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ACCESSIBILITY INTEGRITY OF GRAPHS

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Abstract. In this paper, the concept of accessibility integrity is introduced as a new measure of the stability of a graph G and it is defined as

$$AI(G) = \min\{|S| + m(G - S)\},$$

where S is an accessible set and $m(G - S)$ is the order of a maximum component of $G - S$. First, the accessibility integrity of some graphs is obtained and the relations between accessibility integrity and other parameters are determined. Next AI-changing and AI-stable graphs are studied. The properties of AI-stellar and just AI-stellar graph are discussed. It is shown that the accessibility integrity is much stronger characteristic compared to the integrity in determining the stability of a graph. The effect of vertex deletion on the accessibility integrity of a graph is studied. Also a recent problem called the inverse problem is completely solved for the accessibility integrity of a graph by showing that there exists at least one graph with accessibility integrity equal to r for all integers $r \geq 4$.

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1. Introduction

Throughout this paper, all graphs are finite, simple and undirected. For a graph G, we denote the vertex set and the edge set of G by $V(G)$ and $E(G)$, respectively. We use p to denote the number of vertices and q to denote the number of edges of a graph G. By the neighborhood of a vertex v of G we mean the set $N_G(v) = \{u \in V(G) : uv \in E(G)\}$. The open neighborhood $N_G(v)$ of a vertex v consists of the set of vertices adjacent to v, that is, $N_G(v) = \{u \in V : uv \in E\}$ and the closed neighborhood of v is $N_G[v] = N_G(v) \cup v$. For a set $S \subseteq V$, the open neighborhood $N_G(S)$ is defined to be $\cup_{v\in S} N_G(v)$.

The degree of a vertex v, denoted by $deg(v)$, is the cardinality of its neighborhood. By a pendant vertex we mean a vertex of degree one, while a support vertex is a vertex adjacent to a pendant vertex. For a subset S of $V(G)$, the open S-private neighborhood of v, denoted by $pn(v, S)$, is defined as the set of all vertices in the open neighborhood of v but not in the open neighborhood of $S\backslash\{v\}$, that is $pn(v, S) = N_G(v)\backslash N_G(S\backslash\{v\})$. Given any vertex $v \in V(G)$, the graph obtained from G by removing the vertex v and all of its incident edges is denoted by $G - v$. The reader may follow [8] for graph-theoretical terminology and notation not defined here.

There are various measures of the accuracy of a communication network, a stylish one is called the integrity of the network. Barefoot, Entringer and Swart, [2], introduced this concept as a useful measure of the vulnerability of the graph in 1987. The integrity $I(G)$ of a graph G is defined by

$$I(G) = \min\{|S| + m(G - S) : S \subseteq V(G)\},$$

where $m(G-S)$ denotes the order of the largest component in $G-S$. There is a substantial literature on integrity, but most of the papers are concerned with calculating the integrity of particular graphs. There are results on the interrelations between integrity and other graph parameters, [6]. Clark et. al., [3], proved that the problem of the determination of the integrity of a given graph is NP-complete. Many graph parameters have been introduced to measure the vulnerability of communication networks. They include domination integrity, hub-integrity, hub edge-integrity, global domination integrity and distance majorization integrity [10, 11, 12, 13, 14, 15, 16, 18].

The complement $\overline{G}$ of a graph G has $V(G)$ as its vertex set and two vertices are adjacent in $\overline{G}$ if and only if they are not adjacent in G, [8]. The double star graph $S_{n,m}$ is the graph constructed from two star graphs $K_{1,n-1}$ and $K_{1,m-1}$ by joining their centers v_0 and u_0. That is,

$$V(S_{n,m}) = V(K_{1,n-1}) \cup V(K_{1,m-1})$$

and

$$E(S_{n,m}) = \{v_0u_0, v_0v_i, u_0u_j \ : \ 1 \leq i \leq n-1, \ 1 \leq j \leq m-1\},$$

[7]. The corona product $G_1 \circ G_2$ of two graphs G_1 and G_2 is the graph G obtained by taking one copy of G_1 (which has p_1 vertices) and p_1 copies of G_2, and then joining the i^{th} vertex of G_1 to every vertex in the i^{th} copy of G_2. The symbols $\delta(G)$ and $\Delta(G)$ denote the minimum and maximum degrees in G, respectively. A cut-set in a connected graph is a set of vertices whose removal leaves a disconnected graph. $\lceil x \rceil$ will denote the smallest integer that is greater than or equal to x, and $\lfloor x \rfloor$ will denote the greatest integer that is smaller than or equal to x.

