infty} S_2(n,m)(x)_n, \ \text{(see [5, 7])}. \tag{8}$$

From (6), we can easily derive the following generating function.

$$\frac{1}{m!}(e^t-1)^m = \sum_{n=m}^{\infty} S_2(n,m)\frac{t^n}{n!}, \ \text{(see [5, 7])}. \tag{9}$$

The q-Changhee polynomials $Ch_{n,q}(x)$ are defined by the generating function

$$\int_{\mathbb{Z}_p} (1+t)^{[x+y]_q} d\mu_{-q}(y) = \sum_{n=0}^{\infty} Ch_{n,q}(x)\frac{t^n}{n!}, \ \text{(see [6, 8, 24, 26])}. \tag{10}$$

In the special case $x = 0$, $Ch_{n,q} = Ch_{n,q}(0)$ are called the *q-Changhee numbers.*

Recently, several authors studied the degenerate q-special polynomials related to Carlitz's degenerate exponential function

$$e^x_\lambda(t) = (1+\lambda t)^{\frac{x}{\lambda}},$$

(see [11-25]). Note that $\lim_{\lambda\to 0} e^x_\lambda(t) = e^{xt}$.

In this paper, we introduce the Carlitz's type degenerate q-Changhee numbers and polynomials, and investigate some interesting identities and properties for the Carlitz's type degenerate q-Changhee numbers and polynomials arising from p-adic q-integral on $\mathbb{Z}_p$. In addition, we also define the higher-order q-Changhee polynomials and numbers, find some relations between the Stirling number of the first and second numbers, higher-order q-Euler polynomials and these polynomials.

2. Carlitz's type degenerate q-Changhee polynomials and numbers

In this section, we assume that $\lambda, t \in \mathbb{C}_p$ with $|\lambda t|_p < p^{-\frac{1}{p-1}}$.

In the viewpoint of (1.2), we define the Carlitz's type degenerate q-Changhee polynomials which are given by the generating function to be

$$\int_{\mathbb{Z}_p} \left(1+\log(1+\lambda t)^{\frac{1}{\lambda}}\right)^{[x+y]_q} d\mu_{-q}(y) = \sum_{n=0}^{\infty} Ch_{n,q,\lambda}(x)\frac{t^n}{n!}, \ (n \geq 0). \tag{11}$$

Substituting t by $\frac{1}{\lambda}\log(1+\lambda t)$ in (10), we get

$$\begin{aligned} \int_{\mathbb{Z}_p} \left(1+\log(1+\lambda t)^{\frac{1}{\lambda}}\right)^{[x+y]_q} d\mu_{-q}(y) &= \sum_{n=0}^{\infty} Ch_{n,q}(x)\frac{\lambda^{-n}}{n!}(\log(1+\lambda t))^n \\ &= \sum_{n=0}^{\infty} Ch_{n,q}(x)\lambda^{-n}\sum_{m=n}^{\infty} S_1(m,n)\frac{\lambda^m t^m}{m!} \\ &= \sum_{m=0}^{\infty}\left(\sum_{n=0}^{m} Ch_{n,q}(x)\lambda^{m-n}S_1(m,n)\right)\frac{t^m}{m!}. \end{aligned} \tag{12}$$

Thus, we have the following theorem.

Theorem 2.1. *For $n \geq 0$, we have*

$$Ch_{m,q,\lambda}(x) = \sum_{n=0}^{m} Ch_{n,q}(x)\lambda^{m-n}S_1(m,n). \tag{13}$$

In the viewpoint of inversion formula, Substitute t by $\frac{e^{\lambda t}-1}{\lambda}$ in (11), we consider that

$$\begin{aligned} \int_{\mathbb{Z}_p} (1+t)^{[x+y]_q} d\mu_{-q}(y) &= \sum_{m=0}^{\infty} Ch_{m,q,\lambda}(x)\lambda^{-m}\frac{1}{m!}(e^{\lambda t}-1)^m \\ &= \sum_{m=0}^{\infty} Ch_{m,q,\lambda}(x)\lambda^{-m}\sum_{n=m}^{\infty} S_2(n,m)\frac{\lambda^n t^n}{n!} \\ &= \sum_{n=0}^{\infty}\left(\sum_{m=0}^{n} Ch_{n,q,\lambda}(x)\lambda^{n-m}S_2(n,m)\right)\frac{t^n}{n!} \end{aligned} \tag{14}$$

Thus, we have the following theorem.

