Advanced Studies in Contemporary Mathematics
31 (2021), No. 1, pp. 123 – 137

www.jangjeon.or.kr
http://dx.doi.org/10.17777/ascm2021.31.1.123

A NOTE ON SOLUTIONS FOR AN ITERATIVE FUNCTIONAL EQUATION IN 2-BANACH SPACES

JYOTSANA JAKHAR[1], RENU CHUGH[2], RAMU DUBEY[3,*] AND LAKSHMI NARAYAN MISHRA[4,*]

ABSTRACT. In this work, we explain an iterative equation for convex solutions in 2-Banach spaces. Our aim is to discuss the existence of monotone solutions in the real 2-Banach spaces and obtain the forward condition under which the solutions are convex or concave in order. Using the expansion method, we establish a general construction of the convex solutions for the iterative functional equation.

1. INTRODUCTION

Iterative functional equation given in [2, 9, 22, 27], is one of most important form of functional equations and also referred to as equation of rank one, in which iterates of the unknown function are linked in a linear combination. In energetic systems, many problems like embedding flows and dynamics of a quadratic mapping can be minimized to an iterative equation. The polynomial-like iterative equation

$$\alpha_1\psi(x) + \alpha_2\psi^2(x) + ... + \alpha_l\psi^l(x) = \Psi(x),\ x \in T, \tag{1.1}$$

where T is a nonempty subset of a vector space over $\mathbb{R}$, $\Psi : T \to T$ is a known function, $\alpha_i (i = 1, ..., l)$ are real constants, $\psi : T \to T$ is an unknown function and ψ^i is the i^{th} iterate of ψ, that is, $\psi^i = \psi(\psi^{i-1}(x))$ and $\psi^0(x) = x$ for all $x \in T$. In case of non-linear Ψ, many results had been discovered, for example, [12, 32] for $l = 2$, [25, 28] for any l, [29, 30] for smoothness and [16-19] for analyticity.

The actuality of convex solutions was established by initiating a partial order in Banach spaces in [6]. An extensive iterative equation can be expressed as

$$H(\psi(x), \psi^{k_1}(x), ..., \psi^{k_n}(x)) = \Psi(x),\ x \in T, \tag{1.2}$$

2010 *Mathematics Subject Classification.* Primary: 39B12, 39B52; Secondary: 46B99.

Key words and phrases. Convex Solutions, Increasing Solutions, Iterative Functional Equation, 2-Banach Spaces.

where $n > 0$ and $k_1, k_2, ..., k_n \geq 2$.
The existence, uniqueness and stability of continuous solutions for (1.2) were explored on the unit circle in [14, 15, 26]. A more extensive iterative functional equation

$$\mathcal{F}(\psi) \circ \psi = \Psi \tag{1.3}$$

was investigated in [11, 20, 21] in high dimensional spaces, where $\mathcal{F}$ is a self mapping. Actually, equation (1.3) is a generalization of iterative equation (1.2). If $\mathcal{F}(\psi) = H(\psi^0, \psi^{k_1-1}, ..., \psi^{k_n-1})$, then (1.3) reduces to (1.2). In [11], the existence of Lipschitzian solutions of (1.3) was proved on a compact convex subset of $\mathbb{R}^N$ and by using this result, the existence of Lipschitzian solutions for equation

$$\sum_{k=1}^{\infty} \alpha_k \psi^k(x) = \Psi(x), \tag{1.4}$$

was explored on a compact interval of $\mathbb{R}$ and on a compact convex subset of $\mathbb{R}^N, N > 1$. After that, the results were slightly generalized to an arbitrary closed subset of a Banach space and existence of solutions for iterative functional equations

$$\sum_{k=-\infty}^{\infty} B_k \psi^k(x) = \Psi(x), \tag{1.5}$$

$$B_\circ \psi(x) + \Sigma_{k=1}^{\infty} B_k \psi^k = \Psi(x) \tag{1.6}$$

was established in [21], where $B_k s$ are bounded linear operators.

The idea of convexity for iterative equations was discovered in 1968 when Kuczma and Smajdor [10] explored the convexity of iterative roots. In [24], convexity of solutions for(1.1) was examined on a compact interval and in [23], non-decreasing convex solutions for (1.1) on an open interval were examined. In [6], convexity of solutions for (1.1) was investigated in Banach spaces.

The concept of 2-normed spaces was initially developed by Gaahler [4] in the mid of 1960's. Since, Gunawan and Mashadi [7], Gaurdal [5] and many others had studied this concept and obtained various results. Let X be a real vector space of dimension d, where $d \geq 2$. A real-valued function $||.,.||$ on X^2 satisfying the following conditions:

(1) $||x_1, x_2|| = 0$ iff x_1, x_2 are linearly dependent,
(2) $||x_1, x_2||$ is invariant under permutation,
(3) $||\alpha x_1, x_2|| = |\alpha| ||x_1, x_2||$, for any $\alpha \in \mathbb{R}$,
(4) $||x + x', x_2|| \leq ||x, x_2|| + ||x', x_2||$

is called a 2-norm on X, and the pair $(X, ||.,.||)$ is called a 2-normed space.

