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Fibonacci Graph and its Energies

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Abstract

The energy of a graph is defined as the sum of absolute values of its eigenvalues. In this paper, we compute the spectrum and energy of the Fibonacci graph. Numerous matrices can be associated with a graph and their spectrums provide useful information about the graph. In recent times, various other graph energies are studied, based on eigenvalues of several graph matrices. In the present paper, we also establish relationship between the usual energy of the Fibonacci graph and other energies like Signless Lapalcian energy, Randić energy, maximum degree energy, common-neighborhood energy, 2-distance energy and Seidal energy.

Keywords: Fibonacci graph, energy of a graph, bounds for energy of a graph, graph parameters, Ramanujan graph.

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1 Introduction

Let G be a graph with vertex set $V(G) = \{v_1, v_2, \dots, v_n\}$ and the edge set $E(G)$, where $|V(G)| = n$ and $|E(G)| = m$. The degree of a vertex v_i is the number of edges incident with v_i and is denoted by d_i. A graph in which all the vertices are of same degree d is called a d-regular graph. In this paper the graphs that we consider are, simple (without loops) and undirected. The adjacency matrix $A = A(G)$ of G is an $n \times n$ matrix whose (i,j)-entry is 1 if the vertices v_i and v_j are adjacent and zero otherwise. The adjacency matrix A is a real symmetric matrix and hence the spectrum of A, that is , the set of eigenvalues of A is real. The spectrum of A is called the spectrum of G and is denoted by $Sp(G)$.

Let $\lambda_1(G) \geq \lambda_2(G) \geq \lambda_3(G) \ldots \geq \lambda_n(G)$ denote the eigenvalues of A. If $\lambda_1, \lambda_2, \lambda_3, \ldots, \lambda_s$ are the distinct eigenvalues of G and if m_i is the multiplicity of λ_i, we write,

$$\text{Sp(G)}=\begin{pmatrix} \lambda_1, & \lambda_2, & \ldots, & \lambda_s \\ m_1, & m_2, & \ldots & m_s \end{pmatrix}.$$

As well known, λ_1 is the spectral radius of G,

$$\sum_{i=1}^{n} \lambda_i = 0,$$

$$\sum_{i=1}^{n} \lambda_i^2 = 2m,$$

and

$$detA = \prod_{i=1}^{n} \lambda_i.$$

An extensive literature about adjacency matrix and the notations not defined here can be found in the book by D. Cvetković, M. Doob, H. Sachs [7].

The first mathematical paper on graph spectra was motivated by the membrane vibration problem. Graph spectra have several important applications in computer science such as internet technologies, pattern recognition, computer vision and in many other areas. Recent book by D. Cvetković and Ivan Gutman [8] discusses applications of graph spectra in various fields like Chemistry, Physics and Computer Science.

The inspiration of description energy of a graph happened from quantum chemistry. During 1930s E.Hockel, while finding an approximate solution of the Schrödinger equation of a class of organic molecules, presented chemical applications of graph theory in his molecular orbital theory. In quantum chemistry, the skeleton of an unsaturated hydrocarbon is represented by a graph. The energy levels of electrons in such a molecule are eigenvalues of the graph. The strength of particles is closely identified with the spectrum of its graph. The carbon atoms and chemical bond between them in a hydrocarbon system denote vertices and edges respectively.

The energy of G, denoted by $\varepsilon(G)$ is defined as

$$\varepsilon(G) = \sum_{i=1}^{n} |\lambda_i|.$$

This concept was introduced by I. Gutman [10], and is intensively studied in Chemistry. Extensive research has been done on energy of graphs. Most of these work can be found in the book [14] and in the research paper by V. Brankov et al. [5]. Very recently Ivan Gutman and Boris Furtula [11] published a paper on "Survey of Energy of graphs", which provides basic facts and bibliographic data on graph energies.

