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STABILITY OF GENERALIZED CUBE ROOT FUNCTIONAL (GCRF) EQUATIONS IN RANDOM NORMED SPACES

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Abstract. In this paper, we introduce and investigate the stability of generalized cube root Functional (GCRF) Equation

$$C(al + bm + 3.a^{\frac{2}{3}}.l^{\frac{2}{3}}.b^{\frac{1}{3}}.m^{\frac{1}{3}} + 3.a^{\frac{1}{3}}.l^{\frac{1}{3}}.b^{\frac{2}{3}}.m^{\frac{2}{3}}) = a^{\frac{1}{3}}C(l) + b^{\frac{1}{3}}C(m) \qquad (0.1)$$

having solution as $C(l) = l^{\frac{1}{3}}$ in the setting of random normed spaces using direct and fixed point approach. After that, we study the stability of GCRF equation in intutionistic random normed space (IR-NS) and non-archimedean random normed space (NA-RNS).

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1. **Introduction**

The crucial point from where the concept of investigating HUS results of functional equations, differential equations, difference equations is the problem of Ulam [25]. Hyers [8] presented a partial solution to the problem of Ulam. Later, Hyers' theorem was extended and generalized in various forms by many mathematicians Aoki [23], T. Rassias [24], J. Rassias [12] and Gavruta [18]. These results instigated many mathematicians to investigate stability of various types of functional equations in different types of spaces. For detailed review of literature on this field, one can refer ([6], [17], [16], [11], [10], [3], [19], [20], [21]).

The various fundamental stabilities associated with stability of reciprocal adjoint and difference functional equations were demonstrated in ([14], [15]). In recent times, there are many papers published on the stabilities and applications of some multiplicative inverse functional equations, one can refer ([1], [7], [9]).

In this article, we introduce generalized cube root Functional (GCRF) Equation having solution as $C(l) = l^{\frac{1}{3}}$ and then find stability in different random normed spaces.

2. **Preliminaries**

In this segment, we recall few primary definitions which are used in next segment.

Definition 2.1. [5] A function $\tau : [0,1] \times [0,1] \longrightarrow [0,1]$ defined on unit interval is fornamed as triangular norm ($t-norm$) if τ function is commutative, associative, monotonic and satisfies the boundary condition i.e. $\tau(a,1) = a \ \forall \ a \in [0,1]$.

$\xi_\tau^{(n)}$ is defined for $t - norm\ \tau$ as $\xi_\tau^{(n)} = \begin{cases} 1, & \text{if } n = 0 \\ \tau(\xi_\tau^{(n-1)}, \xi) & \text{if } n \geq 1 \end{cases}$ for every $0 \leq \xi \leq 1$ and $n \in \mathbb{N} \cup \{0\}$. A t-norm τ is said to be of Hadzic- type if the family $(\xi_\tau^{(n)})_{n\in\mathbb{N}}$ is equicontinuous at $\xi = 1$.

Definition 2.2. [5] A random normed space (RNS) is a triplet $(\mathbb{S}, \mu, \tau)$, where $\mathbb{S}$ is a linear space, τ is a continuous t-norm, and $\mu : \mathbb{S} \to D^+$ is a function such that, the subsequent axioms satisfied:

(RNS1) $\mu_l(t) = \epsilon_0(t)\ forall\ t > 0$ iff $l = 0$;
(RNS2) $\mu_{\alpha l}(t) = \mu_l(\frac{t}{|\alpha|})\ \forall\ l \in \mathbb{S},\ \alpha$ is non-zero;
(RNS3) $\mu_{l+m}(t+s) \geq\ \tau(\mu_l(t),\ \mu_m(s))\ \forall\ l, m \in \mathbb{S}$ and $0 \leq s, t$.

Some other fundamental notions are available in [5].

3. Generalized HUS of equation (0.1) in RNS

Throughout this section, let us assume that $\mathbb{S}$ is a real Vector spaces, $(\mathbb{Y}, \mu^{'}, T_M)$ is a RNS and $(\mathbb{Z}, \mu, T_M)$ be a complete RNS. For the sake of proving our main results in a concise manner, let $D_C : \mathbb{S} \longrightarrow \mathbb{Z}$ be difference operators defined as follow

$$D_C(l,m) = C(al + bm + 3a^{\frac{2}{3}}l^{\frac{2}{3}}b^{\frac{1}{3}}m^{\frac{1}{3}} + 3a^{\frac{1}{3}}l^{\frac{1}{3}}b^{\frac{2}{3}}m^{\frac{2}{3}}) - a^{\frac{1}{3}}C(l) - b^{\frac{1}{3}}C(m)$$

for all $l, m \in \mathbb{S}$. We examine the generalized Hyers-Ulam-Stability (HUS) of the GCRF equation beneath the minimum t-norm T_M.

Theorem 3.1. *Assume $\Omega : \mathbb{S}^2 \longrightarrow \mathbb{Y}$ be a function such that, for some $0 < \alpha < a^{\frac{1}{3}}$, where $a > 0$*

$$\mu^{'}_{\Omega(al,am)}(t) \geq \mu^{'}_{\alpha\Omega(l,m)}(t) \tag{3.1}$$

and

$$\lim_{n\to\infty} \mu^{'}_{\Omega(a^n l, a^n m)}(a^{\frac{n}{3}}t) = 1$$

for all $l, m \in \mathbb{S}$ and $t > 0$. If $C : \mathbb{S} \longrightarrow \mathbb{Z}$ is a mapping with $C(0) = 0$ such that

$$\mu_{D_C(l,m)}(t) \geq \mu^{'}_{\Omega(l,m)}(t)\ \forall\ l, m \in \mathbb{S},\ t > 0, \tag{3.2}$$

then there exists one and only one GCRF $C_3 : \mathbb{S} \longrightarrow \mathbb{Z}$ such that

$$\mu_{C(l)-C_3(l)}(t) \geq \mu^{'}_{\Omega(l,0)}\left(t(a^{\frac{1}{3}} - \alpha)\right)\ \forall\ l, m \in \mathbb{S},\ t > 0. \tag{3.3}$$

Proof. After using m=0 in (3.2), we get

$$\mu_{D_C(l,0)}(t) \geq \mu^{'}_{\Omega(l,0)}(t)$$

$$\mu_{\frac{C(al)}{a^{\frac{1}{3}}} - C(l)}(t) \geq \mu^{'}_{\Omega(l,0)}(ta^{\frac{1}{3}}) \tag{3.4}$$

for all $l \in \mathbb{S}$ and $t > 0$. Now replacing l by $a^n l$ in (3.4), we have

$$\mu_{\frac{C(a^{n+1}l)}{a^{\frac{1}{3}(n+1)}} - \frac{C(a^n l)}{a^{\frac{1}{3}(n)}}}(t) \geq \mu'_{\Omega(l,0)}(ta^{\frac{1}{3}}(\frac{a^{\frac{1}{3}}}{\alpha})^n)$$

for all $l \in \mathbb{S}$ and $t > 0$. Since, $\frac{C(a^n l)}{a^{(\frac{1}{3})n}} - C(l) = \sum_{k=0}^{n-1} \frac{C(a^{k+1}l)}{a^{\frac{1}{3}(k+1)}} - \frac{C(a^k l)}{a^{\frac{1}{3}(k)}}$, we get