A sequence (x_k) in a 2-normed space $(X, ||.,.||)$ is said to converge to some $x \in X$ in the 2-norm if

$$lim_{k\to\infty}||x_k - x, u_1|| = 0 \text{ for every } u_1 \in X.$$

A sequence (x_k) in a 2-normed space $(X, ||.,.||)$ is said to be Cauchy with respect to the 2-norm if

$$lim_{k,l\to\infty}||x_k - x_l, u_1|| = 0 \text{ for every } u_1 \in X.$$

If every Cauchy sequence in X converges to some $x \in X$, then X is said to be complete with respect to the 2-norm. Any complete 2-normed space is said to be a 2-Banach space.

In this paper, we investigate monotonicity and convexity of solutions for (1.2) and (1.3) in 2-Banach spaces as in [4, 5, 7]. Using Schauder's fixed point theorem, we analyze increasing (decreasing) solutions for (1.3). Moreover, we give conditions under which those solutions are convex or concave.

1.1. **Preliminaries.** A non-empty subset L of a real vector space X is called a cone if $x \in L$ implies that $ax \in L$ for all $a > 0$.

A non empty and non-trivial subset $L \subset X$ is called an order cone in X if L is a convex cone and satisfies $L \cap (-L) = \{\theta'\}$, where θ' is zero element of L, we can define a partial order $x \leq_L y$ in X, called the L-order as

$$y - x \in L. \tag{1.7}$$

A real vector space X equipped with a L-order is called an ordered vector space and denoted by (X, L) (see also[1, 3, 8, 13]). A real 2-Banach space $(X, ||.,.||)$ associated with a L-order is called an ordered real 2-Banach space, denoted by $(X, L, ||.,.||)$, if L is closed. A mapping $\psi : \Omega \times \Omega \to X$ is said to be increasing (decreasing) in the sense of the L-order if $x \leq_L y$ implies $f(x) \leq_L f(y)$ ($f(x) \geq_L f(y)$). A mapping is a convex in the sense of L-order if $\psi(\alpha x + (1-\alpha)y) \leq_L \alpha\psi(x) + (1-\alpha)\psi(y)$ and is said to be concave if $\psi(\alpha x + (1-\alpha)y) \geq_L \alpha\psi(x) + (1-\alpha)\psi(y), \forall\alpha \in [0, 1]$ and for every pair of different comparable points $x, y \in X$ (i.e., either $x \leq_L y$ or $x \geq_L y$).

Let Ω be a compact convex subset of an ordered real 2-Banach space $(X, L, ||.,.||)$ with non-empty interior and let $C(\Omega, X)$ be collection of all continuous functions mapping from Ω into X.

A mapping $\psi : \Omega \times \Omega \to X$, $(X, L, ||.,.||)$ is a 2-Banach space equipped with the norm $||\psi\psi_1||_{C(\Omega,X)} = Sup_{x\in\Omega}||\psi(x)\psi_1(x)||$.

For $0 \leq e \leq E < +\infty$, define

$$\begin{aligned} F^{+}(\Omega, e, E) &= \{\psi, \psi_1 \in (\Omega, X) : e(y^2 - x^2) \\ &\leq_L \psi(y)\psi_1(y) - \psi(x)\psi_1(x) \\ &\leq_L E(y^2 - x^2) \ if \ x \leq_L y, \\ &\quad ||\psi(y)\psi_1(y) - \psi(x)\psi_1(x)|| \\ &\leq_L E||y^2 - x^2|| \ if \ x \ and \ y \ are \ not \ comparable\}, \end{aligned}$$

$$\begin{aligned} F^{-}(\Omega, e, E) &= \{\psi, \psi_1 \in (\Omega, X) : e(y^2 - x^2) \\ &\leq_L \psi(x)\psi_1(x) - \psi(y)\psi_1(y) \\ &\leq_L E(y^2 - x^2) \ if \ x \leq_L y, \\ &\quad ||\psi(y)\psi_1(y) - \psi(x)\psi_1(x)|| \\ &\leq_L E||y^2 - x^2|| \ if \ x \ and \ y \ are \ not \ comparable\}, \end{aligned}$$

$$F^{+}_{cv}(\Omega, e, E) = \{\psi, \psi_1 \in F^{+}(\Omega, e, E) : \psi, \psi_1 \ are \ convex \ on \ \Omega \ in \ L - order\},$$

$$F^{-}_{cc}(\Omega, e, E) = \{\psi, \psi_1 \in F^{-}(\Omega, e, E) : \psi, \psi_1 \ are \ concave \ on \ \Omega \ in \ L - order\}$$

$$\begin{aligned} C^{+}(\Omega, e, E) &= \{\psi, \psi_1 \in F^{+}(\Omega, e, E) : \psi(\Omega), \psi_1(\Omega) \subset \Omega\} \\ C^{-}(\Omega, e, E) &= \{\psi, \psi_1 \in F^{-}(\Omega, e, E) : \psi(\Omega), \psi_1(\Omega) \subset \Omega\} \\ C^{+}_{cv}(\Omega, e, E) &= \{\psi, \psi_1 \in F^{+}_{cv}(\Omega, e, E) : \psi(\Omega), \psi_1(\Omega) \subset \Omega\} \\ C^{+}_{cc}(\Omega, e, E) &= \{\psi, \psi_1 \in F^{+}_{cc}(\Omega, e, E) : \psi(\Omega), \psi_1(\Omega) \subset \Omega\} \end{aligned}$$

$$\begin{aligned} I(\psi) &= inf\{E : \forall x, y \in \Omega, \theta' \leq_L \psi(y)\psi_1(y) - \psi(x)\psi_1(x) \\ &\leq_L E(y^2 - x^2) if x \leq_L y, \\ &\quad ||\psi(y)\psi_1(y) - \psi(x)\psi_1(x)|| \\ &\leq E||y^2 - x^2||, \ if \ x \ and \ y \ are \ not \ comparable\}. \end{aligned}$$

As in Lemma 2.2 in [6], $C^{+}(\Omega, e, E), C^{-}(\Omega, e, E), C^{+}_{cv}(\Omega, e, E)$ and $C^{+}_{cc}(\Omega, e, E)$ are compact convex subsets of $C(\Omega, X)$.