In 1971, MeClelland [17] obtained the first upper bound for the energy of graphs in terms of n and m as follows:

$$\varepsilon(G) \leq \sqrt{2mn}. \tag{1.1}$$

MeClelland [17] also discovered the following lower bound in terms of n,m and $detA$:

$$\varepsilon(G) \geq \sqrt{2m + n(n-1)|detA|^{2/n}}. \tag{1.2}$$

Since then many research articles have been published on the bounds for the energy of the graph. For works on bounds of the graph energy one can refer [1, 13]. Fibonacci numbers appear unexpectedly often in Mathematics and their applications which include computer algorithms. The Fibonacci sequence $\{F_n\}_{n=0}^{\infty}$ is defined as follows:

$$F_0 = 1,\ F_1 = 1 \text{ and } F_n = F_{n-1} + F_{n-2} \text{ for } n \geq 2.$$

In this paper we are mainly interested in the Fibonacci graph $F_{d,2n}$. Here d is such that F_d is the largest Fibonacci number less than or equal to n. $F_{d,2n}$ is a graph with vertex set $V = V_1 \cup V_2$, where

$$V_1 = \{v_1, v_2, \ldots, v_n\},$$

$$V_2 = \{v_{n+1}, v_{n+2}, \ldots, v_{2n}\}$$

and v_iv_j for $1 \leq i \leq n$ and $n+1 \leq j \leq 2n$ is an edge if $j-i+1$ or $j-i+1-n$ is a member of the set

$$S = \{F_1, F_2, \ldots, F_d\}.$$

The graph shown in the figure below is a Fibonacci graph of $F_{4,12}$ with degree of each vertex being 4 and with 12 vertices.

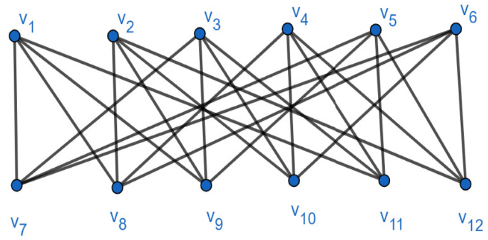

Figure 1: Fibonacci Graph $F_{4,12}$

A Fibonacci pair (f_1, f_2) is a pair with both elements being Fibonacci numbers. For example $(1, 8)$ is a Fibonacci pair as both 1 and 8 are Fibonacci numbers.

Fibonacci graph has been used for the purpose of performing effective communications in networks. They are considered for the 1-way mode(half-duplex) of communication in which the information can flow in one direction at a time, that is, during the call between x and y, only y can receive information from x, or x can receive information from y, not both. For details about the number of rounds in which gossiping can be performed one can refer the paper by Johance Cohen et al.[6].

The rest of the paper is structured as follows. In Section 2, we compute the eigenvalues of the Fibonacci graph $F_{d,2n}$. Section 3 is dedicated to the study of different energies of Fibonacci graphs like signless Laplacian energy, Randić energy, maximum degree energy, 2-distance energy, common-neighbourhood energy and Seidel energy. In the last section we state some open problems about the Fibonacci graph being a Ramanujan graph.

2 Eigenvalues of the Fibonacci Graph

In this section we determine the eigenvalues of the Fibonacci graph. The Fibonacci graph $F_{d,2n}$ is a connected, bipartite, regular graph of degree d and in consequence, its spectra have the following distinctive features:

1. d is an eigenvalue of $F_{d,2n}$.

2. The multiplicity of d is 1.

3. For any eigenvalue λ of $F_{d,2n}$, we have $|\lambda| \leq d$.
4. The spectrum of $F_{d,2n}$ is symmetric with respect to 0, that is , if λ is an eigenvalue of $F_{d,2n}$, then $-\lambda$ is also an eigenvalue of $F_{d,2n}$.