$$\mu_{\frac{C(a^n l)}{a^{(\frac{1}{3})n}} - C(l)} \left(\sum_{k=0}^{n-1} \frac{1}{a^{\frac{1}{3}}} (\frac{\alpha}{a^{\frac{1}{3}}})^k t \right) \geq T_M \left\{ \mu'_{\Omega(l,0)}(t),\ 0 \leq k \leq n-1,\ k \in W \right\} = \mu'_{\Omega(l,0)}(t) \tag{3.5}$$

for all $l \in \mathbb{S}$ and $t > 0$. Letting l by $a^m l$ in (3.5), we obtain

$$\mu_{\frac{C(a^{n+m}l)}{a^{\frac{1}{3}(n+m)}} - \frac{C(a^m l)}{a^{\frac{1}{3}(m)}}}(t) \geq \mu'_{\Omega(l,0)} \left(\frac{ta^{\frac{1}{3}}}{\sum_{k=m}^{m+n-1} (\frac{\alpha}{a^{\frac{1}{3}}})^k} \right) \tag{3.6}$$

for all $l \in \mathbb{S}$, $n,\ m \in \mathbb{Z}$ with $0 \leq m < n$ and $t > 0$. Since $0 < \alpha < a^{\frac{1}{3}}$, the sequence $\{\frac{C(a^n l)}{a^{\frac{n}{3}}}\}$ is a Cauchy sequence but $(\mathbb{Z}, \mu, T_M)$ is complete RNS so converges to some point (say) $C_3(l) \in \mathbb{Z}$. Fix $l \in \mathbb{S}$ and take $m = 0$ in (3.6). We get

$$\mu_{\frac{C(a^n l)}{a^{(\frac{1}{3})n}} - C(l)}(t) \geq \mu'_{\Omega(l,0)} \left(\frac{ta^{\frac{1}{3}}}{\sum_{k=0}^{n-1} (\frac{\alpha}{a^{\frac{1}{3}}})^k} \right)$$

and so, for any $\delta > 0$,

$$\begin{aligned} \mu_{C(l)-C_3(l)}(\delta + t) &\geq T_M \left(\mu_{C_3(l) - \frac{C(a^n l)}{a^{(\frac{1}{3})n}}}(\delta), \mu_{\frac{C(a^n l)}{a^{(\frac{1}{3})n}} - C(l)}(t) \right) \\ &\geq T_M \left(\mu_{C_3(l) - \frac{C(a^n l)}{a^{(\frac{1}{3})n}}}(\delta), \mu'_{\Omega(l,0)} \left(\frac{ta^{\frac{1}{3}}}{\sum_{k=0}^{n-1} (\frac{\alpha}{a^{\frac{1}{3}}})^k} \right) \right) \end{aligned} \tag{3.7}$$

for all $l \in \mathbb{S}$, $t > 0$. Putting $n \longrightarrow \infty$ in (3.7), we get

$$\mu_{C(l)-C_3(l)}(\delta + t) \geq \mu'_{\Omega(l,0)} \left(t(a^{\frac{1}{3}} - \alpha) \right) \tag{3.8}$$

Here, δ is erratic and by picking δ to 0 in (3.8), we get

$$\mu_{C(l)-C_3(l)}(t) \geq \mu'_{\Omega(l,0)} \left(t(a^{\frac{1}{3}} - \alpha) \right) \tag{3.9}$$

for all $l \in \mathbb{S}$, $t > 0$. Hence, we conclude that inequality (3.3) holds.

After changing l and m by $a^n l$ and $a^n m$ in (3.2), respectively, we obtain

$$\mu_{\frac{D_C(a^n l, a^n m)}{a^{\frac{n}{3}}}}(t) \geq \mu'_{\Omega(a^n l, a^n m)}(a^{\frac{n}{3}} t)$$

for all $l,\ m \in \mathbb{S},\ t > 0$. Since $\lim_{n\to\infty} \mu'_{\Omega(a^n l, a^n m)}(a^{\frac{n}{3}}t) = 1$ is given in theorem. After using this, we get C_3 satisfies the equation (0.1) and hence C_3 is GCR mapping.
For proving the uniqueness of the GCRF C_3, suppose there exists another mapping $C_3^\bullet : \mathbb{S} \longrightarrow \mathbb{Z}$ which assures (3.3). For fixed $l \in \mathbb{S}$, $C_3(a^n l) = a^{\frac{n}{3}} C_3(l)$ and $C_3^\bullet(a^n l) = a^{\frac{n}{3}} C_3^\bullet(l)$ $\forall\ n \in \mathbb{Z}^+$. Thus it pursues from (3.3) that

$$\begin{aligned}
\mu_{C_3(l)-C_3^\bullet(l)}(t) &= \mu_{\frac{C_3(a^n l)}{a^{\frac{n}{3}}} - \frac{C_3^\bullet(a^n l)}{a^{\frac{n}{3}}}}(t)\\
&\geq T_M\left(\mu_{\frac{C_3(a^n l)}{a^{\frac{n}{3}}} - \frac{C(a^n l)}{a^{\frac{n}{3}}}}(\frac{t}{2}),\ \mu_{\frac{C(a^n l)}{a^{\frac{n}{3}}} - \frac{C_3^\bullet(a^n l)}{a^{\frac{n}{3}}}}(\frac{t}{2})\right)\\
&\geq \mu'_{\Omega(l,0)}\left(t(a^{\frac{1}{3}} - \alpha)(\frac{a^{1/3}}{\alpha})^n\right)
\end{aligned}$$

As $\lim_{n\to\infty}\left((\frac{a^{1/3}}{\alpha})^n(a^{\frac{1}{3}} - \alpha)t\right) = \infty$, we have $\mu_{C_3(l)-C_3^\bullet(l)}(t) = 1\ \forall\ t > 0$. Hence, we obtain the uniqueness of GCR mapping C_3. This proves the result. □

Theorem 3.2. *Assume $\Omega : \mathbb{S}^2 \longrightarrow \mathbb{Y}$ is a mapping such that, $\alpha > a^{\frac{1}{3}}$, where $a > 0$*

$$\mu'_{\Omega(\frac{l}{a},\frac{m}{a})}(t) \geq \mu'_{\Omega(l,m)}(\alpha t) \tag{3.10}$$

and

$$\lim_{n\to\infty} \mu'_{a^{\frac{n}{3}}\Omega(\frac{l}{a^n},\frac{m}{a^n})}(t) = 1$$

for all $l, m \in \mathbb{S}$ and $t > 0$. If $C : \mathbb{S} \longrightarrow \mathbb{Z}$ is a mapping with $C(0) = 0$ such that

$$\mu_{D_C(l,m)}(t) \geq \mu'_{\Omega(l,m)}(t) \tag{3.11}$$

for all $l, m \in \mathbb{S}$ and $t > 0$, then, there exists one and only one GCR mapping $C_3 : \mathbb{S} \longrightarrow \mathbb{Z}$ such that

$$\mu_{C(l)-C_3(l)}(t) \geq \mu'_{\Omega(l,0)}\left(t(\alpha - a^{\frac{1}{3}})\right) \tag{3.12}$$

for all $l \in \mathbb{S}$ and $t > 0$.