2. Increasing and Decreasing solutions

In this section, we talk about monotonicity and convexity of solutions for iterative functional equation (1.3) in the ordered real 2- Banach space $(X, L, ||., .||)$ such that L is normal and $N(L) \leq 1$ where L is an order cone in X. Examine (1.3) with the following supposition :

$(H1)$ $$\mathcal{F}(x) = x - \mathcal{P}(\psi)(x)$$

Theorem 2.1. *Let* $\mathcal{P} : C^+(\Omega, 0, \infty) \to F^+(\Omega, 0, \infty)$ *such that*

$$\mathcal{P}(f) \in F^+(\Omega, 0, \beta(I(\psi))) \tag{2.1}$$

where $\beta : (0, \infty) \to (0, \infty)$ *is an increasing function and* $\Psi(x) + \mathcal{P}(\psi)(y) \in \Omega$ *for any* $x, y \in \Omega$. *Suppose that* $H1$ *holds and* $\Psi \in F^+(\Omega, 0, q_1)$, *where* $q_1 \in (0, \infty)$ *is a constant. If there exists* $q \in (0, \infty)$ *such that*

$$q_1 \leq q(1 - \beta(q)) \tag{2.2}$$

where $\mathcal{P}|_{C^+(\Omega,0,q)}$ *is continuous, then (1.3) has a solution* $\psi \in C^+(\Omega, 0, q)$.

Proof. Define a mapping $\delta : C^+(\Omega, 0, q) \to C(\Omega, X)$ by

$$\delta\psi(x) = \Psi(x) + \mathcal{P}(\psi) \circ \psi(x). \tag{2.3}$$

We show that δ is a self mapping on $C^+(\Omega, 0, q)$, $\mathcal{P}(\psi) \circ \psi$ is well defined and $\Psi + \mathcal{P}(\psi) \circ \psi \in C(\Omega, X)$. Further, when $x, y \in \Omega$ are not comparable, $x - y \notin L$ and $y - x \notin L$, by (2.1) and $\Psi \in F^+(\Omega, 0, q_1)$, we have

$$\begin{aligned} ||\delta\psi(x)\psi_1(x) - \delta\psi(y)\psi_1(y)|| &= ||\Psi(x)\Psi_1(x) - \Psi(y)\Psi_1(y) \\ &+ \mathcal{P}(\psi\psi_1) \circ \psi(x)\psi_1(x) \\ &- \mathcal{P}(\psi\psi_1) \circ \psi(y)\psi_1(y)|| \\ &\leq ||\Psi(x)\Psi_1(x) - \Psi(y)\Psi_1(y)|| \\ &+ ||\mathcal{P}(\psi\psi_1) \circ \psi(x)\psi_1(x) \\ &- \mathcal{P}(\psi\psi_1) \circ \psi(y)\psi_1(y)|| \\ &\leq q_1||x^2 - y^2|| \\ &+ \beta(I(\psi\psi_1))||\psi(x)\psi_1(x) - \psi(y)\psi_1(y)|| \\ &\leq (q_1 + q\beta(q))||x^2 - y^2|| \end{aligned}$$

where the monotonicity of the function β is engaged, which implies that

$$||\delta\psi(x)\psi_1(x) - \delta\psi(y)\psi_1(y)|| \leq q||x^2 - y^2|| \tag{2.4}$$

as of (2.2). When $x, y \in \Omega$ are comparable, assume that $x \leq_L y$. By the definition of $C^+(\Omega, 0, q), \theta' \leq_L \psi(y)\psi_1(y) - \psi(x)\psi_1(x) \leq_L q(y^2 - x^2)$, thus $\psi(x)\psi_1(x) \leq_L \psi(y)\psi_1(y)$. Hence by (2.1), we get

$$\begin{aligned} \theta' &\leq_L \mathcal{P}(\psi\psi_1) \circ \psi(y)\psi_1(y) - \mathcal{P}(\psi\psi_1) \circ \psi(x)\psi_1(x) \\ &\leq_L \beta(I(\psi\psi_1))(\psi(y)\psi_1(y) - \psi(x)\psi_1(x)). \end{aligned} \tag{2.5}$$