Definition 2.1. *The characteristic polynomial of a graph G is the characteristic polynomial of the adjacency matrix A of G and is denoted by $\chi(G;\lambda)$. Hence,*

$$\chi(G;\lambda) = det(xI - A).$$

Definition 2.2. *A circulant matrix of order 'n' is a square matrix of order 'n' in which all the rows are obtainable by successive cyclic shifts of one of its rows (usually taken as the first row), and so any circulant matrix is determined by its first row.*

Lemma 2.3. *Let C be a circulant matrix of order n with first row $(a_1, a_2, \dots a_n)$ then, the eigenvalues of C are*

$$\lambda_r = \sum_{j=1}^{n} a_j \omega^{(j-1)r}, \quad r = 0, 1, 2, \dots, (n-1).$$

Here, $\omega = exp(2\pi i/n)$. We now compute the eigenvalues of $F_{d,2n}$.

The adjacency matrix A of $F_{d,2n}$ is,

$$\mathrm{A}=\begin{pmatrix} 0 & C \\ C^T & 0 \end{pmatrix}_{2n\times 2n}$$

Here, $C = C_{n\times n}$ is a circulant matrix whose first row is $\{a_1, a_2, \dots, a_n\}$, where

$$a_i = \begin{cases} 1 & \text{if} \quad i = F_1, F_2, \dots, F_d, \\ 0 & \text{otherwise.} \end{cases}$$

The characteristic polynomial of $F_{d,2n}$ is

$$|\lambda I_{2n} - A| = |\lambda^2 I_n - CC^T| = |\lambda^2 I_n - D|,$$

where $D = CC^T$ and I_n is the identity matrix of order n. We have,

$$\mathrm{D}=\begin{pmatrix} a_1 & a_2 & a_3 & \dots & a_{n-1} & a_n \\ a_n & a_1 & a_2 & \dots & a_{n-2} & a_{n-1} \\ . & . & . & \dots & . & . \\ . & . & . & \dots & . & . \\ . & . & . & \dots & . & . \\ . & . & . & \dots & . & . \\ a_2 & a_3 & a_4 & \dots & a_n & a_1 \end{pmatrix} \begin{pmatrix} a_1 & a_n & a_{n-1} & \dots & a_2 \\ a_2 & a_1 & a_n & \dots & a_3 \\ . & . & . & \dots & . \\ . & . & . & \dots & . \\ . & . & . & \dots & . \\ . & . & . & \dots & . \\ a_n & a_{n-1} & a_{n-2} & \dots & a_1 \end{pmatrix}.$$

Observe that C^T is a circulant matrix. Hence $D = CC^T$ is also circulant.

Let $(d_1, d_2, d_3, \ldots, d_n)$ be the first row of the circulant matrix D. Since,

$$a_i{}^2 = \begin{cases} 1 & \text{if} \quad i = F_1, F_2, \ldots, F_d, \\ 0 & \text{otherwise,} \end{cases}$$

it follows that $d_1 = a_1{}^2 + a_2{}^2 + \ldots + a_n{}^2 = d$. For $j \geq 2$,

$$d_j = a_1 a_{n-(j-2)} + a_2 a_{n-(j-3)} + \ldots + a_j a_1 + a_{j+1} a_2 + \ldots + a_n a_{n-(j-1)}. \quad (2.1)$$

Note that each term on the right side of the above equation is either one or zero. It is one, precisely when both suffices of a term are Fibonacci numbers. Thus, d_j is equal to the number of Fibonacci pairs in the set

$$S_1 = \{(n-(j-2),1),(n-(j-3),2),\ldots,(n,j-1),(1,j),\ldots,(n-(j-1),n)\}.$$

Similarly d_{n-j+2} is the number of Fibonacci pairs in the set

$$S_2 = \{(j,1),(j+1,2),\ldots,(n,n-j+1),(1,n-j+2),\ldots,(n,j-1)\}.$$

As $S_1 = S_2$, it follows that

$$d_j = d_{n-j+2}, \quad for \quad j \geq 2. \quad (2.2)$$

Therefore by Lemma 2.3, the eigenvalues of $F_{d,2n}$ are

$$\pm\sqrt{d + d_2\omega^r + d_3\omega^{2r} + \ldots + d_n\omega^{r(n-1)}}, r = 0,1,2,\ldots,n-1, \quad (2.3)$$