Theorem 2.2. *For $n \geq 0$, we have*

$$Ch_{n,q}(x) = \sum_{m=0}^{n} Ch_{m,q,\lambda}(x)\lambda^{n-m}S_2(n,m). \tag{15}$$

From (1), we have the integratal equation,

$$q\int_{\mathbb{Z}_p} f(x+1)d\mu_{-q}(x) + \int_{\mathbb{Z}_p} f(x)d\mu_{-q}(x) = [2]_q f(0). \tag{16}$$

Therefore, by (11) and (16), we obtain

$$\begin{aligned} &q\int_{\mathbb{Z}_p}\left(1+\log(1+\lambda t)^{\frac{1}{\lambda}}\right)^{[x+1+y]_q} d\mu_{-q}(y) + \int_{\mathbb{Z}_p}\left(1+\log(1+\lambda t)^{\frac{1}{\lambda}}\right)^{[x+y]_q} d\mu_{-q}(y) \\ &= [2]_q\left(1+\log(1+\lambda t)^{\frac{1}{\lambda}}\right)^{[x]_q}. \end{aligned} \tag{17}$$

By (11) and (17), we get the following formula.

$$\begin{aligned} &\sum_{n=0}^{\infty}\left(q\ Ch_{n,q,\lambda}(x+1) + Ch_{n,q,\lambda}(x)\right)\frac{t^n}{n!} \\ &= [2]_q\sum_{m=0}^{\infty}\left([x]_q\right)_m \frac{1}{m!}\left(\log(1+\lambda t)^{\frac{1}{\lambda}}\right)^m \\ &= [2]_q\sum_{m=0}^{\infty}\left([x]_q\right)_m \sum_{n=m}^{\infty}\lambda^{n-m}S_1(n,m)\frac{t^n}{n!} \\ &= [2]_q\sum_{n=0}^{\infty}\sum_{m=0}^{n}\left([x]_q\right)_m \lambda^{n-m}S_1(n,m)\frac{t^n}{n!}. \end{aligned} \tag{18}$$

Comparing the coefficients on the both sides of (18), we arrive at the following theorem.

Theorem 2.3. *For $n \geq 0$, we have*

$$q\ Ch_{n,q,\lambda}(x+1) + Ch_{n,q,\lambda}(x) = [2]_q\sum_{m=0}^{n}\left([x]_q\right)_m \lambda^{n-m}S_1(n,m). \tag{19}$$

Now, by (5) and (6), we observe that

(20)
$$\begin{aligned}
\sum_{n=0}^{\infty} Ch_{n,q,\lambda}(x)\frac{t^n}{n!} &= \int_{\mathbb{Z}_p}\left(1+\log(1+\lambda t)^{\frac{1}{\lambda}}\right)^{[x+y]_q} d\mu_{-q}(y)\\
&= \int_{\mathbb{Z}_p}\sum_{l=0}^{\infty}\binom{[x+y]_q}{l}\left(\frac{1}{\lambda}\log(1+\lambda t)\right)^{l} d\mu_{-q}(y)\\
&= \int_{\mathbb{Z}_p}\sum_{l=0}^{\infty}\Big([x+y]_q\Big)_l \frac{\lambda^{-l}}{l!}\left(\log(1+\lambda t)\right)^{l} d\mu_{-q}(y)\\
&= \int_{\mathbb{Z}_p}\sum_{n=0}^{\infty}\left(\sum_{l=0}^{n}\Big([x+y]_q\Big)_l \lambda^{n-l}S_1(n,l)\right) d\mu_{-q}(y)\frac{t^n}{n!}\\
&= \int_{\mathbb{Z}_p}\sum_{n=0}^{\infty}\left(\sum_{l=0}^{n}\sum_{k=0}^{l} S_1(l,k)[x+y]_q^k\lambda^{n-l}S_1(n,l)\right) d\mu_{-q}(y)\frac{t^n}{n!}\\
&= \sum_{n=0}^{\infty}\left(\sum_{l=0}^{n}\sum_{k=0}^{l} S_1(l,k)S_1(n,l)\lambda^{n-l}E_{k,q}(x)\right)\frac{t^n}{n!}.
\end{aligned}$$

Comparing the coefficients on the both sides of (20), we arrive at the following theorem.