In the present work, we introduce the notion of the accessibility integrity of a graph. The exact values of accessibility integrity of some graphs are obtained. The notions of changing and unchanging of accessibility integrity in graphs are studied. A graph G is said to be just AI-stellar graph if for each $v \in V(G)$, there exists a unique AI-set of G containing v. Finally, some properties of just AI-stellar graphs are established.

2. Accessibility integrity of graphs

Definition 2.1. [4] *A subset S of $V(G)$ is called an accessible set of the graph G if each vertex $v \in V(G) - S$ is adjacent to a vertex in $N[S]$, where $N[S]$ is the closed neighborhood of S. The accessibility number of G is defined as the minimum number of vertices over all accessible sets of G and is denoted by $\eta(G)$.*

We define the accessibility integrity of a graph G as follows.

Definition 2.2. *The accessibility integrity of a graph G is defined as the number*

$$AI(G) = \min\{|S| + m(G - S)\}$$

where S is an accessible set and $m(G - S)$ is the order of a maximum component of $G - S$.

Definition 2.3. *An AI-set of G is any subset S of $V(G)$ for which*

$$AI(G) = |S| + m(G - S).$$

Explicitly, $AI(G) \geq I(G)$ for any graph G. Also, the definition shows that $|S| \geq \eta(G)$ and the bound is sharp for $G = \overline{K_p}$ and $AI(G) \geq \eta(G)$. For more details on the accessibility integrity see [9].

First we calculate the accessibility integrity of some standard graphs.

Proposition 2.1. *The accessibility integrity of some specific classes of graphs are as below:*

(a) *For any complete graph K_p, $AI(K_p) = p$.*

(b) *For any path graph P_p with $p \geq 3$,*

$$AI(P_p) = \begin{cases} 2, & \text{if } p = 3; \\ \lfloor \frac{p}{3} \rfloor + 2, & \text{if } 4 \leq p \leq 20; \\ \lfloor \frac{p}{4} \rfloor + 3, & \text{if } p = 21; \\ \lceil \frac{p}{5} \rceil + 4, & \text{if } p \geq 22. \end{cases}$$

(c) *For any cycle graph C_p, $p \geq 4$,*

$$AI(C_p) = \begin{cases} p - 1, & \text{if } p = 4, 5; \\ p - 2, & \text{if } p = 6; \\ \frac{p}{3} + 2, & \text{if } p = 9; \\ \lceil \frac{p}{4} \rceil + 3, & \text{if } 7 \leq p \leq 24, p \neq 9; \\ \lceil \frac{p}{5} \rceil + 4, & \text{if } p \geq 25. \end{cases}$$

(d) *For the star graph $K_{1,p-1}$, $AI(K_{1,p-1}) = 2$.*

(e) *For the double star graph $S_{n,m}$,*

$$AI(S_{n,m}) = 3.$$

(f) *For the complete bipartite graph $K_{n,m}$,*

$$AI(K_{n,m}) = min\{n, m\} + 1.$$

(g) *For the complete k-partite graph $K_{n_1,n_2,\cdots,n_k}$,*

$$AI(K_{n_1,n_2,\cdots,n_k}) = \sum_{i=1}^{k} n_i + 1 - \max_{i=1}^{k} n_i.$$

(h) *For the wheel graph $W_{1,p-1}, p \geq 5$,*

$$AI(W_{1,p-1}) = \lceil 2\sqrt{p-1} \rceil.$$

To illustrate this concept, consider the graphs C_6 and $K_{2,4}$.

Let $V(C_6) = \{u_1, u_2, u_3, u_4, u_5, u_6\}$. We have two cases:

Case 1) If $S = \{u_1, u_4\}, S = \{u_2, u_5\}$ or $S = \{u_3, u_6\}$, then graphs $C_6 - S$ have two components of order 2.

Case 2) If $S = \{u_1, u_3, u_5\}$ or $S = \{u_2, u_4, u_6\}$, then graphs $C_6 - S$ have three components of order 1.