where

$$\omega = e^{2\pi i/n} = cos(2\pi/n) + isin(2\pi/n).$$

From (2.2), we have

$$d_2 = d_n, \quad d_3 = d_{n-1}, \ldots, d_{(n+1)/2} = d_{(n+3)/2}, \text{ for odd n,} \quad (2.4)$$

and

$$d_2 = d_n, d_3 = d_{n-1}, \ldots, d_{n/2} = d_{(n+4)/2}, \text{ for even n.} \quad (2.5)$$

Suppose n is odd. Using (2.4) in (2.3) and the fact that $\omega^n = 1$, we see that the eigenvalues of $F_{d,2n}$ are

$$\pm\sqrt{d + d_2(\omega^r + \omega^{-r}) + d_3(\omega^{2r} + \omega^{-2r}) + \ldots + d_{(n+1)/2}(\omega^{(n-1)r/2} + \omega^{-(n-1)r/2})}$$

$$= \pm\sqrt{d + 2\sum_{j=1}^{(n-1)/2} d_{j+1}cos(2\pi jr/n)}, \quad r = 0,1,2,\ldots,(n-1).$$

Similarly for even n one can show that the eigenvalues of $F_{d,2n}$ are

$$\pm\sqrt{d+2\sum_{j=1}^{(n-2)/2} d_{j+1}cos(2\pi jr/n)+d_{(n+2)/2}(-1)^r}, \quad r=0,1,2,\ldots,(n-1).$$

Thus we have proved the following theorem.

Theorem 2.4. *If 'n' is odd, then the eigenvalues of $F_{d,2n}$ are*

$$\pm\sqrt{d+2\sum_{j=1}^{(n-1)/2} d_{j+1}cos(2\pi jr/n)}, \qquad r=0,1,2,\ldots,(n-1).$$

and, if 'n' is even then the eigenvalues of $F_{d,2n}$ are

$$\pm\sqrt{d+2\sum_{j=1}^{(n-2)/2} d_{j+1}cos(2\pi jr/n)+d_{(n+2)/2}(-1)^r}, \qquad r=0,1,2,\ldots,(n-1).$$

For example, the eigenvalues of $F_{4,12}$ are $\pm4,\pm2,\pm1,\pm1,\pm1,\pm1$ and its energy is $\varepsilon(F_{4,12})=20$.
Eigenvalues of $F_{5,22}$ are $\pm\sqrt{3},\pm\sqrt{3},\pm\sqrt{3},\pm\sqrt{3},\pm\sqrt{3},\pm\sqrt{3},\pm\sqrt{3},\pm\sqrt{3},$ $\pm\sqrt{3},\pm\sqrt{3},\pm5$, and its energy is $\varepsilon(F_{5,22})=10(2\sqrt{3}+1)$.

3 Various Energies of Fibonacci Graph

In recent times, several energies are being considered, based on eigenvalues of different matrices associated with the graph. Some of them provide useful information about the graph[15, 16]. This section is devoted to the computation of various energies of the Fibonacci graph.

3.1 Laplacian and Signless Laplacian Energy of Fibonacci Graph

Let $A(G)$ be the adjacency matrix of a graph G of order n with the vertex set $V(G)=\{v_1,v_2,\ldots,v_n\}$ and the edge set $E(G)$, $|E(G)|=m$. Let d_i be the degree of the vertex v_i for $i=1,2,\ldots,n$ and $D(G)$ be the diagonal matrix of order n whose (i,i)-entry is d_i. Then,

$$L^+(G)=D(G)+A(G)$$

and

$$L(G) = D(G) - A(G)$$

are called the signless Laplacian matrix and Laplacian matrix of G respectively.

