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RESULTS ON UNIQUENESS OF PRODUCT OF CERTAIN TYPE OF DIFFERENCE POLYNOMIALS

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Abstract. In this paper, using the concept of weakly weighted sharing and relaxed weighted sharing, we investigate the uniqueness of product of certain type of difference polynomials. The results of the paper improve and extend some recent results due to Renukadevi S. Dyavanal and Ashwini M. Hattikal.

1. Introduction and main results

In this article, we assume that the reader is familiar with the fundamental results and the standard notation of Nevanlinna value distribution theory (see[3],[9] and [10]). Let $f(z)$ and $g(z)$ be two non-constant meromorphic functions in the complex plane. By $S(r,f)$, we mean any quantity satisfying $S(r,f) = o\{T(r,f)\}$ as $r \to \infty$, possibly outside a set of finite logarithmic measure. We say that the meromorphic function $\alpha(z)$ is a small function of f, if $T(r,\alpha(z)) = S(r,f)$. The order and hyper order of meromorphic function f are defined by

$$\rho(f) = \limsup_{r\to\infty} \frac{logT(r,f)}{logr}, \;\; \rho_2(f) = \limsup_{r\to\infty} \frac{loglogT(r,f)}{logr}.$$

In addition, we need the following definitions.

Definition 1.[11] Let $a \in \mathbb{C} \cup \{\infty\}$. We denote by $N(r,a;f \mid= 1)$ the counting function of simple a points of f. For a positive integer k we denote by $N(r,a;f \mid\leq k)$ the counting function of those a-points of f (counted with proper multiplicities) whose multiplicities are not greater than k. By $\overline{N}(r,a;f \mid\leq k)$ we denote the corresponding reduced counting function. Analogously we can define $N(r,a;f \mid\geq k)$ and $\overline{N}(r,a;f \mid\geq k)$.

Definition 2.[13] Let k be a positive integer or infinity. We denote by $N_k(r,a;f)$ the counting function of a-points of f, where an a-point of multiplicity m is counted m times if $m \leq k$ and k times if $m > k$. Then

$$N_k(r,a;f) = \overline{N}(r,a;f) + \overline{N}(r,a;f \mid\geq 2) + ... + \overline{N}(r,a;f \mid\geq k).$$

Clearly $N_1(r,a;f) = \overline{N}(r,a;f)$.

Recently, A.Banerjee and S.Mukerjee[14] introduced another sharing notion which is also a scaling between IM and CM but weaker than weakly weighted sharing.

Definition 3. Let $a \in \mathbb{C}\cup\{\infty\}$. We denote by $N_E(r,a;f,g)(\overline{N}_E(r,a;f,g))$ the counting function(reduced counting function) of all common zeros of $f-a$ and $g-a$ with same multiplicities and by $N_0(r,a;f,g)(\overline{N}_0(r,a;f,g))$ the counting function(reduced counting function) of all common zeros of $f-a$ and $g-a$ ignoring multiplicities. If

$$\overline{N}\left(r, \frac{1}{f-a}\right) + \overline{N}\left(r, \frac{1}{g-a}\right) - 2\overline{N}_E(r,a;f,g) = S(r,f) + S(r,g),$$

then we say that f and g share the value a "CM". If

$$\overline{N}\left(r, \frac{1}{f-a}\right) + \overline{N}\left(r, \frac{1}{g-a}\right) - 2\overline{N}_0(r,a;f,g) = S(r,f) + S(r,g),$$

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then we say that f and g share the value a "IM".

Definition 4.[12] Let f and g share the value a "IM" and k be a positive integer or infinity. Then $\overline{N}^E_{k)}(r, a; f, g)$ denotes the reduced counting function of those a-points of f whose multiplicities are equal to the corresponding a-points of g, and both of their multiplicities are not greater than k. $\overline{N}^0_{(k}(r, a; f, g)$ denotes the reduced counting function of those a-points of f which are a-points of g and both of their multiplicities are not less than k.