Theorem 2.4. *For $n \geq 0$, we have*

$$Ch_{n,q,\lambda}(x) = \sum_{l=0}^{n}\sum_{k=0}^{l} S_1(l,k)S_1(n,l)\lambda^{n-l}E_{k,q}(x). \tag{21}$$

From (3), we note that

$$\sum_{n=0}^{\infty} E_{n,q}(x)\frac{t^n}{n!} = \int_{\mathbb{Z}_p} e^{[x+y]_q t} d\mu_{-q}(y). \tag{22}$$

Substituting t by $\frac{1}{\lambda}(e^{\lambda(e^t-1)}-1)$ in (11), we get

(23)
$$\begin{aligned}
\sum_{k=0}^{\infty} Ch_{k,q,\lambda}(x)\frac{1}{k!}\left(\frac{1}{\lambda}(e^{\lambda(e^t-1)}-1)\right)^k &= \int_{\mathbb{Z}_p} e^{[x+y]_q t} d\mu_{-q}(y)\\
&= \sum_{n=0}^{\infty} E_{n,q}(x)\frac{t^n}{n!}.
\end{aligned}$$

On the other hand, we observe the left hand of the previous equation (23).

$$
\begin{aligned}
&\sum_{k=0}^{\infty} Ch_{k,q,\lambda}(x)\frac{1}{k!}\left(\frac{1}{\lambda}(e^{\lambda(e^t-1)}-1)\right)^k \\
&=\sum_{k=0}^{\infty} Ch_{k,q,\lambda}(x)\lambda^{-k}\sum_{m=k}^{\infty} S_2(m,k)\frac{1}{m!}\Big(\lambda(e^t-1)\Big)^m \\
&=\sum_{k=0}^{\infty} Ch_{k,q,\lambda}(x)\lambda^{m-k}\sum_{m=k}^{\infty} S_2(m,k)\sum_{n=m}^{\infty} S_2(n,m)\frac{t^n}{n!} \\
&=\sum_{n=0}^{\infty}\left(\sum_{m=0}^{n}\sum_{k=0}^{m} Ch_{k,q,\lambda}(x)\lambda^{m-k}S_2(m,k)S_2(n,m)\right)\frac{t^n}{n!}.
\end{aligned} \tag{24}
$$

Comparing the coefficients on the both sides of (23) and (24), we arrive at the following theorem.

Theorem 2.5. *For $n \geq 0$, we have*

$$
E_{n,q}(x) = \sum_{m=0}^{n}\sum_{k=0}^{m} Ch_{k,q,\lambda}(x)\lambda^{m-k}S_2(m,k)S_2(n,m).
$$

From (1) and (3), we observe that

$$
\begin{aligned}
E_{n,q}(x) &= \int_{\mathbb{Z}_p} [x+y]_q^n d\mu_{-q}(y) \\
&= \lim_{N\to\infty}\frac{1}{[p^N]_{-q}}\sum_{y=0}^{p^N-1}[x+y]_q^n(-q)^y \\
&=\frac{[2]_q}{2(1-q)^n}\sum_{l=0}^{n}\binom{n}{l}(-1)^l q^{lx}\lim_{N\to\infty}\sum_{y=0}^{p^N-1}(-1)^y(q)^{(l+1)y} \\
&=\frac{[2]_q}{(1-q)^n}\sum_{m=0}^{\infty}(-1)^m q^m\sum_{l=0}^{n}\binom{n}{l}(-1)^l q^{l(m+x)} \\
&=[2]_q\sum_{m=0}^{\infty}(-1)^m q^m[m+x]_q^n, \quad \text{(see ([18]).}
\end{aligned} \tag{25}
$$

By Theorem 2.4 and (25), we arrive at the following theorem.

Theorem 2.6. *For $n \geq 0$, we have*

$$
Ch_{n,q,\lambda}(x) = \sum_{m=0}^{\infty}\sum_{l=0}^{n}\sum_{k=0}^{l}[2]_q(-1)^m q^m S_1(l,k)S_1(n,l)\lambda^{n-l}[m+x]_q^k. \tag{26}
$$