If $\mu_1, \mu_2, \ldots, \mu_n$ are the eigenvalues of $L^+(G)$, then the signless Laplacian energy of the graph G is defined as,

$$LE^+(G) = \sum_{i=1}^{n} |\mu_i - \frac{2m}{n}|.$$

If $\eta_1, \eta_2, \ldots, \eta_n$ are the eigenvalues of $L(G)$, then the Laplacian energy of the graph G is defined as

$$LE(G) = \sum_{i=1}^{n} |\eta_i - \frac{2m}{n}|.$$

We begin with the following well known results:

$$LE^+(G) \leq \varepsilon(G) + \sum_{i=1}^{n} |d_i - \frac{2m}{n}|.$$

$$LE(G) \leq \varepsilon(G) + \sum_{i=1}^{n} |d_i - \frac{2m}{n}|.$$

If the graph G is connected, then the equality holds if and only if G is regular.

The Zagreb index of a graph G is defined as:

$$Z_g(G) = \sum_{i=1}^{n} {d_i}^2,$$

Using this invariant and Cauchy-Schwarz inequality, it is easy to show that

$$LE^+(G) \leq \varepsilon(G) + \sqrt{nZ_g(G) - 4m^2},$$

and the equality holds if and only if G is regular.

Theorem 3.1. *We have*

$$LE^+(F_{d,2n}) = \varepsilon(F_{d,2n})$$

and

$$LE(F_{d,2n}) = \varepsilon(F_{d,2n}).$$

Proof. Since $F_{d,2n}$ is a d-regular graph with $2n$ vertices and $m = nd$ edges, we have,

$$\begin{aligned} LE^{+}(F_{d,2n}) &= \varepsilon(F_{d,2n}) + \sqrt{2nZ_g(F_{d,2n}) - 4(nd)^2} \\ &= \varepsilon(F_{d,2n}) + \sqrt{2n\sum_{i=1}^{2n} d^2 - 4n^2d^2} \\ &= \varepsilon(F_{d,2n}). \end{aligned}$$

Proof of the other statement is similar. □

3.2 Randić Energy of Fibonacci Graph

The Randić energy $RE(G)$ of a simple connected graph G is the sum of the absolute values of eigenvalues of the Randić matrix $R(G) = (r_{ij})$, where

$$r_{ij} = \begin{cases} \frac{1}{\sqrt{d_i d_j}}, & \text{if} \quad v_i v_j \epsilon E(G), \\ 0 & \text{otherwise.} \end{cases}$$

Here, d_i and d_j are the degrees of vertices v_i and v_j respectively.

Theorem 3.2.

$$RE(F_{d,2n}) = \frac{1}{d}\varepsilon(F_{d,2n}).$$

Proof. Since $F_{d,2n}$ is a d-regular graph, we have

$$r_{ij} = \begin{cases} \frac{1}{d}, & \text{if} \quad v_i \text{ and } v_j \quad \text{are adjacent,} \\ 0 & \text{otherwise.} \end{cases}$$

Thus,

$$R(F_{d,2n}) = \frac{1}{d}A,$$

where A is the adjacency matrix of $F_{d,2n}$. Hence, Randić eigenvalues are

$$r_i = \frac{\lambda_i}{d}, \qquad i = 1, 2, \ldots, 2n,$$

where, λ_i, $i = 1, 2, \ldots, 2n$ are the eigenvalues of A. Thus,

$$\begin{aligned} RE(F_{d,2n}) &= \sum_{i=1}^{2n} |r_i| \\ &= \frac{1}{d}\sum_{i=1}^{2n} |\lambda_i| \\ &= \frac{1}{d}\varepsilon(F_{d,2n}). \end{aligned}$$

This completes the proof. □

3.3 Maximum Degree Energy of Fibonacci Graph

The maximum degree energy $EM(G)$ of a simple connected graph G [2] is defined as the sum of the absolute values of eigenvalues of the maximum degree matrix $M(G) = (m_{ij})$, where

$$m_{ij} = \begin{cases} max(d_i, d_j), & \text{if} \quad v_i \text{ and } v_j \text{ are adjacent,} \\ 0 & \text{otherwise,} \end{cases}$$

where d_i and d_j are the degrees of v_i and v_j respectively.