Definition 5.[12] For $a \in \mathbb{C} \cup \{\infty\}$, if k is a positive integer or ∞ and

$$\overline{N}_{k)}\left(r, \frac{1}{f-a}\right) - \overline{N}^E_{k)}(r, a; f, g) = S(r, f),$$

$$\overline{N}_{k)}\left(r, \frac{1}{g-a}\right) - \overline{N}^E_{k)}(r, a; f, g) = S(r, g),$$

$$\overline{N}_{(k+1}\left(r, \frac{1}{f-a}\right) - \overline{N}^0_{(k+1}(r, a; f, g) = S(r, f),$$

$$\overline{N}_{(k+1}\left(r, \frac{1}{g-a}\right) - \overline{N}^0_{(k+1}(r, a; f, g) = S(r, g),$$

or if $k = 0$ and

$$\overline{N}\left(r, \frac{1}{f-a}\right) - \overline{N}_0(r, a; f, g) = S(r, f),$$

$$\overline{N}\left(r, \frac{1}{g-a}\right) - \overline{N}_0(r, a; f, g) = S(r, g),$$

then we say f and g weakly share a with weight k. Here we write f, g share "(a, k)" to mean that f, g weakly share a with weight k.

Definition 6.[14] We denote by $\overline{N}(r, a; f \mid= p; g \mid= q)$ the reduced counting function of common a-points of f and g with multiplicities p and q, respectively.

Definition 7.[14] Let f, g share a "IM". Also let k be a positive integer or ∞ and $a \in \mathbb{C} \cup \{\infty\}$. If $\sum_{p,q \leq k} \overline{N}(r, a; f \mid= p; g \mid= q) = S(r)$, then we say f and g share a with weight k in a relaxed manner. Here we write f and g share $(a, k)^*$ to mean that f and g share a with weight k in a relaxed manner.

In 2010, J.F.Xu, F.Lu and H.X.Yi obtained the following result on meromorphic function sharing a fixed point.

Theorem A.[6] Let $f(z)$ and $g(z)$ be two non-constant meromorphic functions and let n, k be two positive integers with $n > 3k + 10$. If $(f^n(z))^{(k)}$ and $(g^n(z))^{(k)}$ share z CM, f and g share ∞ IM, then either $f(z) = c_1 e^{cz^2}, g(z) = c_2 e^{-cz^2}$ where c_1, c_2 and c are three constants satisfying $4n^2(c_1c_2)^n c^2 = -1$, or $f \equiv tg$ for a constant t such that $t^n = 1$.

Further, Fang and Qiu investigated uniqueness for the same functions as in the Theorem A, when $k = 1$.

Theorem B.[2] Let $f(z)$ and $g(z)$ be two non-constant meromorphic functions and let $n \geq 11$ be a positive integer. If $f^n(z)f'(z)$ and $g^n(z)g'(z)$ share z CM, then either $f(z) = c_1 e^{cz^2}, g(z) = c_2 e^{-cz^2}$, where c_1, c_2 and c are three constants satisfying $4(c_1c_2)^{n+1}c^2 = -1$, or $f(z) \equiv tg(z)$ for a constant t such that $t^{n+1} = 1$.

In 2012, Cao and Zhang replaced f' with $f^{(k)}$ and obtained the following theorem.

Theorem C.[1] Let $f(z)$ and $g(z)$ be two transcendental meromorphic functions, whose zeros are of multiplicities atleast k, where k is a positive integer. Let $n > max\{2k-1, 4+4 \mid k+4\}$ be a positive integer. If $f^n(z)f^{(k)}(z)$ and $g^n(z)g^{(k)}(z)$ share z CM, and f and g share ∞ IM, then one of the following two conclusions holds. (1) $f^n(z)f^{(k)}(z) = g^n(z)g^{(k)}(z)$ (2)$f(z) = c_1e^{cz^2}, g(z) = c_2e^{-cz^2}$, where c_1, c_2 and c are constants such that $4(c_1c_2)^{n+1}c^2 = -1$.