Assume that $d \in \mathbb{N}$, with $d \equiv 1 \pmod 2$. Using (2), we get

(27)
$$\begin{aligned}
&\sum_{n=0}^{\infty} Ch_{n,q,\lambda}(x)\frac{t^n}{n!}\\
&=\int_{\mathbb{Z}_p}\left(1+\log(1+\lambda t)^{\frac{1}{\lambda}}\right)^{[x+y]_q} d\mu_{-q}(y)\\
&=\lim_{N\to\infty}\frac{1}{[p^N]_{-q}}\sum_{y=0}^{p^N-1}\left(1+\log(1+\lambda t)^{\frac{1}{\lambda}}\right)^{[x+y]_q}(-q)^y\\
&=\lim_{N\to\infty}\frac{1}{[dp^N]_{-q}}\sum_{a=0}^{d-1}\sum_{y=0}^{p^N-1}\left(1+\log(1+\lambda t)^{\frac{1}{\lambda}}\right)^{[x+a+dy]_q}(-q)^{a+dy}\\
&=\lim_{N\to\infty}\frac{1}{[d]_{-q}}\frac{1}{[p^N]_{-q^d}}\sum_{a=0}^{d-1}\sum_{y=0}^{p^N-1}\left(1+\log(1+\lambda t)^{\frac{1}{\lambda}}\right)^{[d]_q[\frac{a+x}{d}+y]_{q^d}}(-q)^a(-q^d)^y\\
&=\frac{1}{[d]_{-q}}\sum_{a=0}^{d-1}(-q)^a\lim_{N\to\infty}\frac{1}{[p^N]_{-q^d}}\sum_{y=0}^{p^N-1}\sum_{l=0}^{\infty}\binom{[d]_q[\frac{a+x}{d}+y]_{q^d}}{l}\left(\log(1+\lambda t)^{\frac{1}{\lambda}}\right)^l(-q^d)^y\\
&=\frac{1}{[d]_{-q}}\sum_{a=0}^{d-1}(-q)^a\lim_{N\to\infty}\frac{1}{[p^N]_{-q^d}}\\
&\quad\times\sum_{y=0}^{p^N-1}\sum_{n=0}^{\infty}\left(\sum_{l=0}^{n}\sum_{k=0}^{l}S_1(l,k)S_1(n,l)[d]_q^k[\frac{a+x}{d}+y]_{q^d}^k\lambda^{n-l}\right)\frac{t^n}{n!}(-q^d)^y\\
&=\sum_{n=0}^{\infty}\left(\sum_{l=0}^{n}\sum_{k=0}^{l}\frac{1}{[d]_{-q}}\sum_{a=0}^{d-1}(-q)^aS_1(l,k)S_1(n,l)[d]_q^k\lambda^{n-l}E_{k,q^d}\left(\frac{a+x}{d}\right)\right)\frac{t^n}{n!}.
\end{aligned}$$

Comparing the coefficients on the both sides of (27), we obtain the following distribution relation on Carlitz's type degenerate q-Changhee polynomials.

Theorem 2.7. *Let $d \in \mathbb{N}$, with $d \equiv 1$ (mod 2). For $n \geq 0$, we have*

(28)
$$Ch_{n,q,\lambda}(x)=\sum_{l=0}^{n}\sum_{k=0}^{l}\frac{1}{[d]_{-q}}\sum_{a=0}^{d-1}(-q)^aS_1(l,k)S_1(n,l)[d]_q^k\lambda^{n-l}E_{k,q^d}\left(\frac{a+x}{d}\right).$$

For $r \in \mathbb{N}$, the higher-order Carlitz's type degenerate q-Changhee polynomials are given by the multivariate fermionic p-adic q-integral as follows:

$$\begin{aligned}
&\int_{\mathbb{Z}_p}\cdots\int_{\mathbb{Z}_p}\left(1+\log(1+\lambda t)^{\frac{1}{\lambda}}\right)^{[x_1+\cdots+x_r+x]_q} d\mu_{-q}(x_1)\cdots d\mu_{-q}(x_r)\\
&=\sum_{n=0}^{\infty}Ch_{n,q,\lambda}^{(r)}(x)\frac{t^n}{n!},\ (n\geq 0).
\end{aligned} \tag{29}$$

Now, we observe that

$$
\begin{aligned}
(30)\quad & \sum_{n=0}^{\infty} Ch_{n,q,\lambda}^{(r)}(x)\frac{t^n}{n!} \\
&= \int_{\mathbb{Z}_p}\cdots\int_{\mathbb{Z}_p}\left(1+\log(1+\lambda t)^{\frac{1}{\lambda}}\right)^{[x_1+\cdots+x_r+x]_q} d\mu_{-q}(x_1)\cdots d\mu_{-q}(x_r) \\
&= \int_{\mathbb{Z}_p}\cdots\int_{\mathbb{Z}_p}\sum_{l=0}^{\infty}\binom{[x_1+\cdots+x_r+x]_q}{l}\left(\frac{1}{\lambda}\log(1+\lambda t)\right)^l d\mu_{-q}(x_1)\cdots d\mu_{-q}(x_r) \\
&= \int_{\mathbb{Z}_p}\cdots\int_{\mathbb{Z}_p}\sum_{l=0}^{\infty}\left([x_1+\cdots+x_r+x]_q\right)_l\frac{\lambda^{-l}}{l!}\left(\log(1+\lambda t)\right)^l d\mu_{-q}(x_1)\cdots d\mu_{-q}(x_r) \\
&= \int_{\mathbb{Z}_p}\cdots\int_{\mathbb{Z}_p}\sum_{n=0}^{\infty}\left(\sum_{l=0}^{n}\left([x_1+\cdots+x_r+x]_q\right)_l\lambda^{n-l}S_1(n,l)\right) d\mu_{-q}(x_1)\cdots d\mu_{-q}(x_r)\frac{t^n}{n!} \\
&= \int_{\mathbb{Z}_p}\cdots\int_{\mathbb{Z}_p}\sum_{n=0}^{\infty}\left(\sum_{l=0}^{n}\sum_{k=0}^{l}S_1(l,k)[x_1+\cdots+x_r+x]_q^k\lambda^{n-l}S_1(n,l)\right) d\mu_{-q}(x_1)\cdots d\mu_{-q}(x_r)\frac{t^n}{n!} \\
&= \sum_{n=0}^{\infty}\left(\sum_{l=0}^{n}\sum_{k=0}^{l}S_1(l,k)S_1(n,l)\lambda^{n-l}E_{k,q}^{(r)}(x)\right)\frac{t^n}{n!}.
\end{aligned}
$$