Theorem 3.3. *For the Fibonacci graph $F_{d,2n}$,*

$$EM(F_{d,2n}) = d\varepsilon(F_{d,2n}).$$

As the result directly follows from the fact that $M(F_{d,2n}) = dA$, where A is the adjacency matrix of $F_{d,2n}$, we omit the details.

3.4 2-Distance Energy of Fibonacci Graph

Let G be a connected graph with vertex set $\{v_1, v_2, \ldots, v_n\}$. The distance matrix of G, denoted by $D(G)$ is the $n \times n$ matrix whose (i, j)-entry is equal to $d(v_i, v_j)$, the distance between v_i and v_j. Note that $d(v_i, v_i) = 0$, $i = 1, 2, \ldots, n$. The eigenvalues of $D(G)$ are said to be the D-eigenvalues of G. Balaban et al.[4] proposed the use of the largest D-eigenvalue as molecular descriptor, while in [12] it was used to infer the extent of branching and model boiling points of alkanes.

The 2-distance matrix of the Fibonacci graph $F_{d,2n}$ denoted by $D_2(F_{d,2n})$, is the $2n \times 2n$ matrix whose (i,j)-entry is 2 if $d(v_i, v_j) = 2$ and zero otherwise.

The 2-distance energy of $F_{d,2n}$ is the sum of the absolute values of the eigenvalues of $D_2(F_{d,2n})$.

In this section we compute 2-distance energy $ED_2(F_{d,2n})$ of the Fibonacci graph $F_{d,2n}$.

Lemma 3.4. *In the Fibonacci graph $F_{d,2n}$, $d(v_i, v_j) = 2$, for $1 \leq i < j \leq n$.*

Proof. First we prove this Lemma for $i = 1$.

$$\text{Let} \quad N(v) = \{u \mid vu \text{is an edge in} \quad F_{d,2n}\}.$$

We have,

$$N(v_1) = \{v_{n+F_i} \mid i = 1, 2, \ldots, d\}$$

and

$$A_i := N(v_{n+F_i}) = \{v_{(n+F_i)-(F_2-1)}, v_{(n+F_i)-(F_3-1)}, \ldots, v_{(n+F_i)-(F_d-1)}\}$$

for $i = 1, 2, 3, \ldots, d$.

By induction on n, we see that

$$\{v_2, v_3, \ldots, v_n\} \subset A = \cup_{i=1}^{d} A_i (mod n).$$

□

Thus v_j $(1 < j \leq n)$ is adjacent to v_{n+F_i} for some i, $1 \leq i \leq d$. But v_{n+F_i} is adjacent to v_1, which implies $d(v_1, v_j) = 2$, $1 < j < n$. Proofs of other cases are similar.

Following two corollaries directly follow from Lemma 3.4.

Corollary 3.5. *In the Fibonacci graph* $d(v_i, v_k) = 1$ *or 3 for* $1 \leq i \leq n$ *and* $n + 1 \leq k \leq 2n$.

Corollary 3.6. *The diameter of* $F_{d,2n}$ *is 3.*

Lemma 3.7. *[9] Let* $A = \begin{pmatrix} A_0 & A_1 \\ A_1 & A_0 \end{pmatrix}$ *be a symmetric* 2×2 *block matrix. Then the spectrum of* A *is the union of the spectra of* $A_0 + A_1$ *and* $A_0 - A_1$.

Theorem 3.8. *The 2-distance energy* $ED_2(F_{d,2n})$ *of the Fibonacci graph* $F_{d,2n}$ *is equal to* $ED_2(F_{d,2n}) = 8(n-1)$.

