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COMBINATORIAL SUMS INVOLVING FUBINI TYPE NUMBERS AND OTHER SPECIAL NUMBERS AND POLYNOMIALS: APPROACH TRIGONOMETRIC FUNCTIONS AND P-ADIC INTEGRALS

NESLIHAN KILAR AND YILMAZ SIMSEK

ABSTRACT. The aim of this paper is to obtain combinatorial sums, formulas and relations by applications functional equations of generating functions for some well-known special numbers and polynomials, trigonometric functions and p-adic integrals. These combinatorial sums, formulas and relations are related to the Bernoulli numbers, the Euler numbers, the Daehee numbers, the Changhee numbers, the Stirling numbers, the Fubini type numbers and polynomials, two parametric kinds of Fubini-type polynomials, the central factorial numbers, and other special numbers and polynomials.

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1. INTRODUCTION

We start this paper with the following notations, definitions and relations, which can be used:

Let $\mathbb{N}$, $\mathbb{R}$, $\mathbb{C}$ and $\mathbb{Z}_p$ denote the set of natural numbers, the set of real numbers, the set of complex numbers and the set of p-adic integers, respectively, and also $\mathbb{N}_0 = \mathbb{N} \cup \{0\}$. In addition,

$$\binom{\alpha}{n} = \begin{cases} \frac{\alpha(\alpha-1)(\alpha-2)...(\alpha-n+1)}{n!} & n \in \mathbb{N},\ \alpha \in \mathbb{R}(\mathbb{C}) \\ 1 & n = 0 \end{cases}$$

from the above equation, we also have the following falling factorial:

$$(\alpha)_n = \alpha(\alpha-1)(\alpha-2)...(\alpha-n+1)$$

with $(\alpha)_0 = 1$ (*cf.* [1–48]).

In order to deal with and solve some elementary combinatorial problems, one needs many kinds of special numbers and polynomials. Especially, the Stirling numbers and the Fubini numbers have been used to solve or interpret these type problems. Therefore, in this paper, it is aimed to study combinatorial sums and identities that cover

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not only these numbers, but also the Bernoulli numbers and polynomials, the Euler numbers and polynomials, the Changhee numbers, the Daehee numbers, and the central factorial numbers. We now give motivation and method of this paper. Our methods and techniques to be used to give these combinatorial sums are planned to use functional equations of generating functions for special numbers and polynomials, p-adic integrals, and trigonometric identities.

Since the motivation of this paper is to give some combinatorial sums, formulas and relations for Fubini type numbers and polynomials and other special numbers and polynomials, we firstly introduce the Fubini numbers with their combinatorial interpretation.

The Fubini numbers are defined by

$$F_w(z) = \frac{1}{2-e^z} = \sum_{v=0}^{\infty} w_g(v)\frac{z^v}{v!}, \tag{1}$$

(*cf.* [4, 7, 9, 11]).

Using (1), we have the following novel relation between the Fubini numbers $w_g(v)$ and the Stirling numbers of the second kind $S_2(v,j)$:

$$w_g(v) = \sum_{j=0}^{v} j! S_2(v,j), \tag{2}$$

where

$$G_{S2}(z,l) = \frac{(e^z-1)^l}{l!} = \sum_{v=0}^{\infty} S_2(v,l)\frac{z^v}{v!} \tag{3}$$

(*cf.* [1, 4, 7, 9, 11]).

By using (1) and (2), Good [7] gave many combinatorial applications related to the number of orderings of v candidates when ties are permitted. In abstract of his work, Good [7] stated that "in a competition it is customary to rank the candidates permitting ties and it is an interesting elementary combinatorial problem to find the number $w_g(v)$ of such orderings when there are v labelled candidates".

Recently, in [9], we constructed Fubini type numbers and polynomials.

The Fubini type polynomials $a_n^{(l)}(x)$ of order l are defined by

$$G_a(z,l,x) = \frac{2^l}{(2-e^z)^{2l}}e^{xz} = \sum_{n=0}^{\infty} a_n^{(l)}(x)\frac{z^n}{n!}, \tag{4}$$

where $|z| < \ln 2$. Substituting $x=0$ into (4), we have the Fubini type numbers of order l:

$$a_n^{(l)}(0) = a_n^{(l)},$$

(*cf.* [9]).