Proof. Putting m=0 in (3.11) and after that replacing l by $\frac{l}{a}$, we obtain

$$\mu_{C(l)-a^{\frac{1}{3}}C(\frac{l}{a})}(t) \geq \mu'_{\Omega(l,0)}(\alpha t) \tag{3.13}$$

for all $l \in \mathbb{S}$ and $t > 0$. Implementing the triangle inequality, we get

$$\mu_{C(l)-a^{\frac{n}{3}}C(\frac{l}{a^n})}(t) \geq \mu'_{\Omega(l,0)}\left(\frac{t\alpha}{\sum_{k=m}^{m+n-1}(\frac{a^{\frac{1}{3}}}{\alpha})^k}\right)$$

for all $l \in \mathbb{S}$ and $t > 0$. Then the sequence $\{a^{\frac{n}{3}} C(\frac{l}{a^n})\}$ is a Cauchy sequence. but $(\mathbb{Z}, \mu, T_M)$ is complete RNS so converges to some point (say) $C_3 : \mathbb{S} \longrightarrow \mathbb{Z}$ such that

$$C_3(l) = \lim_{n\to\infty} a^{\frac{n}{3}} C(\frac{l}{a^n}) \ \forall\, l \in \mathbb{S}.$$

This mapping C_3 satisfies (0.1) and (3.12). Remaining proof is identical to proof of the previous theorem. This gives the required result. □

Corollary 3.3. *Assume $\Phi \geq 0$ is a real number and n_0 is a fixed unit point of $\mathbb{Y}$. If $C : \mathbb{S} \longrightarrow \mathbb{Z}$ is a mapping with $C(0) = 0$ which satisfies*

$$\mu_{D_C(l,m)}(t) \geq \mu'_{\Phi n_0}(t)$$

for all $l,\ m \in \mathbb{S}$ and $t > 0$, then there exists one and only one GCRF $C_3 : \mathbb{S} \longrightarrow \mathbb{Z}$ such that

$$\mu_{C(l)-C_3(l)}(t) \geq \mu'_{\Phi n_0}(\frac{ta^{\frac{1}{3}}}{2}) \ \textit{for all } l \in \mathbb{S},\ t > 0.$$

Proof. Let $\Phi : \mathbb{S}^2 \longrightarrow \mathbb{Y}$ be described by $\Phi(l,\ m) = \Phi n_0$. and put $\alpha = \frac{a^{\frac{1}{3}}}{2}$ in theorem (3.1), we get the reuired result. □

Corollary 3.4. *Let $0 < p,\ q < \frac{1}{3}$ be real numbers and n_0 be a fixed unit point of $\mathbb{Y}$. If $C : \mathbb{S} \longrightarrow \mathbb{Z}$ is a mapping with $C(0) = 0$ which satisfies*

$$\mu_{D_C(l,m)}(t) \geq \begin{cases} \mu'_{(\|l\|^p+\|m\|^q)n_0}(t) \ \textit{or}, \\ \mu'_{(\|l\|^p\|m\|^q)n_0}(t) \ \textit{where},\ \ p+q < \frac{1}{3} \end{cases}$$

for all $l,\ m \in \mathbb{S}$ and $t > 0$, then there exists one and only one GCRF $C_3 : \mathbb{S} \longrightarrow \mathbb{Z}$ such that

$$\mu_{C(l)-C_3(l)}(t) \geq \begin{cases} \mu'_{n_0\|l\|^p}(t(a^{\frac{1}{3}} - a^p)) \ \textit{or}, \\ \epsilon_0(t(a^{\frac{1}{3}} - a^p)) \end{cases}$$

for all $l \in \mathbb{S}$ and $t > 0$, where $\epsilon_0(t) = \begin{cases} 0, \ \textit{if } 0 \geq t \\ 1, \ \textit{if } 0 < t. \end{cases}$

Proof. Let $\Phi : \mathbb{S}^2 \longrightarrow \mathbb{Y}$ be defined by $\Phi(l,\ m) = (\|l\|^p + \|m\|^q)n_0$ or $(\|l\|^p\|m\|^q)$ and put $\alpha = a^p$ in theorem (3.1), we obtain the required decision. □

Corollary 3.5. *Let $0 < p + q < \frac{1}{3}$ be real numbers and n_0 be a fixed unit point of $\mathbb{Y}$. If $C : \mathbb{S} \longrightarrow \mathbb{Z}$ is a mapping with $C(0) = 0$ which satisfies*

$$\mu_{D_C(l,m)}(t) \geq \mu'_{(\|l\|^p\|m\|^q+\|l\|^{p+q}+\|m\|^{p+q})n_0}(t)$$

for all $l,\ m \in \mathbb{S}$ and $t > 0$, then there exists one and only one GCR mapping $C_3 : \mathbb{S} \longrightarrow \mathbb{Z}$ such that

$$\mu_{C(l)-C_3(l)}(t) \geq \mu'_{n_0\|l\|^{p+q}}(t(a^{\frac{1}{3}} - a^p))$$

for all $l \in \mathbb{S}$ and $t > 0$.

Proof. Let $\Phi : \mathbb{S}^2 \longrightarrow \mathbb{Y}$ be defined by $\Phi(l,\ m) = (\|l\|^p\|m\|^q + \|l\|^{p+q} + \|m\|^{p+q})n_0$. and put $\alpha = a^p$ in theorem (3.1), we obtain the desired result. $\square$

4. Non-Archimedean random normed space (NA-RNS)

A field κ equipped with a function $|.| : \kappa \to [0, \infty]$ is said to be a non-Archimedean field for which $|u| = 0$ iff $u = 0$, $|uv| = |u||v|$, and $|u + v| \leq \max\{|u|, |v|\}$ for all $u,\ v \in \kappa$. It is clear that $|-1| = |1| = 1$ and $|u| \leq 1\ \forall\ u \in \mathbb{N}$.
Suppose that X is a vector space over a field κ with a "non-Archimedean" non-trivial valuation $|.|$.

Definition 4.1. [13] A function $\|.\| : X \to [0, \infty]$ is fornamed to be a "non-Archimedean norm" if following axioms are satisfied:
(i) $\|l\| = 0$ iff $l = 0$
(ii) $\|al\| = \|a\|\|l\|$, for any $a \in \kappa,\ l \in X$;
(iii) $\|l + m\| \leq \max\{\|m\|, \|l\|\}$ for $l, m \in X$ (ultrametric).
Then $(X, \|.\|)$ is fornamed "non-Archimedean normed space".

Definition 4.2. [13] NA-RNS is a triplet (X, μ, T), where X is a vector space over a "non-Archimedean field" κ, T is a continuous t-norm, and $\mu : X \to D^+$ is a mapping that satisfies the axioms given by:
(NA-RNS1) $\mu_l(t) = \epsilon_0(t)\ \forall\ t > 0$ iff $l = 0$;
(NA-RNS2) $\mu_{\alpha l}(t) = \mu_l(\frac{t}{|\alpha|})\ \forall\ l \in X,\ t > 0,\ \alpha \neq 0$;
(NA-RNS3) $\mu_{l+m}(max\{s, t\}) \geq\ T(\mu_l(s),\ \mu_m(t))\ \forall\ l,\ m \in X$ and $t,\ s \geq 0$.

5. **Generalized Ulam-Hyers stability of (0.1) in NA-RNS**

Let $\mathbb{S}$ be a linear space over non-Archimedean field $\mathbb{K}$ and $(\mathbb{X}, \mu, T)$ be a complete NA-RNS over $\mathbb{K}$.

Now, we describe random approximately GCR mapping. Assume $\Phi : \mathbb{S} \times \mathbb{S} \times [0, \infty] \longrightarrow \mathbb{R}$ be a distribution function such that $\Phi(l, m, .)$ is symmetric, nondecreasing and

$$\Phi(cl, cl, t) \geq \Phi(l, l, \frac{t}{|c|})$$

where $l \in \mathbb{S}$, $c \neq 0$.