From Case 1 and Case 2, we have

$$\begin{aligned} AI(C_6) &= min_{S \subset V(C_6)}\{|S| + m(C_6 - S)\} \\ &= min\{2+2, 3+1\} \\ &= 4. \end{aligned}$$

Let $V(K_{2,4}) = \{u_1, u_2, v_1, v_2, v_3, v_4\}$. Consider $S = \{u_1, u_2\}$ as an accessible set of $K_{2,4}$. Then $|S| = 2$ and $m(K_{2,4} - S) = 1$. Also we may consider the set $\{v_1, v_2, v_3, v_4\}$ as an accessible set of $K_{2,4}$. Also there are many other sets as accessible sets of $K_{2,4}$. But the set $\{u_1, u_2\}$ is the only $AI-$set of $K_{2,4}$. So $AI(K_{2,4}) = 3$.

Observation *For any vertex v of a graph G such that v does not belong to an $AI-$set of G, $AI(G) = AI(G - v)$. But the converse is not true. For example, for the graph P_5 with $V(P_5) = \{v_1, v_2, v_3, v_4, v_5\}$, we have $\{v_2, v_4\}$ and $\{v_3\}$ as the AI-sets of P_5. $AI(P_5) = AI(P_5 - v_3) = 3$ and v_3 belongs to an AI-set of P_5.*

We now discuss various properties of $AI(G)$.

Theorem 2.2. *If G is a disconnected graph, then $AI(\overline{G}) = I(\overline{G})$.*

Proof. Since $AI(G) \geq I(G)$ for every graph, we can write $AI(\overline{G}) \geq I(\overline{G})$. Recall that S is an I-set if $|S| + m(G - S) = I(G)$. Let S be an I-set of G and $G = G_1 \cup G_2 \cup \cdots \cup G_k$. For any vertex u in one of the components, we have $N_{\overline{G}}[u] = V(\overline{G}) \setminus N_G(u)$. So, any two vertices u, v from two different components satisfy $N_{\overline{G}}[u] \cup N_{\overline{G}}[v] = V(\overline{G})$. i.e. these vertices are in an I-set of $\overline{G}$. Then any I-set of $\overline{G}$ contains two vertices each from different components, whereas an AI-set of $\overline{G}$ contains exactly two vertices each from different components so that

$$AI(\overline{G}) \leq I(\overline{G}),$$

therefore $AI(\overline{G}) = I(\overline{G})$. □

Proposition 2.3. *For any graph G, $AI(G) = \eta(G)$ if and only if $G \cong \overline{K_p}$.*

Proof. Let S be an AI-set of G, i.e., $AI(G) = |S| + m(G - S)$, assume that $AI(G) = \eta(G)$. Then

$$|S| + m(G - S) = \eta(G).$$

Since $|S| \geq \eta(G)$ by definition of the accessibility integrity of a graph, we have $m(G - S) \leq 0$. Hence as $m(G - S)$ is non-negative, we obtain $m(G -$

$S) = 0$ which implies that all vertices of G form an accessible set of G. So $|S| = |V(G)|$ and this is satisfied if G is $\overline{K_p}$. Therefore, $AI(G) = \eta(G)$ if $G = \overline{K_p}$. Conversely, if $G = \overline{K_p}$, it is easy to show that $AI(G) = \eta(G)$. □

Observation *For any graph G, we have*

(1) $1 \leq AI(G) \leq p$.

(2) *If G is incomplete, then every AI-set of G is a cut-set of G.*

Proposition 2.4. *For any connected graph G, $AI(G) = 2$ if and only if $G \cong K_2$, $\overline{K_2}$ or $K_{1,p-1}$.*

Proof. Let S be an AI-set of G. If $AI(G) = 2$, then $|S| + m(G - S) = 2$ and we have the following cases:

Case 1: $|S| = 0$ and $m(G - S) = 2$, this is impossible, since any graph has at least one nonempty accessible set.

Case 2: $|S| = 1$ and $m(G - S) = 1$, this implies that $G \cong K_{1,p-1}$.

Case 3: $|S| = 2$ and $m(G - S) = 0$, then there exist only two graphs satisfying this equality, $G \cong K_2$ *or* $\overline{K_2}$.