Accordingly, we have

$$\begin{aligned}\theta' &\leq_L \delta\psi(y)\psi_1(y) - \delta\psi(x)\psi_1(x)\\ &= (\Psi(y)\Psi_1(y) - \Psi(x)\Psi_1(x)) + \mathcal{P}(\psi\psi_1)\\ &\quad \circ\psi(y)\psi_1(y) - \mathcal{P}(\psi\psi_1)\circ\psi(x)\psi_1(x)\\ &\leq q_1(y^2 - x^2) + \beta(I(\psi\psi_1))(\psi(y)\psi_1(y) - \psi(x)\psi_1(x))\\ &\leq (q_1 + q\beta(q))(x^2 - y^2)\end{aligned}$$

where the monotonicity of the function β is engaged, which implies that

$$\theta' \leq_L \delta\psi(y)\psi_1(y) - \delta\psi(x)\psi_1(x) \leq_L q(x^2 - y^2) \tag{2.6}$$

as of (2.2). So (2.4) and (2.6) implies that δ is a self mapping on $C^+(\Omega, 0, q)$. The continuity of $\mathcal{P}|_{C^+(\Omega,0,q)}$ implies that δ is continuous on $C^+(\Omega, 0, q)$. Because $C^+(\Omega, 0, q)$ is a compact convex subset, by Schauder fixed point theorem, we note that δ has a fixed point $\delta \in C^+(\Omega, 0, q)$. Hence, ψ is an increasing solution of (1.3). This complete the proof. □

Similarly, we can prove the theorem for decreasing solutions.

2.1. **Convex and Concave Solutions.** In order to prove Theorem 2.3, we use the following lemma.

Lemma 2.2. *(see [6, Lemma 3.1]). Let $(X, L, ||.,.||)$ be an ordered real 2-Banach space. Then composition $\psi \circ \phi$ is convex (concave) if both ψ and ϕ are convex (concave) and increasing. In particular, for increasing convex (concave) operator ψ, the iterate ψ^n is also convex (concave).*

Theorem 2.3. *Let $\mathcal{P} : C^+_{cv}(\Omega, 0, \infty) \to F^+_{cv}(\Omega, 0, \infty)$ such that*

$$\mathcal{P} \in F^+_{cv}(\Omega, 0, \beta(I(\psi))) \tag{2.7}$$

where $\beta : (0, \infty) \to (0, \infty)$ is an increasing function and $\Psi(x) + \mathcal{P}(\psi)(y) \in \Omega$ for any $x, y \in \Omega$. Suppose that $H1$ holds and $\Psi \in F^+_{cv}(\Omega, 0, q_1)$, where $q_1 \in (0, \infty)$ is a constant. If there exists $q \in (0, \infty)$ such that

$$q_1 \leq q(1 - \beta(q)) \tag{2.8}$$

and $\mathcal{P}|_{C^+_{cv}(\Omega,0,q)}$ is continuous, then (1.3) has a solution $\psi \in C^+_{cv}(\Omega, 0, q)$.

Proof. Define a mapping $\delta : C^+(\Omega, 0, q) \to C(\Omega, X)$ given in theorem (2.1). To prove that δ is a self mapping on $C^+_{cv}(\Omega, 0, q)$, it is enough to prove that $\Psi + \mathcal{P}(\psi) \circ \psi$ is convex in the sense of L-order on Ω, by (2.7), we know that $\mathcal{P}(\psi)(x)$ is increasing and convex on Ω. So, by Lemma 2.3 and $\psi \in C^+_{cv}(\Omega, 0, q)$, $\mathcal{P}(\psi) \circ \psi$ is convex in the sense of L-order on Ω. Therefore, by $\Psi \in F^+_{cv}(\Omega, 0, q)$,

$$\begin{aligned}\Psi(rx + (1-r)y) \quad &+ \quad \mathcal{P}(\psi) \circ \psi(rx + (1-r)y)\\ &\leq \quad r\Psi(x) + (1-r)\Psi(y) + r\mathcal{P}(\psi) \circ \psi + (1-r)\mathcal{P}(\psi) \circ \psi(y)\\ &= \quad r(\Psi(x) + \mathcal{P}(\psi) \circ \psi(x))\\ &+ \quad (1-r)(\Psi(y) + \mathcal{P}(\psi) \circ \psi(y))\end{aligned}$$

for every pair of distinct comparable points $x, y \in \Omega$ and $r \in [0, 1]$. So δ is a self mapping on $C^+_{cv}(\Omega, 0, q)$. The remaining part of the proof is same as the proof of above theorem. $\square$

Similarly, we can prove the theorem for concavity of solutions.

3. Iterative equation in 2-Banach spaces

We discuss the monotonicity and convexity of solutions for (1.2) in the ordered real 2-Banach space $(X, L, ||.,.||)$ such that L is normal and $N(L) \leq 1$. Consider (1.2) with the following sopposition:

$(H2)$ $\qquad H(y_0, y_1, ..., y_n) = y_0 + h(y_1, ..., y_n)h_1(y_1, ..., y_n),$
where $h(y_1, ..., y_n), h_1(y_1, ..., y_n) \in C(\Omega^n, X)$. Firstly, we prove the existence of increasing and decreasing solutions of (1.2).

3.1. **Increasing and Decreasing Solutions.** Firstly , we study increasing solutions. Consider (1.2) with the following supposition:

$(H3)$ $\exists\gamma_i > 0$ such that

$$\begin{aligned}\theta \quad &\leq_L \quad h(\overline{y_1}, ..., \overline{y_n})h_1(\overline{y_1}, ..., \overline{y_n}) - h(y_1, ..., y_n)h_1(y_1, ..., y_n)\\ &\leq_L \quad \Sigma_{i=1}^n \gamma_i(y_i^2 - \overline{y_i}^2) \;\; if \;\; \overline{y_i}^2 \leq_L y_i^2\end{aligned}$$

$||h(y_1, ..., y_n)h_1(y_1, ..., y_n) - h(\overline{y_1}, ..., \overline{y_n})h_1(\overline{y_1}, ..., \overline{y_n})|| \leq_L \Sigma_{i=1}^n \gamma_i(y_i^2 - \overline{y_i}^2)$ for any $y_i, \overline{y_i} \in \Omega$.