By using (4), a computation formula for the Fubini type polynomials of order l is given as follows:

$$a_n^{(l)}(x) = \sum_{j=0}^{n} \binom{n}{j} x^j a_{n-j}^{(l)} \tag{5}$$

(*cf.* [9, Theorem 3.2., p. 1612]).

With aid of (1) and (4), we have the following functional equation:

$$G_a(z,1,0) = 2F_w^2(z).$$

Therefore

$$a_n = a_n^{(1)} = 2\sum_{j=0}^{n}\binom{n}{j} w_g(j)w_g(n-j)$$

(*cf.* [9, Theorem 4.7., p. 1616]).

These type numbers and polynomials have been studied by many authors with different methods (*cf.* [12–17, 32, 34, 35, 44, 45, 47]).

The Bernoulli polynomials $B_n(x)$ are defined by

$$\frac{te^{xt}}{e^t-1} = \sum_{n=0}^{\infty} B_n(x)\frac{t^n}{n!},$$

where $|t| < 2\pi$. Putting $x = 0$ in the above generating function, we have the Bernoulli numbers:

$$B_n(0) = B_n,$$

(*cf.* [1–46]).

The Euler polynomials $E_n(x)$ are defined by

$$\frac{2e^{xt}}{e^t+1} = \sum_{n=0}^{\infty} E_n(x)\frac{t^n}{n!},$$

where $|t| < \pi$. Putting $x = 0$ in the above generating function, we have the Euler numbers:

$$E_n(0) = E_n,$$

(*cf.* [1–46]).

The central factorial numbers of the second kind $T(n,l)$ are defined by

$$\frac{\left(e^t + e^{-t} - 2\right)^l}{(2l)!} = \sum_{n=0}^{\infty} T(n,l)\frac{t^{2n}}{(2n)!} \tag{6}$$

(*cf.* [2–5]).

Let $l \in \mathbb{N}_0$ and $\lambda \in \mathbb{C}$. The λ-Stirling numbers of the second kind $S_2(n,l;\lambda)$ are defined by

$$G_S(t,l;\lambda) = \frac{\left(\lambda e^t - 1\right)^l}{l!} = \sum_{n=0}^{\infty} S_2(n,l;\lambda)\frac{t^n}{n!} \tag{7}$$

(*cf.* [40, 43]). Taking $\lambda = 1$ in (7), we have

$$S_2(n,l;1) = S_2(n,l)$$

(*cf.* [1–46]; and the references therein).

The polynomials $C_n(x,y)$ and $S_n(x,y)$ are defined respectively by

$$F_C(t,x,y) = e^{xt}\cos(yt) = \sum_{n=0}^{\infty} C_n(x,y)\frac{t^n}{n!}, \tag{8}$$

and

$$F_S(t,x,y) = e^{xt}\sin(yt) = \sum_{n=0}^{\infty} S_n(x,y)\frac{t^n}{n!} \tag{9}$$

(*cf.* [12–17, 32, 34, 35, 44, 45, 47]).

The polynomials $Y_n^{(k)}(x;\lambda)$ are defined by

$$G_Y(t,k,x;\lambda) = \left(\frac{2}{\lambda(1+\lambda t)-1}\right)^k (1+\lambda t)^x = \sum_{n=0}^{\infty} Y_n^{(k)}(x;\lambda)\frac{t^n}{n!} \tag{10}$$

(*cf.* [33]). Substituting $x=0$ into (10), we have the numbers $Y_n^{(k)}(\lambda)$:

$$Y_n^{(k)}(0;\lambda) = Y_n^{(k)}(\lambda)$$

(*cf.* [33]).

By using (10), a computation formula for the polynomials $Y_n^{(k)}(x;\lambda)$ is given as follows:

$$Y_n^{(k)}(x;\lambda) = \sum_{j=0}^{n}\binom{n}{j}\lambda^{n-j}(x)_{n-j}Y_j^{(k)}(\lambda) \tag{11}$$

(*cf.* [33, Theorem 2.4, p. 2342]).