Definition 5.1. A mapping $C : \mathbb{S} \longrightarrow \mathbb{X}$ is Φ-approximately GCR if

$$\mu_{C(al+bm+3a^{\frac{2}{3}}l^{\frac{2}{3}}b^{\frac{1}{3}}m^{\frac{1}{3}}+3a^{\frac{1}{3}}l^{\frac{1}{3}}b^{\frac{2}{3}}m^{\frac{2}{3}})-a^{\frac{1}{3}}C(l)-b^{\frac{1}{3}}C(m)}(t) \geq \Phi(l, m, t) \tag{5.1}$$

for all $l, m \in \mathbb{S}$, $t > 0$.

Theorem 5.2. *Assume $\mathbb{S}$ be a linear space over non-Archimedean field $\mathbb{K}$ and $(\mathbb{X}, \mu, T)$ be a complete NA-RNS over $\mathbb{K}$. Let $C : \mathbb{S} \longrightarrow \mathbb{X}$ be a Φ-approximately GCR mapping and $C(0) = 0$. If for some $\alpha \in \mathbb{R}^+$ and for some integer k with $|a^k| < \alpha$,*

$$\Phi(a^{-k}l, a^{-k}m, t) \geq \Phi(l, m, \alpha t) \tag{5.2}$$

and

$$\lim_{n\to\infty} T_{i=n}^{\infty} M(l, \frac{\alpha^i t}{|a^{ik}|}) = 1 \tag{5.3}$$

then there exists one and only one GCR mapping $C_3 : \mathbb{S} \longrightarrow \mathbb{X}$ such that

$$\mu_{C(l)-C_3(l)}(t) \geq T_{i=1}^{\infty} M\left(l, \frac{\alpha^{i+1}t}{|a^{ik}|}\right) \;\forall\; l \in \mathbb{S},\; t > 0, \tag{5.4}$$

where

$$M(l, t) := T(\Phi(l, 0, t), \Phi(al, 0, t), ..., \Phi(a^{k-1}l, 0, t)) \;\forall\; l \in \mathbb{S},\; t > 0.$$

Proof. Firstly, with the help of induction applying on j, we show that for each $l \in \mathbb{S}$, $t > 0$ and $j \geq 1$,

$$\mu_{C(a^j l)-a^{\frac{j}{3}}C(l)}(t) \geq M_j(l, t) := T(\Phi(l, 0, t), ..., \Phi(a^{j-1}l, 0, t)). \tag{5.5}$$

Letting $m = 0$ in (5.1), we get

$$\mu_{C(al)-a^{\frac{1}{3}}C(l)}(t) \geq \Phi(l, 0, t)$$

Thus, condition (5.5) holds for j=1. Let us consider that (5.5) is true for all $j \geq 1$. Changing m by 0 and l by $a^j l$ in(5.1), we have

$$\mu_{C(a^{j+1}l)-a^{\frac{1}{3}}C(a^j l)}(t) \geq \Phi(a^j l, 0, t)$$

Since $|a^{\frac{1}{3}}| \leq 1$

$$\begin{aligned}\mu_{C(a^{j+1}l)-a^{\frac{j+1}{3}}C(l)}(t) &\geq T\left(\mu_{C(a^{j+1}l)-a^{\frac{1}{3}}C(a^j l)}(t), \mu_{a^{\frac{1}{3}}C(a^j l)-a^{\frac{j+1}{3}}C(l)}(t)\right)\\ &= T\left(\mu_{C(a^{j+1}l)-a^{\frac{1}{3}}C(a^j l)}(t), \mu_{C(a^j l)-a^{\frac{j}{3}}C(l)}(\frac{t}{|a^{\frac{1}{3}}|})\right)\\ &\geq T\left(\mu_{C(a^{j+1}l)-a^{\frac{1}{3}}C(a^j l)}(t), \mu_{C(a^j l)-a^{\frac{j}{3}}C(l)}(t)\right)\\ &\geq T(\Phi(a^j l, 0, t), M_j(l,t))\\ &= M_{j+1}(l,t)\end{aligned}$$

for all $l \in \mathbb{S}$. Thus (5.5) is true for all $j \geq 1$. Particularly

$$\mu_{C(a^k l)-a^{\frac{k}{3}}C(l)}(t) \geq M(l,t) \tag{5.6}$$

Replacing l by $a^{-(kn+k)}l$ in (5.6) and using (5.2), we get

$$\mu_{C(\frac{l}{a^{nk}})-a^{\frac{k}{3}}C(\frac{l}{a^{kn+k}})}(t) \geq M\left(\frac{l}{a^{kn+k}}, t\right) \geq M\left(l, \alpha^{n+1}t\right) \tag{5.7}$$

for all $l \in \mathbb{S}$, $t > 0$ and n=0,1,2,... Then

$$\mu_{(a^{\frac{k}{3}})^n C(\frac{l}{a^{nk}})-(a^{\frac{k}{3}})^{n+1}C(\frac{l}{a^{kn+k}})}(t) \geq M\left(l, \frac{\alpha^{n+1}t}{|(a^{\frac{k}{3}})^n|}\right)$$

for all $t > 0$, $l \in \mathbb{S}$ and n=0,1,2,... . Therefore,

$$\begin{aligned}\mu_{(a^{\frac{k}{3}})^n C(\frac{l}{a^{nk}})-(a^{\frac{k}{3}})^{n+p}C(\frac{l}{a^{k(n+p)}})}(t) &\geq T_{j=n}^{n+p}\left(\mu_{(a^{\frac{k}{3}})^j C(\frac{l}{a^{jk}})-(a^{\frac{k}{3}})^{j+1}C(\frac{l}{a^{k(j+1)}})}(t)\right)\\ &\geq T_{j=n}^{n+p} M\left(l, \frac{\alpha^{j+1}t}{|a^{\frac{kj}{3}}|}\right)\\ &\geq T_{j=n}^{n+p} M\left(l, \frac{\alpha^{j+1}t}{|a^{kj}|}\right)\end{aligned}$$

Since $\lim_{n\to\infty} T_{j=n}^{\infty} M(l, \frac{\alpha^{j+1}t}{|a^{kj}|}) = 1$ for all $l \in \mathbb{S}$, $t > 0$, $\{(a^{\frac{k}{3}})^n C(\frac{l}{a^{nk}})\}$ is a Cauchy sequence in the complete NA-RNS $(\mathbb{X}, \mu, T)$. Hence, a mapping $C_3 : \mathbb{S} \longrightarrow \mathbb{X}$ can be defined for which

$$\lim_{n\to\infty}\mu_{(a^{\frac{k}{3}})^n C(\frac{l}{a^{nk}})-C_3(l)}(t)=1 \tag{5.8}$$

for all $l\in\mathbb{S}$, $t>0$. Now, for each $n\geq 1$, $l\in\mathbb{S}$ and $t>0$,

$$\begin{aligned}\mu_{C(l)-(a^{\frac{k}{3}})^n C(\frac{l}{a^{nk}})}(t) &= \mu_{\sum_{i=0}^{n-1}(a^{\frac{k}{3}})^i C(\frac{l}{a^{ki}})-(a^{\frac{k}{3}})^{i+1}C(\frac{l}{a^{k(i+1)}})}(t)\\ &\geq T_{i=0}^{n-1}\left(\mu_{(a^{\frac{k}{3}})^i C(\frac{l}{a^{ki}})-(a^{\frac{k}{3}})^{i+1}C(\frac{l}{a^{k(i+1)}})}(t)\right)\\ &\geq T_{i=0}^{n-1}M\left(l,\frac{\alpha^{i+1}t}{|a^{\frac{k}{3}}|^i}\right)\end{aligned}$$

Therefore,

$$\begin{aligned}\mu_{C(l)-C_3(l)}(t) &\geq T\left(\mu_{C(l)-(a^{\frac{k}{3}})^n C(\frac{l}{a^{nk}})}(t),\mu_{(a^{\frac{k}{3}})^n C(\frac{l}{a^{nk}})-C_3(l)}(t)\right)\\ &\geq T\left(T_{i=0}^{n-1}M\left(l,\frac{\alpha^{i+1}t}{|a^{\frac{k}{3}}|^i}\right),\mu_{(a^{\frac{k}{3}})^n C(\frac{l}{a^{nk}})-C_3(l)}(t)\right)\end{aligned}$$

by letting $n\longrightarrow\infty$, we obtain

$$\mu_{C(l)-C_3(l)}(t)\geq T_{i=1}^{\infty}M\left(l,\frac{\alpha^{i+1}t}{|a^{ki}|}\right).$$

This gives (5.4) is true.