Recently, X.B.Zhang reduced the lower bond of n and relax the condition on multiplicity of zeros in Theorem C and proved the below result.

Theorem D.[16] Let $f(z)$ and $g(z)$ be two transcendental meromorphic functions and n, k two positive integers with $n > k+6$. If $f^n(z)f^{(k)}(z)$ and $g^n(z)g^{(k)}(z)$ share z CM, and f and g share ∞ IM, then one of the following two conclusions holds. (1) $f^n(z)f^{(k)}(z) = g^n(z)g^{(k)}(z)$, (2)$f(z) = c_1e^{cz^2}, g(z) = c_2e^{-cz^2}$, where c_1, c_2 and c are constants such that $4(c_1c_2)^{n+1}c^2 = -1$.

In 2016, Renukadevi S. Dyavanal and Ashwini M. Hattikal proved the following theorem.

Theorem E.[8] Let f and g be two transcendental meromorphic functions of hyper order $\rho_2(f) < 1$ and $\rho_2(g) < 1$. Let k, n, d, λ be positive integers and $n > max\{2d(k+2) + \lambda(k+3) + 7, \lambda_1, \lambda_2\}$. If $F(z) = f(z)^n \left[\prod_{j=1}^{d} f(z+c_j)^{s_j}\right]^{(k)}$ and $G(z) = g(z)^n \left[\prod_{j=1}^{d} g(z+c_j)^{s_j}\right]^{(k)}$ share z CM and f, g share ∞ IM, then one of the following two conclusions holds.
(1)$F(z) \equiv G(z)$
(2)$\prod_{j=1}^{d} f(z+c_j)^{s_j} = C_1e^{Cz^2}$, $\prod_{j=1}^{d} g(z+c_j)^{s_j} = C_2e^{Cz^2}$ where C_1, C_2 and C are constants such that $4(C_1C_2)^{n+1}C^2 = -1$.

Theorem E, motivated us to think that whether there exists a similar result, if $f(z)^n[\prod_{j=1}^{d} f(z+c_j)^{s_j}]^{(k)}$ is replaced in Theorem E by $f(z)^nP(f)[\prod_{j=1}^{d} f(z+c_j)^{s_j}]^{(k)}$. In this paper we prove significant results which improve as well as extend Theorem E.
In this paper, we assume $c_j \in \mathbb{C}\setminus\{0\} (j = 1, 2, ..., d)$ are distinct constants, $n, k, d, s_j (j = 1, 2..., d)$ are positive integers and $\lambda = s_1 + s_2 + ...s_d$. Let

$$F(z) = f(z)^nP(f)\left[\prod_{j=1}^{d} f(z+c_j)^{s_j}\right]^{(k)} \tag{1.1}$$

and

$$F_1(z) = f(z)^nP(f)\prod_{j=1}^{d} f(z+c_j)^{s_j}, \tag{1.2}$$

where $P(z) = a_nz^n + a_{n-1}z^{n-1} + ... + a_0$ be a nonzero polynomial, where $a_0, ..., a_n(\neq 0)$ are complex constants, and n is an integer, let $\Gamma_0 = m_1 + m_2$ and $\Gamma_1 = m_1 + 2m_2$, where m_1 is the number of the simple zero of $P(z)$, and m_2 is the number of multiple zeros of $P(z)$.

We consider the uniqueness problems on product of difference polynomials and obtain the following results, which improve the Theorem E.