Theorem 3.1. *Assume that (H2) and (H3) hold and $\Psi \in F^+(\Omega, \circ, E_1)$, where $E_1 \in (0, \infty)$ is a constant. If $\Psi(x) - h(y_1, ..., y_n)h_1(y_1, ..., y_n) \in \Omega$ for any $x_i, y_i \in \Omega$ and*

$$E_1 \leq E - \Sigma_{i=1}^n \gamma_i E^{k_i} \tag{3.1}$$

for any constant $E \in (0, \infty)$, then (1.2) has a solution $\psi \in C^+(\Omega, \circ, E)$. Additionally, if

$$\Sigma_{i=1}^n \gamma_i \Sigma_{j=0}^{k_i - 1} E^j < 1, \tag{3.2}$$

then the solution ψ is unique in $C^+(\Omega, \circ, E)$ and depends continuously on Ψ.

In order to prove Theorem 3.1, we prove the following lemma.

Lemma 3.2. *Let $(X, L, ||.,.||)$ be an ordered real 2-Banach space such that L normal and let $\psi, h \in C^+(\Omega, e, E)$[respectively $C^-(\Omega, e, E)$, $C^+_{cv}(\Omega, e, E)$, and $C^+_{cc}(\Omega, e, E)$], where $0 \leq e \leq E \leq \infty$. Then*

$$||(\psi\psi_1)^k - (hh_1)^k||_{c(\Omega,X)} \leq \Sigma_{j=0}^{n-1} E_0^i ||(\psi\psi_1) - (hh_1)||_{c(\Omega,X)} \quad (3.3)$$

for all n = 1,2,...

Proof. Define

$$\mathcal{P}(\psi)(x) = -h(\psi^{k_1-1}(x), ..., \psi^{k_n-1}(x))h_1(\psi^{k_1-1}(x), ..., \psi^{k_n-1}(x)) \quad (3.4)$$

by (H2), $\mathcal{P}(\psi) \in C(\Omega, X)$. Now we prove that $\mathcal{P}(\psi)$ is an operator from $C^+(\Omega, \circ, \infty)$ to $F^+(\Omega, \circ, \infty)$ such that

$$\mathcal{P}(\psi) \in F^+(\Omega, \circ, \beta(I(\psi))). \quad (3.5)$$

By definition of $C^+(\Omega, \circ, \infty)$, for $\psi\psi_1 \in F^+(\Omega, \circ, \infty)$, we have

$$\theta \leq_L \psi(y)\psi_1(y) - \psi(x)\psi_1(x) \leq_L I(\psi\psi_1)(y^2 - x^2) \quad (3.6)$$

for $x \leq_L y$ and

$$||\psi(x)\psi_1(x) - \psi(y)\psi_1(y)|| \leq I(\psi\psi_1)||x^2 - y^2|| \quad (3.7)$$

if $x, y \in \Omega$ are not comparable, i.e. $x - y \notin L$ and $y - x \notin L$. Notice that

$$\begin{aligned} \theta &\leq_L \psi^{k_i-1}(y)\psi_1^{k_i-1}(y) - \psi^{k_i-1}(x)\psi_1^{k_i-1}(x) \\ &\leq_L I(\psi\psi_1)^{k_i-1}(y^2 - x^2), \ i = 1, 2, ..., n \end{aligned} \quad (3.8)$$

if $x \leq_L y$ and

$$||\psi^{k_i-1}(x)\psi_1^{k_i-1}(x) - \psi^{k_i-1}(y)\psi_1^{k_i-1}(y)|| \leq I(\psi\psi_1)^{k_i-1}||x^2 - y^2|| \quad (3.9)$$

if $x, y \in \Omega$ are not comparable, i.e. $x - y \notin L$ and $y - x \notin L$. By(H3), we get

$$\begin{aligned} \theta &\leq_L \mathcal{P}(\psi\psi_1)(y) - \mathcal{P}(\psi\psi_1)(x) \\ &= h(\psi^{k_1-1}(x)\psi_1^{k_1-1}(x), ..., \psi^{k_n-1}(x)\psi_1^{k_n-1}(x)) \\ &\quad\ h_1(\psi^{k_1-1}(x)\psi_1^{k_1-1}(x), ..., \psi^{k_n-1}(x)\psi_1^{k_n-1}(x)) \\ &- h(\psi^{k_1-1}(y)\psi_1^{k_1-1}(y), ..., \psi^{k_n-1}(y)\psi_1^{k_n-1}(y)) \\ &\quad\ h_1(\psi^{k_1-1}(y)\psi_1^{k_1-1}(y), ..., \psi^{k_n-1}(y)\psi_1^{k_n-1}(y)) \\ &\leq_L \Sigma_{i=1}^n \gamma_i(\psi^{k_i-1}(y)\psi_1^{k_i-1}(y) - \psi^{k_i-1}(x)\psi_1^{k_i-1}(x)) \\ &\leq_L \Sigma_{i=1}^n \gamma_i((\psi\psi_1)^{k_i-1})(y^2 - x^2), \end{aligned} \quad (3.10)$$