The two parametric kinds of Fubini-type polynomials $a_n^{(c,l)}(x,y)$ and $a_n^{(s,l)}(x,y)$ are defined respectively by

$$F_{ac}(t,l,x,y) = \frac{2^l e^{xt}}{(2-e^t)^{2l}}\cos(yt) = \sum_{n=0}^{\infty} a_n^{(c,l)}(x,y)\frac{t^n}{n!}, \tag{12}$$

and

$$F_{as}(t,l,x,y) = \frac{2^l e^{xt}}{(2-e^t)^{2l}}\sin(yt) = \sum_{n=0}^{\infty} a_n^{(s,l)}(x,y)\frac{t^n}{n!} \tag{13}$$

(*cf.* [47]).

Using (4), (8), (9), (12) and (13), computation formulas for the two parametric kinds of Fubini-type polynomials are given as follows:

$$a_n^{(c,l)}(x,y) = \sum_{j=0}^{n}\binom{n}{j}a_j^{(l)}C_{n-j}(x,y)$$

and

$$a_n^{(s,l)}(x,y) = \sum_{j=0}^{n}\binom{n}{j}a_j^{(l)}S_{n-j}(x,y)$$

(*cf.* [47]).

We now summarize our paper results as follows:

In Section 2, by using functional equations of the generating functions, we give some combinatorial sums including the Fubini type polynomials, the Stirling numbers, and the polynomials $Y_n^{(l)}(x;\lambda)$.

In Section 3, with the aid of some well-known trigonometric identities, we derive some combinatorial sums including the two parametric Fubini-type polynomials, the central factorial numbers of the second kind, the Bernoulli numbers, the λ-Stirling numbers of the second kind, the polynomials $C_n(x,y)$, the polynomials $S_n(x,y)$.

In Section 4, using p-adic integrals formulas, we derive some combinatorial sums including the Bernoulli numbers, the Euler numbers, the Changhee numbers, the Daehee numbers, and also the Fubini type numbers.

2. Combinatorial sums involving Fubini type polynomials, combinatorial polynomials and Stirling numbers

In this section, we give a functional equation for the Stirling numbers, the Fubini type polynomials, combinatorial polynomials. Using this equation, we give combinatorial sum including the Fubini type polynomials, the Stirling numbers of the second kind, and the polynomials $Y_n^{(l)}(x;\lambda)$.

Putting $k=2l$, $\lambda=\frac{1}{2}$ and $t=2\left(e^t-1\right)$ in (10), then combining (3) and (4), we get the following functional equation:

$$G_a(t,l,x)=\sum_{m=0}^{\infty}Y_m^{(2l)}\left(x;\frac{1}{2}\right)2^{m-3l}G_{S2}(t,m). \tag{14}$$

By using the above functional equation, we give the following theorem:

Theorem 2.1. *Let $n\in\mathbb{N}_0$. Then we have*

$$\sum_{m=0}^{n}Y_m^{(2l)}\left(x;\frac{1}{2}\right)2^mS_2(n,m)=2^{3l}a_n^{(l)}(x). \tag{15}$$

Proof. By using (14), we have

$$2^{3l}\sum_{n=0}^{\infty}a_n^{(l)}(x)\frac{t^n}{n!}=\sum_{n=0}^{\infty}\sum_{m=0}^{n}Y_m^{(2l)}\left(x;\frac{1}{2}\right)2^mS_2(n,m)\frac{t^n}{n!}.$$

Comparing the coefficients of $\frac{t^n}{n!}$ on both sides of the aforementioned equation, we arrive at the desired result. □

3. Combinatorial sums derived from trigonometric functions

In this section, using trigonometric identities, we give some combinatorial sums and relations involving the two parametric Fubini-type polynomials, the central factorial numbers of the second kind, the Bernoulli numbers, the λ-Stirling numbers of the second kind, the polynomials $C_n(x,y)$, and the polynomials $S_n(x,y)$.

Theorem 3.1. *Let $n\in\mathbb{N}_0$. Then we have*

$$\sum_{j=0}^{n}\binom{n}{j}a_j^{(c,l)}(x,y)S_2\left(n-j,2l;\frac{1}{2}\right)=\frac{C_n(x,y)}{2^l(2l)!}.$$

Proof. Using (7), (8) and (12), we have

$$F_C(t,x,y)=2^l(2l)!F_{ac}(t,l,x,y)\,G_S\left(t,2l;\frac{1}{2}\right).$$

With the help of the above functional equation, we get

$$\sum_{n=0}^{\infty}C_n(x,y)\frac{t^n}{n!}=2^l(2l)!\sum_{n=0}^{\infty}a_n^{(c,l)}(x,y)\frac{t^n}{n!}\sum_{n=0}^{\infty}S_2\left(n,2l;\frac{1}{2}\right)\frac{t^n}{n!}.$$