Theorem 1.1. Let f and g be two transcendental meromorphic functions of finite order, and $\alpha(z)$ be a small function with respect to both $f(z)$ and $g(z)$. Let k, n, d, λ be positive integers and $n > 2\Gamma_1 + 2(k+2)d + \lambda(5+k) + 8 - m$. If $f(z)^nP(f)\left[\prod_{j=1}^{d} f(z+c_j)^{s_j}\right]^{(k)}$ and $g(z)^nP(g)\left[\prod_{j=1}^{d} g(z+c_j)^{s_j}\right]^{(k)}$ share "$(\alpha, 2)$", then $f(z) \equiv g(z)$.

Theorem 1.2. Let f and g be two transcendental meromorphic functions of finite order, and $\alpha(z)$ be a small function with respect to both $f(z)$ and $g(z)$. Let k, n, d, λ be positive integers and $n > 2\Gamma_1 + \Gamma_0 + (3k+5)d + (6+k)\lambda + 10 - m$. If $f(z)^n P(f)\left[\prod_{j=1}^{d} f(z+c_j)^{s_j}\right]^{(k)}$ and $g(z)^n P(g)\left[\prod_{j=1}^{d} g(z+c_j)^{s_j}\right]^{(k)}$ share $(\alpha, 2)^*$, then $f(z) \equiv g(z)$.

Theorem 1.3. Let f and g be two transcendental entire functions of finite order, and $\alpha(z)$ be a small function with respect to both $f(z)$ and $g(z)$. Let k, n, d, λ be positive integers and $n > 2\Gamma_1 + 2\Gamma_0 + 2(2k+3)d + (k+1)\lambda + 6 - m$.
If $\overline{E}_{2)}(\alpha(z), f^n P(f)(\prod_{j=1}^{d} f(z+c_j)^{s_j})^{(k)}) = \overline{E}_{2)}(\alpha(z), g^n P(g)(\prod_{j=1}^{d} g(z+c_j)^{s_j})^{(k)})$, then $f(z) \equiv g(z)$.

Theorem 1.4. Let f and g be two transcendental entire functions of finite order, and $\alpha(z)$ be a small function with respect to both $f(z)$ and $g(z)$. Let k, n, d, λ be positive integers and $n > 2\Gamma_1 + 2(k+2)d + (k+1)\lambda - m + 4$. If $f(z)^n P(f)\left[\prod_{j=1}^{d} f(z+c_j)^{s_j}\right]^{(k)}$ and $g(z)^n P(g)\left[\prod_{j=1}^{d} g(z+c_j)^{s_j}\right]^{(k)}$ share "$(\alpha, 2)$", then $f(z) \equiv g(z)$.

Theorem 1.5. Let f and g be two transcendental entire functions of finite order, and $\alpha(z)$ be a small function with respect to both $f(z)$ and $g(z)$. Let k, n, d, λ be positive integers and $n > 2\Gamma_1 + (2k+3)d + (k+1)\lambda - m + 4$. If $f(z)^n P(f)\left[\prod_{j=1}^{d} f(z+c_j)^{s_j}\right]^{(k)}$ and $g(z)^n P(g)\left[\prod_{j=1}^{d} g(z+c_j)^{s_j}\right]^{(k)}$ share $(\alpha, 2)^*$, then $f(z) \equiv g(z)$.

2. **Lemmas**

Let F and G be two non-constant meromorphic functions defined in $\mathbb{C}$. We denote by H the function as follows.

$$H = \left(\frac{F''}{F'} - \frac{2F'}{F-1}\right) - \left(\frac{G''}{G'} - \frac{2G'}{G-1}\right) \tag{2.1}$$

Lemma 2.1.[5] Let $f(z)$ be a transcendental meromorphic functions of hyper order $\rho_2(f) < 1$, and let c be a non-zero complex constant Then we have
$T(r, f(z+c)) = T(r, f(z)) + S(r, f(z))$,
$N(r, f(z+c)) = N(r, f(z)) + S(r, f(z))$,
$N(r, \frac{1}{f(z+c)}) = N(r, \frac{1}{f(z)}) + S(r, f(z))$.