if $x \leq_L y$ and

$$\begin{aligned}
|| \quad & \mathcal{P}(\psi\psi_1)(y) - \mathcal{P}(\psi\psi_1)(x)|| \\
= \quad & ||h(\psi^{k_1-1}(x)\psi_1^{k_1-1}(x), ..., \psi^{k_n-1}(x)\psi_1^{k_n-1}(x)) \\
& h_1(\psi^{k_1-1}(x)\psi_1^{k_1-1}(x), ..., \psi^{k_n-1}(x)\psi_1^{k_n-1}(x)) \\
- \quad & h(\psi^{k_1-1}(y)\psi_1^{k_1-1}(y), ..., \psi^{k_n-1}(y)\psi_1^{k_n-1}(y)) \\
& h_1(\psi^{k_1-1}(y)\psi_1^{k_1-1}(y), ..., \psi^{k_n-1}(y)\psi_1^{k_n-1}(y))|| \\
\leq_L \quad & \Sigma_{i=1}^n \gamma_i ||(\psi^{k_i-1}(y)\psi_1^{k_i-1}(y) - \psi^{k_i-1}(x)\psi_1^{k_i-1}(x))|| \\
\leq_L \quad & \Sigma_{i=1}^n \gamma_i ((\psi\psi_1)^{k_i-1})||(y^2 - x^2)|| \qquad (3.11)
\end{aligned}$$

if $x, y \in \Omega$ are not comparable, i.e. $x - y \notin L$ and $y - x \notin L$, where $N(L) \leq 1$ is engaged. So,

$$\mathcal{P}(\psi\psi_1) \in F^+(\Omega, \circ, \Sigma_{i=1}^n \gamma_i (I(\psi\psi_1)^{k_i-1})).$$

Let $\beta(I(\psi\psi_1) = \Sigma_{i=1}^n \gamma_i (I(\psi\psi_1)^{k_i-1}))$ therefore, function β is increasing on $(0, \infty)$. So, we have $\psi^{k_i-1}(x) \in \Omega$, $i = 1, 2, ..., n$ for any $\psi \in C^+(\Omega, \circ, \infty)$. Therefore $\Psi(x) + \mathcal{P}(\psi)(y) \in \Omega$ for all $x, y \in \Omega$. Let $q = E$ and $q_1 = E_1$ by (3.1)

$$q(1 - \beta(q)) = E(1 - \Sigma_{i=1}^n \gamma_i E^{k_i-1}) = E - \Sigma_{i=1}^n \gamma_i E^{k_i} \geq E_1. \qquad (3.12)$$

Next we prove that $\mathcal{P}(\psi)|_{C^+(\Omega,\circ,E)}$ is continuous using (H3), for any $\psi', \psi'' \in C^+(\Omega, \circ, E)$, by Lemma 3.2, we get

$$\begin{aligned}
|| \quad & \mathcal{P}(\psi') - \mathcal{P}(\psi'')||_{C^+(\Omega,X)} \\
= \quad & sup_{x\in\Omega}||h(\psi'^{k_1-1}(x)\psi'^{k_1-1}_1(x), ..., \psi'^{k_n-1}(x)\psi'^{k_n-1}_1(x)) \\
& h_1(\psi'^{k_1-1}(x)\psi'^{k_1-1}_1(x), ..., \psi'^{k_n-1}(x)\psi'^{k_n-1}_1(x)) \\
- \quad & h(\psi''^{k_1-1}(x)\psi''^{k_1-1}_1(x), ..., \psi''^{k_n-1}(x)\psi''^{k_n-1}_1(x)) \\
& h_1(\psi''^{k_1-1}(y)\psi''^{k_1-1}_1(x), ..., \psi''^{k_n-1}(x)\psi''^{k_n-1}_1(x))|| \\
\leq \quad & \Sigma_{i=1}^n \gamma_i ||(\psi'^{k_i-1}(x))^2 - (\psi''^{k_i-1}(x))^2||_{C^+(\Omega,X)} \\
\leq \quad & \Sigma_{i=1}^n \gamma_i \Sigma_{j=0}^{k_i-2} E^j ||\psi'^2 - \psi''^2)||_{C^+(\Omega,X)}. \qquad (3.13)
\end{aligned}$$

By Theorem 1.1, there exists an $\psi \in C^+(\Omega, \circ, X)$ such that

$$\mathcal{F}(\psi) \circ \psi = \Psi. \qquad (3.14)$$

By $(H2)$, we get

$$H(\psi, \psi^{k_1}, ..., \psi^{k_n}) = \Psi \qquad (3.15)$$

and

by (3.2), we have

$$||\Psi' - \Psi''||_{C(\Omega,X)} \leq \left(\frac{1}{1 - \Sigma_{i=1}^{n}\gamma_i\Sigma_{j=1}^{k_{i-1}}E^j}\right)||\Psi' - \Psi''||_{C(\Omega,X)},$$

which implies that the solution of (1.2) depends continuously on Ψ. The proof is completed. □

Similarly, we can prove the theorem for decreasing solutions. We need the following supposition:

$(H4)$ $\exists$ $\gamma_i > 0$ such that

$$\begin{aligned} \theta &\leq_L h(y_1, y_2, ..., y_n)h_1(y_1, y_2, ..., y_n) \\ &\leq_L \Sigma_{i=1}^{n}\gamma_i(y_i^2 - \overline{y}_i^2) \ if \ \overline{y}_i \leq_L y_i, \end{aligned}$$

$||h(y_1, y_2, ..., y_n)h_1(y_1, y_2, ..., y_n) - h(\overline{y_1}, \overline{y_2}, ..., \overline{y_n})h_1(\overline{y_1}, \overline{y_2}, ..., \overline{y_n})||$
$\leq \Sigma_{i=1}^{n}\gamma_i(y_i^2 - \overline{y}_i^2)$ for any $y_i, \overline{y}_i \in \Omega$.