Therefore

$$\sum_{n=0}^{\infty}C_n(x,y)\frac{t^n}{n!}=2^l(2l)!\sum_{n=0}^{\infty}\sum_{j=0}^{n}\binom{n}{j}a_j^{(c,l)}(x,y)\,S_2\left(n-j,2l;\frac{1}{2}\right)\frac{t^n}{n!}.$$

Comparing the coefficients of $\frac{t^n}{n!}$ on both sides of the aforementioned equation, we arrive at the desired result. □

Theorem 3.2. *Let $n\in\mathbb{N}_0$. Then we have*

$$\sum_{j=0}^{n}\binom{n}{j}a_j^{(s,l)}(x,y)\,S_2\left(n-j,2l;\frac{1}{2}\right)=\frac{S_n(x,y)}{2^l(2l)!}.$$

Proof. By using (7), (9) and (13), we obtain the following functional equation:

$$F_S(t,x,y)=2^l(2l)!F_{as}(t,l,x,y)\,G_S\left(t,2l;\frac{1}{2}\right).$$

From the above equation, we have the following yields

$$\sum_{n=0}^{\infty}S_n(x,y)\frac{t^n}{n!}=2^l(2l)!\sum_{n=0}^{\infty}\sum_{j=0}^{n}\binom{n}{j}a_j^{(s,l)}(x,y)\,S_2\left(n-j,2l;\frac{1}{2}\right)\frac{t^n}{n!}.$$

Comparing the coefficients of $\frac{t^n}{n!}$ on both sides of the aforementioned equation, we arrive at the desired result. □

Here note that the following Lemma, which has very important applications in theory of double series and its applications:

Lemma 3.3.

$$\sum_{n=0}^{\infty}\sum_{k=0}^{\infty}A(k,n)=\sum_{n=0}^{\infty}\sum_{k=0}^{\left[\frac{n}{2}\right]}A(k,n-2k),\tag{16}$$

where $[x]$ denotes the greatest integer function (cf. [37, p. 57, Lemma 11, Eq. (7)]*).*

Theorem 3.4. *Let $n\in\mathbb{N}$. Then we have*

$$\begin{aligned}a_n^{(s,l)}(x,y) &= \sum_{v=0}^{\left[\frac{n-1}{2}\right]}\sum_{j=0}^{v}(-1)^{j+v}\binom{n}{2v+1}\binom{j-\frac{1}{2}}{j}2^{2v-2j}y^{2v+1}\\ &\quad\times(2j+1)!a_{n-1-2v}^{(c,l)}(x,y)\,T(2v+1,2j+1).\end{aligned}$$

Proof. Replacing t by $2yt$ and setting $m=0$ in the following identity

$$\left(\tan\left(\frac{t}{2}\right)\right)^{2m+1} = \sum_{k=m}^{\infty}(-1)^k \frac{t^{2k+1}}{(2k+1)!} \times \sum_{j=m}^{k}(-1)^j \frac{(2j+1)!}{2^{2j+1}}\binom{j-\frac{1}{2}}{j-m} T(2k+1,2j+1)$$

(*cf.* [3, p. 442, Eq. (4.1.9)]), we get

$$\tan(yt) = \sum_{k=0}^{\infty}(-1)^k \frac{(2yt)^{2k+1}}{(2k+1)!} \times \sum_{j=0}^{k}(-1)^j \frac{(2j+1)!}{2^{2j+1}}\binom{j-\frac{1}{2}}{j} T(2k+1,2j+1). \tag{17}$$

With the aid of (12) and (13), we have the following functional equation:

$$F_{as}(t,l,x,y) = F_{ac}(t,l,x,y)\tan(yt). \tag{18}$$

Combining (18) with (17), we obtain

$$\sum_{n=0}^{\infty} a_n^{(s,l)}(x,y)\frac{t^n}{n!} = \sum_{n=0}^{\infty} a_n^{(c,l)}(x,y)\frac{t^n}{n!}\sum_{n=0}^{\infty}\sum_{j=0}^{n}(-1)^{n+j}\binom{j-\frac{1}{2}}{j}\frac{(2j+1)!}{2^{2j+1}} \times T(2n+1,2j+1)\frac{(2yt)^{2n+1}}{(2n+1)!}.$$