Lemma 2.2.[10] Let f be a non-constant meromorphic function, let $P(f) = a_0 + a_1 f + a_2 f^2 + ... + a_n f^n$, where $a_0, a_1, a_2, ...a_n$ are constants and $a_n \neq 0$. Then $T(r, P(f)) = nT(r, f) + S(r, f)$.

Lemma 2.3.[10] Let f be a non-constant meromorphic function and p, k be positive integers. Then

$$T(r, f^{(k)}) \leq T(r, f) + k\overline{N}(r, f) + S(r, f), \tag{2.2}$$

$$N_p(r, \frac{1}{f^{(k)}}) \leq T(r, f^{(k)}) - T(r, f) + N_{p+k}(r, \frac{1}{f}) + S(r, f), \tag{2.3}$$

$$N_p(r, \frac{1}{f^{(k)}}) \leq N_{p+k}(r, \frac{1}{f}) + k\overline{N}(r, f) + S(r, f), \tag{2.4}$$

$$N(r, \frac{1}{f^{(k)}}) \leq N(r, \frac{1}{f}) + k\overline{N}(r, f) + S(r, f). \tag{2.5}$$

Lemma 2.4.[14] H be defined as above. If F and G share "$(1,2)$" and $H \not\equiv 0$,

$$T(r,F) \le N_2(r,\frac{1}{F}) + N_2(r,\frac{1}{G}) + N_2(r,F) + N_2(r,G) - \sum_{p=3}^{\infty} \overline{N}_{(p}(r,\frac{G}{G'}) + S(r,F) + S(r,G),$$

and the same inequality holds for $T(r,G)$.

Lemma 2.5.[14] H be defined as above. If F and G share $(1,2)^*$ and $H \not\equiv 0$, then

$$T(r,F) \le N_2(r,\frac{1}{F}) + N_2(r,\frac{1}{G}) + N_2(r,F) + N_2(r,G) + \overline{N}(r,\frac{1}{F}) + \overline{N}(r,F) - m(r,\frac{1}{G-1}) + S(r,F) + S(r,G),$$

and the same inequality holds for $T(r,G)$.

Lemma 2.6.[15] Let F and G be two non-constant entire functions, and $p \ge 2$ an integer. If $\overline{E}_{p)}(1,F) = \overline{E}_{p)}(1,G)$ and $H \not\equiv 0$, then

$$T(r,F) \le N_2(r,\frac{1}{F}) + N_2(r,\frac{1}{G}) + 2\overline{N}(r,\frac{1}{F}) + \overline{N}(r,\frac{1}{G}) + S(r,F) + S(r,G).$$

Lemma 2.7. Let $f(z)$ be a transcendental meromorphic function of hyper order $\rho_2(f) < 1$ and $F_1(z)$ be stated as in (1.2). Then

$$(n+m-\lambda)T(r,f) + S(r,f) \le T(r,F_1(z)) \le (n+m+\lambda)T(r,f) + S(r,f).$$

Proof. Since f is a meromorphic function with $\rho_2(f) < 1$. From Lemma 2.2 and Lemma 2.3, we have

$$\begin{aligned} T(r,F_1(z)) &\le T(r,f(z)^n) + T(r,P(f)) + T\left(r, \prod_{j=1}^{d} f(z+c_j)^{s_j}\right) + S(r,f) \\ &\le (n+m+\lambda)T(r,f) + S(r,f) \end{aligned}$$

On the other hand, from Lemma 2.1 and Lemma 2.2, we have

$$\begin{aligned} (n+m+\lambda)T(r,f) &= T(r,f^n f^m f^\lambda) + S(r,f) \\ &\le m\left(r, \frac{F_1(z)f^\lambda}{\prod_{j=1}^{d} f(z+c_j)^{s_j}}\right) + N\left(r, \frac{F_1(z)f^\lambda}{\prod_{j=1}^{d} f(z+c_j)^{s_j}}\right) + S(r,f) \\ &\le T(r,F_1(z)) + T\left(r, \frac{F_1(z)f^\lambda}{\prod_{j=1}^{d} f(z+c_j)^{s_j}}\right) + S(r,f) \\ &\le T(r,F_1(z)) + 2\lambda T(r,f) + S(r,f) \\ (n+m+\lambda-2\lambda)T(r,f) &\le T(r,F_1(z)) + S(r,f) \\ (n+m-\lambda)T(r,f) + S(r,f) &\le T(r,F_1(z)). \end{aligned}$$

Hence we get Lemma 2.7.