Convex and Concave solutions

Now, we study convexity of solutions for (1.2). Consider (1.2) with the following supposition:

$(H5)$ if $y_i \leq_L \overline{y}_i$ or $\overline{y}_i \leq_L y_i$ in Ω, then

$rh(y_1, y_2, ..., y_n) + (1-r)h(\overline{y_1}, \overline{y_2}, ..., \overline{y_n}) \leq_L h(ry_1 + (1-r)\overline{y_1}), ..., h(ry_n + (1-r)\overline{y_n})$

Theorem 3.3. *Suppose that (H2), (H3) and (H5) holds and* $\Psi \in F_{cv}^{+}(\Omega, \circ, E_1)$ *where* $E_1 \in (0, \infty)$ *is a constant. If* $\Psi(x) - h(y_1, y_2, ..., y_n) \in \Omega$ *for any* $x_i, y_i \in \Omega$ *and*

$$E_1 \subseteq E - \Sigma_{i=1}^{n}\gamma_i M^{k_i} \tag{3.16}$$

for a constant $E \in (0, \infty)$*, then (1.2) has a solution on* $\psi \in C_{cv}^{+}(\Omega, \circ, E)$*. Additionally, if*

$$\Sigma_{i=1}^{n}\gamma_i\Sigma_{j=1}^{k_{i-1}}E^j < 1 \tag{3.17}$$

then the solution ψ *is unique in* $C_{cv}^{+}(\Omega, \circ, E)$ *and depends continuously on* Ψ*.*

Proof. Similar to Theorem 2.1, by applying Theorem 2.3, it is sufficient to prove that $\mathcal{P}(\psi) = -h(\psi^{k_1-1}, ..., \psi^{k_n-1})$ is convex in the sense of L-order. In fact, each ψ^{k_i-1}, $i = 1, 2, ..., n$ is convex in the sense of L-order because ψ is increasing and convex by Lemma 2.2.

Therefore, for every distinct comparable point $x, y \in \Omega$, suppose $x \leq_L y$; by $(H3)$ and $(H5)$, we have

$$\begin{aligned}
\mathcal{P}(\psi)(rx &+ (1-r)y) \\
&= -h(\psi^{k_1-1}(rx+(1-r)y), ..., \psi^{k_n-1}(rx+(1-r)y)) \\
&\leq_L -h(r\psi^{k_1-1}(x)) \\
&+ (1-r)\psi^{k_1-1}(y), ..., (r\psi^{k_n-1}(x)) + (1-r)\psi^{k_n-1}(y) \\
&\leq_L -rh(\psi^{k_1-1}(x), ..., \psi^{k_n-1}(x)) \\
&- (1-r)h(\psi^{k_1-1}(x), ..., \psi^{k_n-1}(x)) \\
&= r\psi x + (1-r)\psi(y).
\end{aligned}$$

This completes the proof. □

In a similar manner, we can obtain the theorem for concavity of solutions. We need the following supposition:

$(H6)$ if $y_i \leq_L \overline{y_i}$ or $\overline{y_i} \leq_L y_i$ in Ω, then

$$\begin{aligned}
h &\quad (ry_1 + (1-r)\overline{y_1}, ..., ry_n + (1-r)\overline{y_n}) \\
\leq_L &\quad rh(y_1, ...y_n) + (1-r)h(\overline{y_1}, ..., \overline{y_n})
\end{aligned} \tag{3.18}$$

Availability of data and material. Not applicable

Competing interests. The authors has no competing interests.

Funding. The authors have no funding agency.

Authors' contributions. All authors read and approved the final article.

Acknowledgements. Suggestions of the referee(s), in forming the paper in the present form, are gratefully acknowledged. The authors has no Competing interests.

References

[1] Amann H., Fixed point equations and nonlinear eigenvalue problems in ordered Banach spaces, SIAM Review, vol. 18, no. 4, pp. 620-709, 1976.

[2] Baron K. and Jarczyk W., Recent results on functional equations in a single variable, perspectives and open problems, Aequationes Mathematicae, 61, 1-2, 1-48, 2001.

[3] Dhombres J. G., Iteration lineaire dordre deux, Publicationes Mathematicae Debrecen, 24, 3-4, 177-187, 1977.

[4] Gaahler S., Linear 2-normietre Raume, Math. Nachr. 28, 1-43, 1965.

[5] Gaurdal M., On ideal convergent sequences in 2-normed spaces, Thai J. Math. 4, 85-91, 2006.

[6] Gong X. and Zhang W., Convex solutions of the polynomial-like iterative equation in Banach spaces, Publicationes Mathematicae Debrecen, 82, 2, 341-358, 2013.

[7] Gunawan H. and Mashadi, On finite dimensional 2-normed spaces, Soo-chow J. of Math. 27, 631-639, 2001.

[8] Jarczyk W., On an equation of linear iteration, Aequationes Mathematicae, 51, 3, 303-310, 1996.