Proof. By Lemma 3.4, we have,

$$D_2(F_{d,2n}) = \begin{pmatrix} 2J - 2I & 0 \\ 0 & 2J - 2I \end{pmatrix},$$

where J is the $n \times n$ matrix with all entries equal to 1 and I is the $n \times n$ identity matrix. From Lemma 3.7 it follows that the spectrum of $D_2(F_{d,2n})$ is

$$\begin{pmatrix} (2n-2), & -2 \\ 2, & (2n-2) \end{pmatrix}.$$

Hence,

$$ED_2(F_{d,2n}) = 8(n-1).$$

□

3.5 Common-Neighborhood Energy of Fibonacci Graph

For $i \neq j$, the common-neighborhood of the vertices v_i and v_j denoted by $\Gamma(v_i, v_j)$, is the set of vertices, different from v_i and v_j which are adjacent to both v_i and v_j. The common-neighborhood matrix of $F_{d,2n}$ is

$$CN(F_{d,2n}) = (\gamma_{ij})_{2n\times 2n},$$

where

$$\gamma_{ij} = \begin{cases} |\Gamma(v_i, v_j)|, & \text{if} \quad i \neq j \\ 0 & \text{otherwise.} \end{cases}$$

If $\gamma_1, \gamma_2, \ldots, \gamma_{2n}$ are eigenvalues of $CN(F_{d,2n})$, then the common-neighborhood energy of $F_{d,2n}$ is

$$ECN(F_{d,2n}) = \sum_{i=1}^{2n} |\gamma_i|.$$

Lemma 3.9. *[3] Let $D(G) = diag(deg(v_1), deg(v_2), \ldots, deg(v_n))$. Then $CN(G) = A(G)^2 - D(G)$, where $A(G)$ is the adjacency matrix of the graph G.*

Theorem 3.10. *The common-neighborhood spectrum of $F_{d,2n}$ is*

$$\begin{pmatrix} d^2 - d, & {\lambda_2}^2 - d, \ldots, & {\lambda_n}^2 - d \\ 2, & 2, \quad \ldots, & 2 \end{pmatrix}.$$

Proof. Since $F_{d,2n}$ is a d-regular graph, by Lemma 3.9 , we have

$$CN(F_{d,2n}) = A^2(F_{d,2n}) - D(F_{d,2n}) = A^2(F_{d,2n}) - dI_{2n}.$$

If $\pm d, \pm\lambda_2, \ldots, \pm\lambda_n$ are the eigenvalues of $F_{d,2n}$then, the common-neighborhood eigenvalues of $F_{d,2n}$ are

$$d^2 - d, \lambda_2^2 - d, \lambda_3^2 - d, \ldots, \lambda_n^2 - d$$

with each multiplicity two. □

Theorem 3.11. *We have*

$$ECN(F_{d,2n}) = 2(d^2 - d) + 2\sum_{i=2}^{n} |\lambda_i^2 - d|.$$

Proof. Directly follows from Theorem 3.10. □

Theorem 3.12. *We have, $ECN(F_{d,2n}) \leq 4nd$*

Proof. By the above theorem

$$\begin{aligned} ECN(F_{d,2n}) &= 2\sum_{i=1}^{n} |\lambda_i^2 - d|, \quad where \quad \lambda_1 = d \\ &\leq 2\sum_{i=1}^{n} |\lambda_i|^2 + 2\sum_{i=1}^{n} d \\ &= 2m + 2nd \quad (m \quad is \quad the \quad number \quad of \quad edges \quad in \quad F_{d,2n}) \\ &= 2nd + 2nd \\ &= 4nd. \end{aligned}$$

□

3.6 Seidal Energy of Fibonacci Graph

The Seidal energy $ES(G)$ of a simple connected graph G is the sum of the absolute values of eigenvalues of the Seidal matrix $S(G) = (s_{ij})$, where

$$s_{ij} = \begin{cases} -1, & \text{if} \quad v_i \quad \text{and} \quad v_j \quad \text{are adjacent for} \quad i \neq j, \\ 1, & \text{if} \quad v_i \quad \text{and} \quad v_j \quad \text{are non adjacent for} \quad i \neq j, \\ 0 & \text{otherwise.} \end{cases}$$

We can easily verify that, $S(G) = A(\overline{G}) - A(G)$.