T being continuous, from a famous result in probabilistic metric space ([4]), it follows that

$$\begin{aligned}&\lim_{n\to\infty}\mu_{(a^{\frac{k}{3}})^n C(\frac{al+bm+3a^{\frac{2}{3}}.l^{\frac{2}{3}}.b^{\frac{1}{3}}.m^{\frac{1}{3}}+3a^{\frac{1}{3}}.l^{\frac{1}{3}}.b^{\frac{2}{3}}.m^{\frac{2}{3}}}{(a^{\frac{k}{3}})^n}).-(a^{\frac{k}{3}})^n.a^{\frac{1}{3}}.C(\frac{l}{(a^{\frac{k}{3}})^n})-(a^{\frac{k}{3}})^n.b^{\frac{1}{3}}.C(\frac{l}{(a^{\frac{k}{3}})^n})}(t)\\ &=\mu_{C(al+bm+3a^{\frac{2}{3}}.l^{\frac{2}{3}}.b^{\frac{1}{3}}.m^{\frac{1}{3}}+3a^{\frac{1}{3}}.l^{\frac{1}{3}}.b^{\frac{2}{3}}.m^{\frac{2}{3}})-a^{\frac{1}{3}}C(l)-b^{\frac{1}{3}}C(m)}(t)\end{aligned}$$

for almost all $t>0$.

Moreover, replacing l by $a^{-kn}l$ and m $a^{-kn}m$ in (5.1) and using (NA-RNS2) and (5.2), we obtain

$$\begin{aligned}&\mu_{(a^k)^n C(\frac{al+bm+3a^{\frac{2}{3}}.l^{\frac{2}{3}}.b^{\frac{1}{3}}.m^{\frac{1}{3}}+3a^{\frac{1}{3}}.l^{\frac{1}{3}}.b^{\frac{2}{3}}.m^{\frac{2}{3}}}{a^{kn}}).-(a^k)^n.a^{\frac{1}{3}}.C(\frac{l}{(a^k)^n})-(a^k)^n.b^{\frac{1}{3}}.C(\frac{l}{(a^k)^n})}(t)\\ &\geq\Phi(a^{-kn}l,\ a^{-kn}m,\frac{t}{|a^k|^n})\geq\Phi(l,\ m,\frac{\alpha^n t}{|a^k|^n})\end{aligned}$$

for all $l,\ m\in\mathbb{S}$ and all $t>0$. Since $\lim_{n\to\infty}\Phi(l,\ m,\frac{\alpha^n t}{|a^k|^n})=1$, we deduce that C_3 is a GCR mapping.

If $C_3^\bullet : \mathbb{S} \longrightarrow \mathbb{X}$ is another GCR mapping such that $\mu_{C_3^\bullet(l)-C(l)}(t) \geq M(l,t)$ for all $l \in \mathbb{S}$ and $t > 0$, then for each $n \in \mathbb{N}$, $l \in \mathbb{S}$ and $t > 0$

$$\mu_{C_3^\bullet(l)-C(l)}(t) \geq T\left(\mu_{C_3^\bullet(l)-(a^{\frac{k}{3}})^n C(\frac{l}{(a^k)^n})}(t), \mu_{(a^{\frac{k}{3}})^n C(\frac{l}{(a^k)^n})-C_3(l)}(t)\right)$$

By using (5.8), we conclude that $C_3^\bullet = C_3$. $\square$

Corollary 5.3. *Let $\mathbb{S}$ be a vector space over non-Archimedean field $\mathbb{K}$ and $(\mathbb{X}, \mu, T)$ be a complete NA-RNS over $\mathbb{K}$ under a t-norm $T \in \mathcal{H}$. Suppose that $C : \mathbb{S} \longrightarrow \mathbb{X}$ is a Φ-approximately GCR mapping. If*

$$\Phi(a^{-k}l, a^{-k}m, t) \geq \Phi(l, m, \alpha t) \text{ for } x \in X,\ t > 0, \tag{5.9}$$

for some $\alpha \in \mathbb{R}^+$ and for some integer k with $|a^k| < \alpha$, then there exists one and only one GCR mapping $C_3 : \mathbb{S} \longrightarrow \mathbb{X}$ such that

$$\mu_{C(l)-C_3(l)}(t) \geq T_{i=1}^{\infty} M\left(l, \frac{\alpha^{i+1}t}{|a|^{ik}}\right) \tag{5.10}$$

for all $l \in \mathbb{S}$, $t > 0$, where

$$M(l,t) := T(\Phi(l,0,t), \Phi(al,0,t), ..., \Phi(a^{k-1},0,t))$$

for all $l \in \mathbb{S}$, $t > 0$.

Proof. Since $\lim_{i\to\infty} M\left(l, \frac{\alpha^i t}{|a|^{ik}}\right) = 1$, for all $l \in \mathbb{S},\ t > 0$ and T is of Hadzic type, hence, we obtain

$$\lim_{n\to\infty} T_{i=n}^{\infty} M\left(l, \frac{\alpha^{i+1}t}{|a^{\frac{1}{3}}|^{ik}}\right) = 1 \text{ for all } l \in \mathbb{S},\ t > 0.$$

Now we can apply Theorem (5.2) to get the result. $\square$

6. **Fixed point approach**

This segment is dedicated to the fixed point approach for finding "random stability" of the GCRF Equation. Luxemburg ([26]) introduced the notion of "generalized metric space".

Lemma 6.1. [13] *Let (ψ, d) be a "complete generalized metric space" and $T : \psi \longrightarrow Y$ be a strict contraction with the "Lipschitz constant" k such that $d(l_0,\ A(l_0)) < +\infty$ for some $l_0 \in X$. Then T has a unique fixed point in the set $Y := \{m \in \psi, d(l_0, m) < \infty\}$ and the* *sequence $\{T^n(l)\}$ converges to the fixed point l^* for every $l \in Y$. Also, $d(l_0, T(l_0)) \leq \omega$ gives $d(l^*, l_0) \leq \frac{\omega}{1-k}$.*

let $H : \mathbb{S} \times R \longrightarrow [0,1]$ be a function where $\mathbb{S}$ is a linear space, $(\mathbb{X}, \nu, T_M)$ is a complete RNS such that $H(l,.) \in D^+$ for all $x \in \mathbb{S}$. Suppose set $F := \{k : \mathbb{S} \longrightarrow \mathbb{X},\ k(0) = 0\}$ and the mapping $d_H : F \times F \longrightarrow R^+$ as

$$d_H(h,k) = \inf\{b \in R^+, \nu_{h(l)-k(l)}(bt) \geq H(l,t),\ \forall\ l \in \mathbb{S},\ t > 0\}$$