By using (16) in the above equation, we get

$$\sum_{n=0}^{\infty} a_n^{(s,l)}(x,y)\frac{t^n}{n!} = \sum_{n=0}^{\infty}\sum_{v=0}^{\left[\frac{n-1}{2}\right]}\sum_{j=0}^{v}(-1)^{j+v}\binom{n}{2v+1}\binom{j-\frac{1}{2}}{j}2^{2v-2j}y^{2v+1} \times (2j+1)!\,a_{n-1-2v}^{(c,l)}(x,y)\,T(2v+1,2j+1)\frac{t^n}{n!}.$$

Comparing the coefficients of $\frac{t^n}{n!}$ on both sides of the aforementioned equation, we arrive at the desired result. □

Theorem 3.5. *Let $n \in \mathbb{N}_0$. Then we have*

$$\sum_{j=0}^{\left[\frac{n}{2}\right]}(-1)^j\binom{n}{2j}a_{n-2j}^{(c,l)}(x,m-1) = \frac{a_n^{(c,l)}(x,m)+a_n^{(c,l)}(x,m-2)}{2}.$$

Proof. Multiplying both sides of the following well-known identity by $\frac{2^l e^{xt}}{(2-e^t)^{2l}}$,

$$\cos(mt) = 2\cos(t)\cos((m-1)t) - \cos((m-2)t),$$

(*cf.* [48, p. 430]) and using (12), we have the following functional equation:

$$F_{ac}(t,l,x,m) = 2\cos(t)F_{ac}(t,l,x,m-1) - F_{ac}(t,l,x,m-2).$$

From the above equation, we have

$$\sum_{n=0}^{\infty} a_n^{(c,l)}(x,m)\frac{t^n}{n!} = 2\sum_{n=0}^{\infty}(-1)^n\frac{t^{2n}}{(2n)!}\sum_{n=0}^{\infty} a_n^{(c,l)}(x,m-1)\frac{t^n}{n!} - \sum_{n=0}^{\infty} a_n^{(c,l)}(x,m-2)\frac{t^n}{n!}.$$

By using (16) in the above equation, we obtain

$$\sum_{n=0}^{\infty} a_n^{(c,l)}(x,m)\frac{t^n}{n!} = 2\sum_{n=0}^{\infty}\sum_{j=0}^{\left[\frac{n}{2}\right]}(-1)^j\binom{n}{2j} a_{n-2j}^{(c,l)}(x,m-1)\frac{t^n}{n!} - \sum_{n=0}^{\infty} a_n^{(c,l)}(x,m-2)\frac{t^n}{n!}.$$

Comparing the coefficients of $\frac{t^n}{n!}$ on both sides of the aforementioned equation, we arrive at the desired result. □

Theorem 3.6. *Let $n \in \mathbb{N}_0$. Then we have*

$$\sum_{j=0}^{\left[\frac{n}{2}\right]}(-1)^j\binom{n}{2j} a_{n-2j}^{(s,l)}(x,m-1) = \frac{a_n^{(s,l)}(x,m) + a_n^{(s,l)}(x,m-2)}{2}.$$

Proof. Multiplying both sides of the following well-known identity by $\frac{2^l e^{xt}}{(2-e^t)^{2l}}$,

$$\sin(mt) = 2\cos(t)\sin((m-1)t) - \sin((m-2)t),$$

(*cf.* [48, p. 430]) and using (13), we have the following functional equation:

$$F_{as}(t,l,x,m) = 2\cos(t)F_{as}(t,l,x,m-1) - F_{as}(t,l,x,m-2).$$

By using (16) in the above equation, we obtain

$$\sum_{n=0}^{\infty} a_n^{(s,l)}(x,m)\frac{t^n}{n!} = 2\sum_{n=0}^{\infty}\sum_{j=0}^{\left[\frac{n}{2}\right]}(-1)^j\binom{n}{2j} a_{n-2j}^{(s,l)}(x,m-1)\frac{t^n}{n!} - \sum_{n=0}^{\infty} a_n^{(s,l)}(x,m-2)\frac{t^n}{n!}.$$

Comparing the coefficients of $\frac{t^n}{n!}$ on both sides of the aforementioned equation, we arrive at the desired result. □