Conversely, let $G \cong K_2, \overline{K_2}$ *or* $K_{1,p-1}$. The proof is clear. □

Lemma 2.5. *If G is a connected graph, then $AI(G - S) = m(G - S)$ if and only if $G \cong K_p$, where S is an AI-set of G.*

Proof. Let $AI(G - S) = m(G - S)$ and suppose that $G \neq K_p$. Let S be an AI-set of G such that $AI(G) = |S| + m(G - S)$ and $S^\star$ be an AI-set of $G - S$ such that

$$AI(G - S) = |S^\star| + m((G - S) - S^\star).$$

Since $G \neq K_p$, S is a cut-set of G. So $|S^\star| > |S|$ and $m((G - S) - S^\star) \geq 1$. Then $AI(G - S) > |S| + 1$ implying that $m(G - S) > |S| + 1$, a contradiction. Then $G \cong K_p$.

Conversely, let $G \cong K_p$ and S be an AI-set of K_p. If $|S| = 1$, then $K_p - S = K_{p-1}$. Therefore $AI(K_p - S) = p - 1$ and $m(K_p - S) = p - 1$. So $AI(K_p - S) = m(K_p - S)$. In general if $|S| = k$, $1 \leq k \leq p$, then $K_p - S = K_{p-k}$ and $AI(K_{p-k}) = p - k$ and $m(K_p - S) = p - k$. Thus $AI(K_p - k) = m(K_p - k)$, hence the result. □

Proposition 2.6. *For any graph G, $AI(G) = p$ if and only if $G \cong K_p$ or $\overline{K_p}$.*

Proof. Suppose $AI(G) = p$. Then $|S| + m(G - S) = p$. The following cases can be considered:

Case 1) If $|S| = k$, $1 \leq k \leq p - 2$, then $m(G - S) = p - k$. So there is only one graph achieving this value, that is $G \cong K_p$.

Case 2) If $|S| = j$, $j \geq p - 1$ then $m(G - S) = p - j$. The graphs that achieve these values are K_p or $\overline{K_p}$.

Conversely, assume that $G \cong K_p$ *or* $\overline{K_p}$. By Proposition 2.1, the proof follows. □

Definition 2.4. [17] *A galaxy graph G is a forest in which each component is a star.*

Lemma 2.7. *If G is a galaxy graph, then $AI(G) = k + 1$, where k is the number of components of G.*

Proof. Since every component of G is a star $K_{1,p-1}$ and $AI(K_{1,p-1}) = 2$, then it is enough to take one vertex of every component to form an accessible set, so $m(G - S) = 1$. Therefore, $AI(G) = k + 1$. □

3. Effect of vertex removal on the accessibility integrity of graphs

In many combinatorial calculations in graph theory, especially in those with topological graph indices and several invariants related to graphs, it is a very practical way to add or remove a vertex or edge and follow the change in the number under consideration. Here we deal with what we need, which is vertex removal. Removing a vertex from a graph may cause its accessibility integrity to increase, to decrease, or to remain the same. As examples to these three cases, first let us take $G = K_{1,p-1},\ p \geq 4$ and let v be the central vertex of G. Then $AI(K_{1,p-1}) = 2$ and $AI(K_{1,p-1} - v) = p - 1$. So $K_{1,p-1}$ is an example for the increase of $AI(G)$. Let $G = K_p$. For every $v \in K_p$, we have

$$AI(K_p - v) < AI(K_p)$$

giving an example for the decrease of $AI(G)$. To give an example to that the accessibility integrity remains the same, let $G = C_p,\ 23 \leq p \leq 25$. Then $AI(G - v) = AI(G)$ for every vertex of G.

Consequently, it is wise to define the following partition of the vertex set $V(G)$ of any graph G.

Definition 3.1. *For a graph G, we define three sets as below:*

$$\begin{aligned} V^0(G) &= \{v \in V(G) : AI(G - v) = AI(G)\}, \\ V^+(G) &= \{v \in V(G) : AI(G - v) > AI(G)\}, \\ V^-(G) &= \{v \in V(G) : AI(G - v) < AI(G)\}. \end{aligned}$$

The following result gives the place of the isolated vertices if any:

Proposition 3.1. *If v is an isolated vertex of G, then $v \in V^-(G)$.*

Proof. Let v be an isolated vertex of G. Let S be an AI-set of G such that $|S| + m(G - S) = AI(G)$. Since v is an isolated vertex, we know that $v \in S$. Put $S^\star = S - \{v\}$. It is then clear that $S^\star$ is an accessible set of $G - v$. Thus

$$\begin{aligned} AI(G - v) &= |S^\star| + m((G - v) - S^\star) \\ &= |S| - 1 + m(G - S) \\ &= AI(G) - 1. \end{aligned}$$

Then, $AI(G - v) < AI(G)$ and hence $v \in V^-(G)$. □

Observation *The converse of the Proposition 3.1 is not true. For example, for* $G = K_p$*, we know that* $AI(K_p) = p$ *and* $AI(K_p - v_i) = p-1$ *for* $1 \leq i \leq p$*. But* v_i *is not an isolated vertex.*

Lemma 3.2. *If* u *is an isolated vertex of* G *or there exists an* AI*-set* S *of* G *containing* u *and* $S - \{u\}$ *is an accessible set of* $G - u$*, then* $u \in V^-(G)$*.*

Proof. If u is an isolated vertex of G, then by Proposition 3.1, we have $u \in V^-(G)$. Assume that u is not an isolated vertex of G. Then there exists an AI-set S of G such that $S - \{u\}$ is an accessible set of $G - u$. If $S^\star = S - \{u\}$, it is clear that $S^\star$ is an accessible set of $G - u$, and also

$$AI(G-u) \leq |S^\star| + m((G-u) - S^\star).$$

Since $S^\star = S - \{u\}$, we have $|S^\star| = |S| - 1$ and so we obtain

$$\begin{aligned} AI(G-u) &\leq |S| - 1 + m((G-u) - (S - \{u\})) \\ &= |S| - 1 + m(G - S) \\ &= AI(G) - 1. \end{aligned}$$

Then, $AI(G-u) < AI(G)$. So $u \in V^-(G)$. □

Remark 3.1. *If* uv *is an edge in a tree* T *and if* $u \in V^+$*, then* $v \in V^0$*.*

Definition 3.2. *A graph* G *is called* AI*-changing if* $AI(G-v) \neq AI(G)$ *for every* $v \in V(G)$*. A graph* G *is* AI*-stable if* $AI(G-v) = AI(G)$ *for every* $v \in V(G)$*.*

If G is an AI-changing graph, then $V(G) = V^+(G) \cup V^-(G)$, but there does not exist any graph G such that $V(G) = V^+(G) \cup V^-(G)$. Therefore AI-changing graphs satisfy $V(G) = V^-(G)$. For example, the graphs K_p and C_p are AI-changing graphs and P_3, P_5, $2K_2$ and $2K_3$ are examples of AI-stable graphs.

A recent problem in graph theory is the inverse problem dealing with the question that which non-negative integers can be attained by a graph theoretical quantity, generally a graph theoretical index. This problem has been solved for several indices in [1], [19], [20] and [5]. The following result answers this problem for accessibility integrity of graphs:

Proposition 3.3. *For any positive integer* $r \geq 4$*, there exists a graph* G *with* $AI(G) = r$*.*

Proof. Consider the graph $G = K_{r-1} \circ K_1$. Then G is the graph that satisfies this property. □

Definition 3.3. *A vertex of* G *is called* AI*-useful if it is contained in some* AI*-set of* G*. A vertex of* G *is called* AI*-poor if it does not belong to any* AI*-set of* G*. A graph* G *is called* AI*-stellar if every vertex in* G *is* AI*-useful.*

Example 3.1. *Let the graph* G *be as in Fig. 1.*

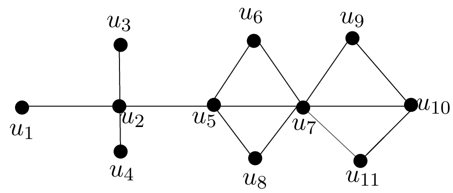

Figure 1: G

$AI(G) = 5$ and $\{u_2, u_7\}$ and $\{u_2, u_5, u_7, u_{10}\}$ are two AI-sets of G. So u_2, u_5, u_7, u_{10} are AI-useful vertices of G and $u_1, u_3, u_4, u_6, u_8, u_9, u_{11}$ are $AI-$poor vertices of G.