Theorem 6.2. *Let $\mathbb{S}$ be a real Vector space, $(\mathbb{X}, \mu, T_M)$ a complete RNS and $C : \mathbb{S} \longrightarrow \mathbb{X}$ is a mapping with $C(0) = 0$ and $\Phi : \mathbb{S}^2 \longrightarrow D^+$ be a mapping with the inequality*

$$\text{for } 0 < \alpha < a^{\frac{1}{3}} : \Phi_{al,am}(\alpha t) \geq \Phi_{l,m}(t),\ \forall\ l,\ m \in \mathbb{S} \text{ and } t > 0 \tag{6.1}$$

If

$$\mu_{C(al+bm+3a^{\frac{2}{3}}.l^{\frac{2}{3}}.b^{\frac{1}{3}}.m^{\frac{1}{3}}+3a^{\frac{1}{3}}.l^{\frac{1}{3}}.b^{\frac{2}{3}}.m^{\frac{2}{3}})-a^{\frac{1}{3}}C(l)-b^{\frac{1}{3}}C(m)}(t) \geq \Phi_{l,m}(t) \tag{6.2}$$

$\forall\ l,\ m \in \mathbb{S}$, then there exists one and only one GCR mapping $C_3 : \mathbb{S} \longrightarrow \mathbb{X}$ such that

$$\mu_{C_3(l)-C(l)}(t) \geq \Phi_{l,0}((a^{\frac{1}{3}} - \alpha)t),\ \forall\ l,\ m \in \mathbb{S},\ t > 0. \tag{6.3}$$

Moreover, $C_3(l) = \lim_{n\to\infty} \frac{C(a^n l)}{a^{\frac{n}{3}}}$

Proof. Put $m = 0$ in (6.2), we get

$$\mu_{C(al)-a^{\frac{1}{3}}C(l)}(t) \geq \Phi_{l,0}(t) \text{ or}$$
$$\mu_{\frac{C(al)}{a^{\frac{1}{3}}}-C(l)}(t) \geq \Phi_{l,0}(a^{\frac{1}{3}}t)$$

for all $l \in \mathbb{S}$, $t > 0$. Let $H(l,t) := \Phi_{l,0}(a^{\frac{1}{3}}t)$. Suppose set

$$F := \{k : \mathbb{S} \longrightarrow \mathbb{X},\ k(0) = 0\}$$

and the mapping $d_H : F \times F \longrightarrow R^+$ as

$$d_H(h,k) = \inf\{b \in R^+, \nu_{h(l)-k(l)}(bt) \geq H(l,t),\ \forall\ l \in \mathbb{S},\ t > 0\}$$

Using above lemma, (F, d_H) is a "complete generalized metric space". Next, suppose a linear mapping $J : F \longrightarrow F$ for which $Jh(l) := \frac{1}{a^{\frac{1}{3}}}h(al)$.

We prove J to be a strictly contractive with the Lipschitz constant $k = \frac{\alpha}{a^{\frac{1}{3}}}$.
For this, let $h,\ k \in F$ be mappings such that $d_H(h,k) < \epsilon$. Then

$$\mu_{h(l)-k(l)}(\epsilon t) \geq H(l,t),\ \forall\ l \in \mathbb{S},\ t > 0,$$

hence

$$\mu_{Jh(l)-Jk(l)}(\frac{\alpha}{a^{\frac{1}{3}}}\epsilon t) = \mu_{\frac{h(al)-k(al)}{a^{\frac{1}{3}}}}(\frac{\alpha}{a^{\frac{1}{3}}}\epsilon t) = \mu_{h(al)-k(al)}(\alpha\epsilon t) \geq H(al,\alpha t)$$

for all $l \in \mathbb{S}$, $t > 0$. Since $H(al,\alpha t) \geq H(l,t)$, then

$$\mu_{Jh(l)-Jk(l)}(\frac{\alpha}{a^{\frac{1}{3}}}\epsilon t) \geq H(l,t),$$

$$d_H(h,k) < \epsilon \implies d_H(Jh,Jk) < \frac{\alpha}{a^{\frac{1}{3}}}\epsilon.$$

$$d_H(Jh,Jk) < \frac{\alpha}{a^{\frac{1}{3}}} d_H(h,k)\ \forall\ h,\ k \in F.$$

Now, from

$$\mu_{C(l)-\frac{C(al)}{a^{\frac{1}{3}}}}(t)\ \geq H(l,t)$$

it gives that $d_H(C,JC) \leq 1$. Using the [26] theorem, there exists a fixed point of J, i.e, there exists a mapping $C_3 : \mathbb{S} \longrightarrow \mathbb{X}$ such that $C_3(al) = a^{\frac{1}{3}}C_3(l)$ for all $l \in \mathbb{S}$.
Therefore, for any $l \in \mathbb{S}$ and $t > 0$,

$$d_H(u,v) < \epsilon \quad \implies \mu_{u(l)-v(l)}(t) \geq H(l,\frac{t}{\epsilon})$$

from $d_H(J^nC,C_3) \longrightarrow 0$, it follows that $\lim_{n\to\infty} \frac{C(a^n l)}{a^{\frac{n}{3}}} = C_3(l)$ for any $l \in \mathbb{S}$.
Also $d_H(C,C_3) \leq \frac{1}{1-L}d(C,JC) \implies d_H(C,C_3) \leq \frac{1}{1-\frac{\alpha}{a^{\frac{1}{3}}}}$ from which it immediately follows $\mu_{C_3(l)-C(l)}\left(\frac{a^{\frac{1}{3}}t}{a^{\frac{1}{3}}-\alpha}\right) \geq H(l,t)\ \forall\ t > 0$ and all $l \in \mathbb{S}$. This means that

$$\mu_{C_3(l)-C(l)}(t) \geq H\left(l, \frac{a^{\frac{1}{3}}-\alpha}{a^{\frac{1}{3}}}t\right) \forall\ t > 0 \text{ and } \ l \in \mathbb{S}.$$

It follows that

$$\mu_{C_3(l)-C(l)}(t) \geq \Phi_{l,0}((a^{\frac{1}{3}}-\alpha)t)\ \forall\ t > 0 \text{ and } \ l \in \mathbb{S}.$$

Also, C_3 is the unique fixed point of J with the property: there exists a real number $k \geq 0$ such that $\mu_{C_3(l)-C(l)}(kt) \geq H(l,t)$ for all $t > 0$ and all $l \in \mathbb{S}$, as desired. □

7. **Intutionistic random normed spaces (IR-NS)**

In 2010, Chang et al. [22] introduced the notation of IR-NS.

Lemma 7.1. [22] *Suppose that the set* L^* *and operation* $\leq_{L^*}$ *described by:*

$$L^* = \{(l_1,\ l_2) : (l_1,\ l_2) \in [0,1]^2 \textit{and}\ l_1 + l_2 \leq 1\},$$

$$(l_1, l_2) \leq_{L^*} (m_1, m_2) \Leftrightarrow\ l_1 \leq m_1,\ l_2 \geq m_2,\ \forall (l_1, l_2),\ (m_1, m_2) \in L^*.$$

Then $(L^*, \leq_{L^*})$ *is a "complete lattice".*

Units are indicated by $0_{L^*} = (0,1)$ and $1_{L^*} = (1,0)$. Using the lattice $(L^*, \leq_{L^*})$, definition of t-norm can be developed.