Theorem 3.7. *Let $n \in \mathbb{N}_0$. Then we have*

$$\sum_{j=0}^{\left[\frac{n+1}{2}\right]}(-1)^{j+1}\binom{n+1}{2j} y^{2j-1}\left(2^{2j-1}-1\right)B_{2j}a_{n+1-2j}^{(s,l)}(x,2y) = (n+1)a_n^{(c,l)}(x,y).$$

Proof. Multiplying both sides of the following well-known trigonometric identity by $\frac{2^l e^{xt}}{(2-e^t)^{2l}}$,

$$\sin(2yt) = 2\sin(yt)\cos(yt),$$

and using (12), and (13), we have the following functional equation:

$$2F_{ac}(t,l,x,y) = \csc(yt)\, F_{as}(t,l,x,2y).$$

From the previous equation, we get

$$2\sum_{n=0}^{\infty} a_n^{(c,l)}(x,y)\frac{t^n}{n!} = \csc(yt)\sum_{n=0}^{\infty} a_n^{(s,l)}(x,2y)\frac{t^n}{n!}. \tag{19}$$

Combining (19) with the following well-known relation

$$\csc(t) = \sum_{n=0}^{\infty}(-1)^{n+1}\left(2^{2n}-2\right)B_{2n}\frac{t^{2n-1}}{(2n)!},$$

(*cf.* [6, Eq. (9)], see also [1, 42]), we obtain

$$\begin{aligned} & 2\sum_{n=0}^{\infty} a_n^{(c,l)}(x,y)\frac{t^n}{n!} \\ = & \sum_{n=0}^{\infty}(-1)^{n+1}\left(2^{2n}-2\right)B_{2n}\frac{(yt)^{2n-1}}{(2n)!}\sum_{n=0}^{\infty} a_n^{(s,l)}(x,2y)\frac{t^n}{n!}. \end{aligned}$$

By using (16) in the above equation, we get

$$\begin{aligned} & \sum_{n=0}^{\infty} a_n^{(c,l)}(x,y)\frac{t^n}{n!} \\ = & \sum_{n=0}^{\infty}\sum_{j=0}^{\left[\frac{n+1}{2}\right]}(-1)^{j+1}\binom{n+1}{2j}\frac{y^{2j-1}\left(2^{2j-1}-1\right)}{n+1}B_{2j}a_{n+1-2j}^{(s,l)}(x,2y)\frac{t^n}{n!}. \end{aligned}$$

Comparing the coefficients of $\frac{t^n}{n!}$ on both sides of the aforementioned equation, we arrive at the desired result. □

4. Combinatorial sums derived from p-adic integrals

In this section, by applying p-adic integrals formulas, we give some formulas and combinatorial sums including the Fubini type numbers, the Bernoulli numbers, the Euler numbers, the Changhee numbers, and the Daehee numbers. Therefore, we need some definitions and identities for these integrals.

4.1. p-adic integrals.

Here, we give some definitions and properties of p-adic integrals, which have important role, especially, in p-adic calculus, in mathematics, and in mathematical physics.

Let $\mathbb{K}$ be a field with a complete valuation. Let $C^1(\mathbb{Z}_p \to \mathbb{K})$ be a set of continuous derivative functions:

$$\left\{ f : \mathbb{Z}_p \to \mathbb{K} : f(x) \text{ is differentiable and } \frac{d}{dx}f(x) \text{ is continuous} \right\}.$$

The Volkenborn integral (or the bosonic p-adic integral) of the continuous derivative function f on $\mathbb{Z}_p$ is given by

$$\int_{\mathbb{Z}_p} f(x)\, d\mu_1(x) = \lim_{N\to\infty} \frac{1}{p^N} \sum_{x=0}^{p^N-1} f(x), \tag{20}$$

where $\mu_1(x)$ denotes the Haar distribution:

$$\mu_1(x) = \mu_1\left(x + p^N \mathbb{Z}_p\right) = \frac{1}{p^N}$$

(*cf.* [20, 26, 27, 38, 41]; and the references therein).

By using (20), the Bernoulli numbers B_n is also given as follows:

$$\int_{\mathbb{Z}_p} x^n d\mu_1(x) = B_n \tag{21}$$

(*cf.* [38]; and the references therein).