3. **Proof of the Theorems**

Proof of Theorem 1.1. Let $F(z) = \frac{f(z)^n P(f)\left[\prod_{j=1}^{d} f(z+c_j)^{s_j}\right]^{(k)}}{\alpha(z)}$, $G(z) = \frac{g(z)^n P(g)\left[\prod_{j=1}^{d} g(z+c_j)^{s_j}\right]^{(k)}}{\alpha(z)}$. Then $F(z)$ and $G(z)$ share "$(1,2)$" except the zeros or poles of $\alpha(z)$.
Case 1. Suppose Lemma 2.4 holds

$$T(r,F) \le N_2\left(r,\frac{1}{F}\right) + N_2\left(r,\frac{1}{G}\right) + N_2(r,F) + N_2(r,G) + S(r,F) + S(r,G). \tag{3.1}$$

From Lemma 2.1 and Lemma 2.7, we have $S(r,F)=S(r,f)$ and $S(r,G)=S(r,g)$.
From (3.1), we have

$$\begin{aligned} T(r,F) &\le N_2\left(r,\frac{1}{F}\right)+N_2\left(r,\frac{1}{G}\right)+N_2(r,F)+N_2(r,G)+S(r,f)+S(r,g)\\ &\le N_2\left(r,\frac{1}{f^n}\right)+N_2\left(r,\frac{1}{P(f)}\right)+N_2\left(r,\frac{1}{(\prod_{j=1}^d f(z+c_j)^{s_j})^{(k)}}\right)+N_2\left(r,\frac{1}{g^n}\right)+N_2\left(r,\frac{1}{P(g)}\right)\\ &\quad+N_2\left(r,\frac{1}{(\prod_{j=1}^d g(z+c_j)^{s_j})^{(k)}}\right)+2\overline{N}(r,f^nP(f))+2\overline{N}\left(r,(\prod_{j=1}^d f(z+c_j)^{s_j})^{(k)}\right)\\ &\quad+2\overline{N}(r,g^nP(g))+2\overline{N}\left(r,(\prod_{j=1}^d g(z+c_j)^{s_j})^{(k)}\right)+S(r,f)+S(r,g). \end{aligned} \tag{3.2}$$

Using (2.2) of Lemma 2.3 in (3.1) we have

$$\begin{aligned} T(r,F) &\le 2\overline{N}_{(2}(r,\frac{1}{f^n})+N_2(r,\frac{1}{P(f)})+T\left(r,(\prod_{j=1}^d f(z+c_j)^{s_j})^{(k)}\right)-T\left(r,\prod_{j=1}^d f(z+c_j)^{s_j}\right)+N_{k+2}\left(r,\frac{1}{\prod_{j=1}^d f(z+c_j)^{s_j}}\right)\\ &\quad+2\overline{N}_{(2}(r,\frac{1}{g^n})+N_2(r,\frac{1}{P(g)})+T\left(r,(\prod_{j=1}^d g(z+c_j)^{s_j})^{(k)}\right)-T\left(r,\prod_{j=1}^d g(z+c_j)^{s_j}\right)+N_{k+2}\left(r,\frac{1}{\prod_{j=1}^d g(z+c_j)^{s_j}}\right)\\ &\quad+2N(r,f)+2N(r,\prod_{j=1}^d f(z+c_j)^{s_j})+2N(r,g)+2N(r,\prod_{j=1}^d g(z+c_j)^{s_j})+S(r,f)+S(r,g) \end{aligned}$$