Definition 7.2. [22] A mapping $T : (L^*)^2 \to L^*$ is fornamed to be triangular norm (t-norm) if:
(i) $T(l, 1_{L^*}) = x\ \forall l \in L^*$;
(ii) $T(l, m) = T(m, l)\ \forall (l, m) \in (L^*)^2$;
(iii) $T(l, T(m, n)) = T(T(l, m), n)\ \forall\ (l,\ m,\ n) \in (L^*)^3$;
(iv) $l \leq_{L^*} l',\ m \leq_{L^*} m' \Rightarrow T(l, m) \leq_{L^*} T(l', m')\ \forall (l, l', m, m') \in (L^*)^4$.

Definition 7.3. [22] Let $\mu,\ \nu : X \times (0, \infty) \longrightarrow [0,1]$ be "measure and non-measure distribution functions" such that $\mu_l(t) + \nu_l(t) \leq 1\ \forall\ l \in X,\ t > 0$. The triplet $(X, P_{\mu,\nu}, T)$ is fornamed to be IR-NS if X is a linear space, T is continuous t-norm and $P_{\mu,\nu} : X \times (0, \infty) \to L^*$ is a mapping satisfying
(i) $P_{\mu,\nu}(l, 0) = 0_{L^*}$;
(ii) $P_{\mu,\nu}(l, t) = 1_{L^*}$ iff $l = 0$;
(iii) $P_{\mu,\nu}(\alpha l, t) = P_{\mu,\nu}(l, \frac{t}{|\alpha|})\ \forall\ \alpha \neq 0$;
(iv) $P_{\mu,\nu}(l + m, t + s) \geq_{L^*} T(P_{\mu,\nu}(l, t), P_{\mu,\nu}(m, s))$
for all $l,\ m \in X$ and $t,\ s > 0$. Then $P_{\mu,\nu}$ is called an IR-NS, where $P_{\mu,\nu}(l, t) = (\mu_l(t), \nu_l(t))$.

8. **Generalized Ulam- Hyers stability of equation** (0.1) **in IR-NS**

This segment deals with the "generalized Ulam- Hyers stability" of the GCRF equation in IR-NS.

Theorem 8.1. *Let $C : \mathbb{S} \longrightarrow \mathbb{X}$ be a mapping with $C(0) = 0$ where $\mathbb{S}$ is a linear space and $(\mathbb{X},\ \mathbb{P}_{\mu,\nu},\ T)$ is a complete IR-NS, for which there are $\xi,\ \Omega : \mathbb{S}^2 \longrightarrow D^+$, where $\xi(l,\ m)$ is represented by $\xi_{l,\ m}$ and $\Omega(l,\ m)$ is represented by $\Omega_{l,\ m}$, further $(\xi_{l,\ m}(t),\ \Omega_{l,\ m}(t))$ is represented by $Q_{\xi,\Omega}(l, m, t)$, with inequality:*

$$\mathbb{P}_{\mu,\nu}(C(al + bm + 3a^{\frac{2}{3}}.l^{\frac{2}{3}}.b^{\frac{1}{3}}.m^{\frac{1}{3}} + 3a^{\frac{1}{3}}.l^{\frac{1}{3}}.b^{\frac{2}{3}}.m^{\frac{2}{3}}) - a^{\frac{1}{3}}C(l) - b^{\frac{1}{3}}C(m), t) \geq^*_L Q_{\xi,\Omega}(l, m, t) \tag{8.1}$$

If

$$T^{\infty}_{i=1}(Q_{\xi,\Omega}(a^{m+i-1}l,\ 0,\ a^{\frac{-2(i-\frac{m}{2}+1)}{3}}t)) = 1_{L^*} \tag{8.2}$$

and

$$\lim_{n\to\infty} Q_{\xi,\Omega}(a^n l, a^n m, a^{\frac{n}{3}}t) = 1_{L^*}\ \forall\ l,\ m \in \mathbb{S} \text{ and } t > 0, \tag{8.3}$$

then there exists one and only one GCRF $C_3 : \mathbb{S} \longrightarrow \mathbb{X}$ such that

$$\mathbb{P}_{\mu,\nu}(C(l) - C_3(l), t) \geq_{L^*} T^{\infty}_{i=1}Q_{\xi,\Omega}(a^{i-1}l, 0, a^{\frac{-2}{3}(i+1)}t). \tag{8.4}$$

Proof. Choosing m to be zero in (8.1), we have

$$\mathbb{P}_{\mu,,\nu}\left(C(al) - a^{\frac{1}{3}}C(l), t\right) \geq_{L^*} Q_{\xi,\Omega}(l, 0, t)$$
$$\mathbb{P}_{\mu,,\nu}\left(\frac{1}{a^{\frac{1}{3}}}C(al) - C(l), t\right) \geq_{L^*} Q_{\xi,\Omega}(l, 0, a^{\frac{1}{3}}t) \tag{8.5}$$

Therefore, it follows that

$$\mathbb{P}_{\mu,,\nu}\left(\frac{C(a^{k+1}l)}{a^{\frac{1}{3}(k+1)}} - \frac{C(a^k l)}{a^{\frac{1}{3}(k)}}, \frac{t}{a^{\frac{k}{3}}}\right) \geq_{L^*} Q_{\xi,\Omega}(a^k l, 0, a^{\frac{1}{3}}t) \tag{8.6}$$

which implies that

$$\mathbb{P}_{\mu,,\nu}\left(\frac{C(a^{k+1}l)}{a^{\frac{1}{3}(k+1)}} - \frac{C(a^k l)}{a^{\frac{k}{3}}}, t\right) \geq_{L^*} Q_{\xi,\Omega}(a^k l, 0, a^{\frac{1}{3}(k+1)}t) \tag{8.7}$$

that is,

$$\mathbb{P}_{\mu,,\nu}\left(\frac{C(a^{k+1}l)}{a^{\frac{1}{3}(k+1)}} - \frac{C(a^k l)}{a^{\frac{k}{3}}}, \frac{t}{a^{k+1}}\right) \geq_{L^*} Q_{\xi,\Omega}(a^k l, 0, a^{\frac{-2}{3}(k+1)}t) \tag{8.8}$$

$\forall\ k \in \mathbb{N}$ and $t > 0$. Using triangular Inequality, it follows

$$\mathbb{P}_{\mu,,\nu}\left(\frac{C(a^k l)}{a^{\frac{k}{3}}} - C(l), t\right) \geq_{L^*} T^{n-1}_{k=0}\left(\mathbb{P}_{\mu,,\nu}\left(\frac{C(a^{k+1}l)}{a^{\frac{1}{3}(k+1)}} - \frac{C(a^k l)}{a^{\frac{k}{3}}}, \sum_{k=0}^{n-1}\frac{t}{a^{k+1}}\right)\right)$$
$$\geq_{L^*} T^{n}_{k=1}\left(Q_{\xi,\Omega}(a^{k-1}l, 0, a^{\frac{-2}{3}(k+1)}t)\right) \tag{8.9}$$

Now, to prove convergence of the $\{\frac{C(a^k l)}{a^{\frac{k}{3}}}\}$, replace l by $a^{\frac{k}{3}} l$ in (8.9), we have,

$$\mathbb{P}_{\mu,,\nu}\left(\frac{C(a^{k+m}l)}{a^{\frac{1}{3}(k+m)}} - \frac{C(a^m l)}{a^{\frac{m}{3}}}, t\right) \geq_{L^*} T_{i=1}^{n} Q_{\xi,\Omega}(a^{i+m-1}l, 0, a^{\frac{-2}{3}(i-\frac{m}{2}+1)}t). \tag{8.10}$$

for $m,\ n > 0$. Also, as $m \longrightarrow \infty$ R.H.S. of the equation (8.10) tends to 1_L^*, therefore $\{\frac{C(a^k l)}{a^{\frac{k}{3}}}\}$ is a Cauchy sequence. So, we define a mapping C_3 such that $C_3(l) = \lim_{n\to\infty} \frac{C(a^k l)}{a^{\frac{k}{3}}}$ for all $l \in \mathbb{S}$.