By using (20), Kim et al. [27] gave the following relation between the bosonic p-adic integral and the Daehee numbers D_n:

$$\int_{\mathbb{Z}_p} (x)_n\, d\mu_1(x) = \frac{(-1)^n n!}{n+1} = D_n. \tag{22}$$

Recently, with aid of (20) and (22), various properties of the function $(x)_n$ were given by Simsek [41].

The fermionic p-adic integral of function f on $\mathbb{Z}_p$ is given by

$$\int_{\mathbb{Z}_p} f(x)\, d\mu_{-1}(x) = \lim_{N\to\infty} \sum_{x=0}^{p^N-1} (-1)^x f(x), \tag{23}$$

where

$$\mu_{-1}(x) = \mu_{-1}\left(x + p^N \mathbb{Z}_p\right) = (-1)^x$$

(*cf.* [21–24], for detail, see also [28, 41]; and the references therein).

By using (23), the Euler numbers E_n is also given as follows:

$$\int_{\mathbb{Z}_p} x^n d\mu_{-1}(x) = E_n \tag{24}$$

(*cf.* [21–24]; and the references therein).

By using (23), Kim et al. [28] gave the following relation between the fermionic p-adic integral and the Changhee numbers Ch_n:

$$\int_{\mathbb{Z}_p} (x)_n\, d\mu_{-1}(x) = \frac{(-1)^n n!}{2^n} = Ch_n. \tag{25}$$

Recently, with aid of (23) and (25), various properties of the function $(x)_n$ were given by Simsek [41].

4.2. **Formulas and combinatorial identities.** By using p-adic integral formulas, we give combinatorial sums involving the Fubini type numbers, the Bernoulli numbers, the Euler numbers, the Changhee numbers, the Daehee numbers.

Theorem 4.1. *Let $n \in \mathbb{N}_0$. Then we have*

$$\sum_{j=0}^{n}\binom{n}{j}a_{n-j}^{(l)}B_j = 2^{-3l}\sum_{m=0}^{n}\sum_{j=0}^{m}\binom{m}{j}2^j D_{m-j}Y_j^{(2l)}\left(\frac{1}{2}\right)S_2\left(n,m\right).$$

Proof. Using (5), (11) and (15), then by applying the Volkenborn integral to the final equation, we have

$$\begin{aligned}&\sum_{j=0}^{n}\binom{n}{j}a_{n-j}^{(l)}\int_{\mathbb{Z}_p}x^j d\mu_1\left(x\right)\\ =\;& 2^{-3l}\sum_{m=0}^{n}\sum_{j=0}^{m}\binom{m}{j}2^j Y_j^{(2l)}\left(\frac{1}{2}\right)S_2\left(n,m\right)\int_{\mathbb{Z}_p}\left(x\right)_{m-j}d\mu_1\left(x\right).\end{aligned}$$

Combining the above equation with (21) and (22), we arrive at the desired result. □

Theorem 4.2. *Let $n \in \mathbb{N}_0$. Then we have*

$$\sum_{j=0}^{n}\binom{n}{j}a_{n-j}^{(l)}E_j = 2^{-3l}\sum_{m=0}^{n}\sum_{j=0}^{m}\binom{m}{j}2^j Ch_{m-j}Y_j^{(2l)}\left(\frac{1}{2}\right)S_2\left(n,m\right).$$

Proof. Using (5), (11) and (15), then by applying the fermionic p-adic integral to the final equation, we have

$$\begin{aligned}&\sum_{j=0}^{n}\binom{n}{j}a_{n-j}^{(l)}\int_{\mathbb{Z}_p}x^j d\mu_{-1}\left(x\right)\\ =\;& 2^{-3l}\sum_{m=0}^{n}\sum_{j=0}^{m}\binom{m}{j}2^j Y_j^{(2l)}\left(\frac{1}{2}\right)S_2\left(n,m\right)\int_{\mathbb{Z}_p}\left(x\right)_{m-j}d\mu_{-1}\left(x\right).\end{aligned}$$

Combining the above equation with (24) and (25), we arrive at the desired result. □

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Department of Mathematics, Faculty of Science University of Akdeniz TR-07058 Antalya-TURKEY
Email address: neslihankilar@gmail.com

Department of Mathematics, Faculty of Science University of Akdeniz TR-07058 Antalya-TURKEY
Email address: ysimsek@akdeniz.edu.tr