$$\begin{aligned} &\le (2+\Gamma_1)T(r,f)+T\left(r,(\prod_{j=1}^d f(z+c_j)^{s_j})^{(k)}\right)+T(r,f^n)+T(r,P(f))-T(r,f^n)-T(r,P(f))\\ &-T\left(r,\prod_{j=1}^d f(z+c_j)^{s_j}\right)+(k+2)dT(r,f)+(2+\Gamma_1)T(r,g)+T\left(r,\prod_{j=1}^d g(z+c_j)^{s_j}\right)+k\overline{N}\left(r,\prod_{j=1}^d g(z+c_j)^{s_j}\right)\\ &-T\left(r,\prod_{j=1}^d g(z+c_j)^{s_j}\right)+(k+2)dT(r,g)+(2+2\lambda)\{T(r,f)+T(r,g)\}+S(r,f)+S(r,g) \end{aligned}$$

$$\begin{aligned} T(r,F) \le&(2+\Gamma_1)T(r,f)+T(r,F)-T(r,F_1)+(k+2)dT(r,f)+(2+\Gamma_1)T(r,g)+k\lambda T(r,g)\\ &+(k+2)dT(r,g)+(2+2\lambda)\{T(r,f)+T(r,g)\}+S(r,f)+S(r,g) \end{aligned}$$

$$T(r,F_1)\le\{2+\Gamma_1+(k+2)d\}\{T(r,f)+T(r,g)\}+k\lambda T(r,g)+(2+2\lambda)\{T(r,f)+T(r,g)\}+S(r,f)+S(r,g).$$

From Lemma 2.4, we have

$$(n+m-\lambda)T(r,f)\le\{2+\Gamma_1+(k+2)d\}\{T(r,f)+T(r,g)\}+k\lambda T(r,g)+(2+2\lambda)\{T(r,f)+T(r,g)\}+S(r,f)+S(r,g). \tag{3.3}$$

Similarly for $T(r,g)$, we obtain the following

$$\begin{aligned} (n+m-\lambda)T(r,g)\le&\{2+\Gamma_1+(k+2)d\}\{T(r,f)+T(r,g)\}+k\lambda T(r,f)+(2+2\lambda)\{T(r,f)+T(r,g)\}\\ &+S(r,f)+S(r,g). \end{aligned} \tag{3.4}$$

From (3.3) and (3.4), we have

$$\begin{aligned} (n+m-\lambda)\{T(r,f)+T(r,g)\}\le&\,2\{2+\Gamma_1+(k+2)d\}\{T(r,f)+T(r,g)\}+k\lambda\{T(r,f)+T(r,g)\}\\ &+2(2+2\lambda)\{T(r,f)+T(r,g)\}+S(r,f)+S(r,g), \end{aligned}$$

which is contradiction to $n \geq 2\Gamma_1 + 2(k+2)d + \lambda(5+k) + 8 - m$.

Case 2. If $F \equiv G$

$$i.e,\ \ f(z)^n P(f)\left[\prod_{j=1}^{d} f(z+c_j)^{s_j}\right]^{(k)} \equiv g(z)^n P(g)\left[\prod_{j=1}^{d} g(z+c_j)^{s_j}\right]^{(k)}$$

This proves the Theorem 1.1.