Comparing the coefficients on the both sides of (30), we arrive at the following theorem.

Theorem 2.8. *For $n \geq 0$, we have*

$$
(31)\qquad Ch_{n,q,\lambda}^{(r)}(x) = \sum_{l=0}^{n}\sum_{k=0}^{l}S_1(l,k)S_1(n,l)\lambda^{n-l}E_{k,q}^{(r)}(x).
$$

Substituting t by $\frac{1}{\lambda}(e^{\lambda(e^t-1)}-1)$ in (29), we get

$$
\begin{aligned}
(32)\quad & \sum_{k=0}^{\infty} Ch_{k,q,\lambda}^{(r)}(x)\frac{1}{k!}\left(\frac{1}{\lambda}(e^{\lambda(e^t-1)}-1)\right)^k = \int_{\mathbb{Z}_p} e^{[x_1+\cdots+x_r+x]_q t} d\mu_{-q}(x_1)\cdots d\mu_{-q}(x_r) \\
&\qquad = \sum_{n=0}^{\infty} E_{n,q}^{(r)}(x)\frac{t^n}{n!}.
\end{aligned}
$$

On the other hand, we observe the left hand (32) is given by.

$$
\begin{aligned}
&\sum_{k=0}^{\infty} Ch_{k,q,\lambda}^{(r)}(x)\frac{1}{k!}\left(\frac{1}{\lambda}(e^{\lambda(e^t-1)}-1)\right)^k \\
&=\sum_{k=0}^{\infty} Ch_{k,q,\lambda}^{(r)}(x)\lambda^{-k}\sum_{m=k}^{\infty} S_2(m,k)\frac{1}{m!}\Big(\lambda(e^t-1)\Big)^m \\
&=\sum_{k=0}^{\infty} Ch_{k,q,\lambda}^{(r)}(x)\lambda^{m-k}\sum_{m=k}^{\infty} S_2(m,k)\sum_{n=m}^{\infty} S_2(n,m)\frac{t^n}{n!} \\
&=\sum_{n=0}^{\infty}\left(\sum_{m=0}^{n}\sum_{k=0}^{m} Ch_{k,q,\lambda}^{(r)}(x)\lambda^{m-k}S_2(m,k)S_2(n,m)\right)\frac{t^n}{n!}.
\end{aligned} \tag{33}
$$

Comparing the coefficients on the both sides of (32) and (33), we arrive at the following theorem.

Theorem 2.9. *For $n \geq 0$, we have*

$$
E_{n,q}^{(r)}(x) = \sum_{m=0}^{n}\sum_{k=0}^{m} Ch_{k,q,\lambda}^{(r)}(x)\lambda^{m-k}S_2(m,k)S_2(n,m). \tag{34}
$$

3. Conclusion

The Changhee polynomials are one of the important special polynomials and have been investigated some interesting properties of those polynomials and numbers by many researchers (see [1, 6, 8, 24, 26]). In particular, they found that the Changhee polynomials are related closely to the Euler polynomials, the Stirling numbers of the first and second kind.

In this paper, we define the Carlitz's type degenerate q-Changhee polynomials and show that these polynomials can be represented the linear combinations of the q-Changhee polynomials, the Stirling numbers of the first and second kinds and q-Euler polynomials. In addition, we also define the higher-order Carlitz's type degenerate q-Changhee polynomials by using p-adic q-integal on $\mathbb{Z}_p$ and find some interesting properties of those polynomials.

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