Now, we prove that C_3 is a GCR mapping. Changing $l,\ m$ with $a^n l$ and $a^n m$, respectively, in (8.1), we get

$$\mathbb{P}_{\mu,\nu}(\frac{C(a^n(al + bm + 3a^{\frac{2}{3}}.l^{\frac{2}{3}}.b^{\frac{1}{3}}.m^{\frac{1}{3}} + 3a^{\frac{1}{3}}.l^{\frac{1}{3}}.b^{\frac{2}{3}}.m^{\frac{2}{3}}))}{a^{\frac{n}{3}}} - a^{\frac{1}{3}}\frac{C(a^n l)}{a^{\frac{n}{3}}} - b^{\frac{1}{3}}\frac{C(a^n m)}{a^{\frac{n}{3}}}, t)$$
$$\geq_L^* Q_{\xi,\Omega}(a^n l, a^n m, a^{\frac{n}{3}}t) \tag{8.11}$$

Hence, C_3 satisfies (0.1) for all $l,\ m \in \mathbb{S}$ as n tends to ∞ and using the limit $n \longrightarrow \infty$ in (8.9), we find (8.4).

Now, for proving the uniqueness of GCR mapping C_3 satisfying (8.4), let us consider that there exists another GCR mapping $C_3^\bullet$ subject to (8.4). Hence it follows from (8.4) that

$$\begin{aligned}
&\mathbb{P}_{\mu,\nu}(C_3(l) - C_3^\bullet(l), t)\\
&\geq_L^* \mathbb{P}_{\mu,\nu}(C_3(a^n l) - C_3^\bullet(a^n l), a^{\frac{n}{3}}t)\\
&\geq_L^* T\left(\mathbb{P}_{\mu,\nu}(C_3(a^n l) - C(a^n l), a^{\frac{n}{3}-1}t), \mathbb{P}_{\mu,\nu}(C(a^n l) - C_3^\bullet(a^n l), a^{\frac{n}{3}-1}t)\right)\\
&\geq_L^* T\left(T_{i=1}^{\infty} Q_{\xi,\Omega}(a^{n+i-1}l, 0, a^{\frac{-2}{3}(i+1)+n}t), T_{i=1}^{\infty} Q_{\xi,\Omega}(a^{n+i-1}l, 0, a^{\frac{-2}{3}(i+1)+n}t)\right)
\end{aligned}$$

for all $l \in \mathbb{S}$. We prove the uniqueness of C_3 by taking $n \longrightarrow \infty$ in (8.4). This gives the desired result. □

Theorem 8.2. *Let $C : \mathbb{S} \longrightarrow \mathbb{X}$ be a mapping where $(\mathbb{S}, \mathbb{P}'_{\mu',\nu'},\ T)$ is an IR-NS and $(\mathbb{X},\ \mathbb{P}_{\mu,\nu},\ T)$ is a complete IR-NS, such that*

$$\mathbb{P}_{\mu,\nu}(C(al+bm+3a^{\frac{2}{3}}.l^{\frac{2}{3}}.b^{\frac{1}{3}}.m^{\frac{1}{3}}+3a^{\frac{1}{3}}.l^{\frac{1}{3}}.b^{\frac{2}{3}}.m^{\frac{2}{3}})-a^{\frac{1}{3}}C(l)-b^{\frac{1}{3}}C(m), t) \geq_{L^*} \mathbb{P}'_{\mu',\nu'}(l+m, t)$$

for all $t > 0$ in which

$$\lim_{m\to\infty} T_{i=1}^{\infty}(\mathbb{P}'_{\mu',\nu'}(l,\ a^{\frac{-2(i-\frac{m}{2}+1)}{3}}t)) = 1_{L^*}$$

for all $l,\ m \in \mathbb{S}$ *and all* $t > 0$*, then there exists one and only one GCRF* $C_3 : \mathbb{S} \longrightarrow \mathbb{X}$ *such that*

$$\mathbb{P}_{\mu,\nu}(C(l) - C_3(l), t) \geq_{L^*} T_{i=1}^{\infty} \mathbb{P}\,'_{\mu',\nu'}(l,\ a^{\frac{-2}{3}(i-1)}t).$$

Example 8.3. *Let* $(\mathbb{S}, \|.\|)$ *be a Banach algebra,* $(\mathbb{S}, \mathbb{P}_{\mu,\nu},\ M)$ *an IR-NS in which*

$$\mathbb{P}_{\mu,\nu}(l,t) = \Big(\frac{t}{t+\|l\|}, \frac{\|l\|}{t+\|l\|}\Big)$$

and let $(\mathbb{X}, \mathbb{P}_{\mu,\nu},\ M)$ *be a complete IR-NS* $\forall\ l \in \mathbb{S}$*. Define* $C : \mathbb{S} \longrightarrow \mathbb{S}$ *by* $C(l) = l^{\frac{1}{3}}$*. we get*

$$\mathbb{P}_{\mu,\nu}(C(al+bm+3a^{\frac{2}{3}}.l^{\frac{2}{3}}.b^{\frac{1}{3}}.m^{\frac{1}{3}}+3a^{\frac{1}{3}}.l^{\frac{1}{3}}.b^{\frac{2}{3}}.m^{\frac{2}{3}})-a^{\frac{1}{3}}C(l)-b^{\frac{1}{3}}C(m),t) \geq_{L^*} \mathbb{P}\,'_{\mu',\nu'}(l+m,t),$$

$\forall\ t > 0$ *and* $a < 0$*. Also,*

$$\begin{aligned}\lim_{m\to\infty} M_{i=1}^{\infty}(\mathbb{P}_{\mu,\nu}(a^{m+i-1}l,\ a^{\frac{-2(i-\frac{m}{2}+1)}{3}})) &= \lim_{m\to\infty}\lim_{n\to\infty} M_{i=1}^{n}(\mathbb{P}_{\mu,\nu}(l,\ a^{\frac{-2(i-\frac{m}{2}+1)}{3}-m-i+1}t))\\ &= \lim_{m\to\infty}\lim_{n\to\infty}(\mathbb{P}_{\mu,\nu}(l,\ a^{\frac{-2}{3}(m+2)}t))\\ &= \lim_{m\to\infty}(\mathbb{P}_{\mu,\nu}(l,\ a^{\frac{-2}{3}(m+2)}t))\\ &= 1_{L^*}\end{aligned}$$

Thus, all the conditions of (8.1) *are true and so there exists one and only one GCRF* $C_3 : \mathbb{S} \longrightarrow \mathbb{X}$ *such that*

$$\mathbb{P}_{\mu,\nu}(C(l) - C_3(l), t) \geq_{L^*} \mathbb{P}_{\mu,\nu}(l,\ a^{\frac{-4}{3}}t).$$

9. Conclusion

We wind up this paper with a conclusion that we have proved stability results of GCRF equation associating a constant, sum of powers of norms, general control function, mixed product-sum of powers of norm and product of different powers of norms appropriate to the results established by Gavruta [18] and Rassias [12] in RNS using direct and fixed point method. We have proved the stability results in NA-RNS. Also, with the help of an example, we study stability results in IR-NS.

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