Proof of Theorem 1.2. Let

$$F(z) = \frac{f(z)^n P(f)\left[\prod_{j=1}^{d} f(z+c_j)^{s_j}\right]^{(k)}}{\alpha(z)}, \quad G(z) = \frac{g(z)^n P(g)\left[\prod_{j=1}^{d} g(z+c_j)^{s_j}\right]^{(k)}}{\alpha(z)}.$$

Then $F(z)$ and $G(z)$ share $(1,2)^*$ except the zeros or poles of $\alpha(z)$. Obviously

$$\begin{aligned}
&2N_2(r,\frac{1}{F}) + 2N_2(r,\frac{1}{G}) + 2N_2(r,F) + 2N_2(r,G) + \overline{N}(r,\frac{1}{F}) + \overline{N}(r,\frac{1}{G}) + \overline{N}(r,F) + \overline{N}(r,G) + S(r,F) + S(r,G)\\
&\quad \leq (2\Gamma_1 + \Gamma_0 + (3k+5)d + (6+k)\lambda + 10 - m)T(r,f)\\
&\quad + (2\Gamma_1 + \Gamma_0 + (3k+5)d + (6+k)\lambda + 10 - m)T(r,g) + S(r,f) + S(r,g).
\end{aligned} \tag{3.5}$$

According to (3.3) and Lemma 2.5, we can prove Theorem 1.2 in a similar way as in proof of Theorem 1.1.

Proof of Theorem 1.3. Let

$$F(z) = \frac{f(z)^n P(f)\left[\prod_{j=1}^{d} f(z+c_j)^{s_j}\right]^{(k)}}{\alpha(z)}, \quad G(z) = \frac{g(z)^n P(g)\left[\prod_{j=1}^{d} g(z+c_j)^{s_j}\right]^{(k)}}{\alpha(z)}.$$

Then $\overline{E}_{2)}\left(1, f(z)^n P(f)\left[\prod_{j=1}^{d} f(z+c_j)^{s_j}\right]^{(k)}\right) = \overline{E}_{2)}\left(1, g(z)^n P(g)\left[\prod_{j=1}^{d} g(z+c_j)^{s_j}\right]^{(k)}\right)$ except the zeros or poles of $\alpha(z)$. Obviously

$$\begin{aligned}
&2N_2(r,\frac{1}{F}) + 2N_2(r,\frac{1}{G}) + 3\overline{N}(r,\frac{1}{F}) + 3\overline{N}(r,\frac{1}{G}) + S(r,F) + S(r,G)\\
&\quad \leq (2\Gamma_1 + 2\Gamma_0 + 2(2k+3)d + (k+1)\lambda + 6 - m)T(r,f)\\
&\quad + (2\Gamma_1 + 2\Gamma_0 + 2(2k+3)d + (k+1)\lambda + 6 - m)T(r,g) + S(r,f) + S(r,g).
\end{aligned} \tag{3.6}$$

Using to (3.5) and Lemma 2.6, we can prove Theorem 1.3 in a similar way as in proof of Theorem 1.1.

Proof of Theorem 1.4. Since f and g are entire functions we have $\overline{N}(r,f) = \overline{N}(r,g) = 0$. Proceeding in the proof of Theorem 1.1 we can easily prove Theorem 1.4.

Proof of Theorem 1.5. Since f and g are entire functions we have $\overline{N}(r,f) = \overline{N}(r,g) = 0$. Proceeding in the proof of Theorem 1.2 we can easily prove Theorem 1.5.

4. **Open Questions**

Question. Can the difference polynomials in Theorem 1.1 - 1.3 be replaced by difference polynomials of the form $f^n(z)(f(z)-1)^m[\prod_{j=1}^{d} f(z+c_j)^{s_j}]^{(k)} \prod_{j=1}^{k} f^{(i)}(z)$?

Conclusions. Using the notion of weakly weighted sharing and relaxed weighted sharing, in this paper, we provide five results, which extend the main results that were derived in the paper [8]. Obtaining our results from more general hypotheses without complicated calculations is probably the most interesting feature of this paper. Finally, in this paper, we pose one more general open question for further studies.

Application. Nevanlinna theory has been used as a detector of integrability of difference equations. In this paper we study meromorphic and entire solutions of difference polynomials and extend some key results from Nevanlinna theory. These results include a difference analogue of the value distribution of solutions.

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