If $s_1, s_2, \ldots, s_{2n}$, are the eigenvalues of $S(F_{d,2n})$, then the Seidal energy of $F_{d,2n}$ is

$$ES(F_{d,2n}) = \sum_{i=1}^{2n} |s_i|.$$

Lemma 3.13. *Let G be a d-regular graph with ordinary eigenvalues*

$$d, \lambda_2, \lambda_3, \ldots, \lambda_n.$$

Then the eigenvalues of the complement of G are

$$n - d - 1, -1 - \lambda_2, -1 - \lambda_3, \ldots, -1 - \lambda_n.$$

Theorem 3.14. *The Seidal spectrum of $F_{d,2n}$ is*

$$\begin{pmatrix} 2(n-d) - 1, & -1 - 2\lambda_2, & -1 - 2\lambda_3, & \ldots, & -1 - 2\lambda_{2n} \\ 1, & 1, & 1, & \ldots, & 1 \end{pmatrix}.$$

Proof. Since $F_{d,2n}$ is a d-regular graph, we know that $S(G) = A(\overline{G}) - A(G)$, hence by Lemma 3.13, the Seidal eigenvalues of $F_{d,2n}$ are

$$2(n-d) - 1, -1 - 2\lambda_2, \ldots, -1 - 2\lambda_{2n}.$$

and hence its spectrum. □

Theorem 3.15. *The Seidal energy $ES(F_{d,2n})$ of the Fibonacci graph $F_{d,2n}$ is given by,*

$$ES(F_{d,2n}) = 2(n-d) - 1 + \sum_{i=2}^{2n} |1 + 2\lambda_i|, \quad where \quad d = \lambda_1$$

Proof. Directly follows from Theorem 3.14. □

Theorem 3.16. *We have,*

$$ES(F_{d,2n}) \leq 4(n-d) + 2(\varepsilon(F_{d,2n}) - 1).$$

Proof. By Theorem 3.15

$$\begin{aligned} ES(F_{d,2n}) &= 2(n-d) - 1 + \sum_{i=2}^{2n} |1 + 2\lambda_i| \\ &\leq 2(n-d) - 1 + \sum_{i=2}^{2n} 1 + 2\sum_{i=2}^{2n} |\lambda_i| \\ &\leq 4(n-d) + 2(\varepsilon(F_{d,2n}) - 1). \end{aligned}$$

□

4 Conjectures

In recent times, lot of interest has been shown on Ramanujan graphs by mathematicians in various fields like Number Theory and Communication Theory. For more details one may refer the paper by Rammurthy [18] and the reference cited therein.

In Spectral graph theory, a "Ramanujan Graph" is a regular graph whose spectral gap is almost as large as possible and they are excellent spectral expanders. In other words if $\lambda_1 \geq \lambda_2 \geq \lambda_3 \geq \ldots \geq \lambda_n$ are the eigenvalues of a connected d-regular graph and if,

$$\lambda(G) = max_{i \neq 1}|\lambda_i| = max(|\lambda_2|, |\lambda_3|, \ldots, |\lambda_n|),$$

then G is Ramanujan if

$$\lambda(G) \leq 2\sqrt{(d-1)}.$$

Ramanujan graphs resolves extremal problems in communication network theory and they are also helpful in cryptography.
For $n \leq 54$, $F_{d,2n}$ is a Ramanujan graph.
If $n \geq 55$, then $F_{d,2n}$ need not be a Ramanujan graph. For example, we have verified using MATLAB software that, for $F_{d,2n}$ is not Ramanujan for n=55 and 60. Similarly there can be values for $n > 62$ for which $F_{d,2n}$ is not Ramanujan.

Conjecture 4.1. *There are infinitely many 'n' such that $F_{d,2n}$ is a Ramanujan graph.*

Conjecture 4.2. *There are infinitely many 'n' such that $F_{d,2n}$ is not a Ramanujan graph.*

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