[9] Kuczma M., Choczewski B. and Ger R., Iterative Functional Equations, Encyclopedia of Mathematics and its Applications, Cambridge University Press, Cambridge, UK, vol. 32, 1990.

[10] Kuczma M. and Smajdor A., Fractional iteration in the class of convex functions, Bulletin delAcademie Polonaise des Sciences. Serie des Sciences Mathematiques, Astronomiques et Physiques, 16, 717-720, 1968.

[11] Kulczycki M. and Tabor J., Iterative functional equations in the class of Lipschitz functions, Aequationes Mathematicae, 64, 1-2, 24-33, 2002.

[12] Malenica M., On the solutions of the functional equation $\phi(x) + \phi^2(x) = F(x)$, Matematicki Vesnik, 6, 3, 301-305, 1982.

[13] Matkowski J. and Zhang W., On linear dependence of iterates, Journal of Applied Analysis, 6, 1, 149-157, 2000.

[14] Mukherjea A. and Ratti J. S., On a functional equation involving iterates of a bijection on the unit interval, Nonlinear Analysis, 7, 8, 899-908, 1983.

[15] Mukherjea A. and Ratti J. S., A functional equation involving iterates of a bijection on the unit interval. II, Nonlinear Analysis: Theory, Methods and Applications, 31, 3-4, 459-464, 1998.

[16] Si J. G., The existence of local analytic solutions to the iterated equation $\Sigma_{i=1}^{n}\lambda_i f^i(x) = F(x)$, Acta Mathematica Sinica, 37, 5, 590-600, 1994 (Chinese).

[17] Si J. G., Discussion on the C^0 solution of the iterated equation $G(f^{n0}(x); f^{n1}(x), ..., f^{nk}(x)) = F(x)$, Journal of Mathematical Research and Exposition, 15, 149-150, 1995 (Chinese).

[18] Si J. G., C^1-solutions of a class of iterative equations, Acta Mathematica Sinica, Chinese Series, 39, 2, 247-256, 1996 (Chinese).

[19] Si J. G. and Zhang W. N., On C^2 solutions of a functional equations, Acta Mathematica Sinica, 41, 5, 1061-1064, 1998 (Chinese).

[20] Tabor J., On a linear iterative equation, Results in Mathematics, 27, 3-4, 412-421, 1995.

[21] Tabor J. and Zodak M., Iterative equations in Banach spaces, Journal of Mathematical Analysis and Applications, 29, 2, 651-662, 2004.

[22] Targonski G., Topics in Iteration Theory, vol. 6, Vandenhoeck and Ruprecht, Gottingen, Germany, 1981.

[23] Trif T., Convex solutions to polynomial-like iterative equations on open intervals, Aequationes Mathematicae, 79, 3, 315-325, 2010.

[24] Xu B. and Zhang W., Decreasing solutions and convex solutions of the polynomial-like iterative equation, Journal of Mathematical Analysis and Applications, 329, 1, 483-497, 2007.

[25] Yang D. and Zhang W., Characteristic solutions of polynomial-like iterative equations, Aequationes Mathematicae, 67, 1-2, 80-105, 2004.

[26] Zdun M. C. and Zhang W., A general class of iterative equations on the unit circle, Czechoslovak Mathematical Journal, 57, 3, 809-829, 2007.

[27] Zhang J., Yang L. and Zhang W., Some advances on functional equations, Advances in Mathematics, 24, 5, 385-405, 1995.

[28] Zhang W., Discussion on the iterated equation on $\Sigma_{i=1}^{n} \lambda_i f^i(x) = F(x)$, Chinese Science Bulletin, 32, 1444-1451, 1987.

[29] Zhang W., Discussion on the differentiable solutions of the iterated equation $\Sigma_{i=1}^{n}\lambda_i f^i(x) = F(x)$, Nonlinear Analysis Theory, Methods and Applications, 15, 4, 387-398, 1990.

[30] Zhang W., Solutions of equivariance for a polynomial-like iterative equation, Proceedings of the Royal Society of Edinburgh A, 130, 5, 1153-1163, 2000.

[31] Zhang W., Nikodem K. and Xu B., Convex solutions of polynomial-like iterative equations, Journal of Mathematical Analysis and Applications, 315, 1, 29-40, 2006.

[32] Zhao L. R., Existence and uniqueness theorem for the solution of the functional equation $\lambda_1 f(x)+\lambda_2 f^{[2]}(x) = F(x)$, Journal of China University of Science and Technology, 13, 21-27, 1983 (Chinese).

Jyotsana Jakhar[1], Renu Chugh[2], Ramu Dubey[3,*] and Lakshmi Narayan Mishra[4,*]

[1,2]Department of Mathematics, M.D. University, Rohtak-124001, Haryana (INDIA), [3,*]Department of Mathematics, J.C. Bose University of Science and Technology, YMCA, Faridabad 121 006, Haryana, INDIA, [4,*]Department of Mathematics, School of Advanced Sciences, Vellore Institute of Technology (VIT) University, Vellore 632 014, Tamil Nadu, India

E-mail address: dahiya.jyotsana.j@gmail.com (Jyotsana Jakhar); chugh.r1@gmail.com (Renu Chugh); rdubeyjiya@gmail.com, ramudubey@jcboseust.ac.in (Ramu Dubey) and lakshminarayanmishra04@gmail.com (Lakshmi Narayan Mishra)