Remark 3.2. *(1) The center vertex of $K_{1,p-1}$ is AI-useful and every other vertex is AI-poor.*

(2) In a double star graph $S_{n,m}$, the two supporting vertices are AI-useful and all the pendant vertices are AI-poor. Hence both $K_{1,p-1}$ and $S_{n,m}$ are not AI-stellar graphs.

Proposition 3.4. *The path graph P_p is AI-stellar if and only if $p = 4$ or $p \equiv 0 \pmod 3$ and $6 \le p \le 18$ or $p = 22$ or $p \equiv 1 \pmod 5$ and $p \ge 25$.*

Theorem 3.5. *The complete graph K_p is AI-stellar.*

Proof. The AI-sets of K_p are $\{v_1\}, \{v_2\}, \cdots, \{v_p\}, \{v_1, v_2\}, \{v_2, v_3\}, \{v_3, v_4\}, \cdots, \{v_{p-1}, v_p\}$, $\{v_1, v_2, v_3\}, \{v_4, v_5, v_6\}, \cdots, \{v_{p-2}, v_{p-1}, v_p\}$ or $\{v_1, v_2, v_3, \cdots, v_p\}$, so all vertices of K_p are AI-useful. Thus K_p is AI-stellar. □

Theorem 3.6. *$\overline{K_p}$ is AI-stellar.*

Proof. Since AI-set of $\overline{K_p}$ is $\{v_1, v_2, v_3, ..., v_p\}$, then every vertex of $\overline{K_p}$ is AI-useful, then the result follows. □

Proposition 3.7. *The cycle graph C_p is AI-stellar.*

Proof. For every vertex $u \in V(C_p)$, there is an AI-set containing it. Hence C_p is $AI-$stellar. □

Corollary 3.8. *The wheel graph $W_{1,p-1}$ is AI-stellar.*

Proof. Since

$$W_{1,p-1} = K_1 + C_{p-1}$$

and C_{p-1} is AI-stellar, we get the result. □

Proposition 3.9. *$K_{n,m}$ is AI-stellar if $n = m$.*

Proof. Since $AI(K_{n,n}) = n+1$, we can say that $\{v_1, v_2, \cdots, v_n\}$ and $\{u_1, u_2, \cdots, u_n\}$ are two AI-sets. Then all vertices of $K_{n,n}$ are contained in AI-sets, thus $K_{n,n}$ is AI-stellar. □

Theorem 3.10. *Suppose a (p, q) graph G has a unique AI-set. Then G is AI-stellar if and only if $G = \overline{K_p}$.*

Proof. Suppose G has a unique AI-set. Then there exists only one set S as AI-set of G such that $AI(G) = |S| + m(G - S)$. Assume that G is AI-stellar.

Then for every $v \in V(G)$, $v \in S$ and this implies that $S = \{v_1, v_2, \cdots, v_p\}$ and therefore $|S| = p$. Then

$$AI(G) = |S| + m(G - S) = p.$$

By Proposition 2.6, $G \cong K_p$ or $\overline{K_p}$. If $G \cong K_p$, from Theorem 3.5, K_p is AI-stellar, but K_p has more than one AI-sets of G. Hence $G \cong \overline{K_p}$.

Conversely, if $G \cong \overline{K_p}$, clearly $\overline{K_p}$ has a unique AI-set and hence it is AI-stellar. □

Corollary 3.11. *We have*

i) *If a connected graph G is AI-changing, then $\delta(G) \geq 2$. That is, there are no pendant vertices.*

ii) *If G is AI-changing, then G is AI-stellar and $V(G) = V^-(G)$.*

Remark 3.3. *If G is an AI-stellar graph, G is not necessarily AI-changing. For example, $G \cong P_6$.*

Proposition 3.12. *A connected graph G of order at least 3 is AI-changing if $\delta(G) \geq 2$, and for every $v \in V(G)$, there exists an AI-set S such that $v \in S$ and the following conditions are satisfied:*

(1) *There exists some AI-set S of G such that $|N(v) \cap S| = 0$.*

(2) *If v does not belong to at least one of AI-sets, then $|N(v) \cap S| \geq 2$.*

Proof. Suppose that G is AI-changing. From above observation, $\delta(G) \geq 2$ and G is AI-stellar. Thus every vertex of G is contained in some AI-set.

Let S be an AI-set such that $AI(G) = |S| + m(G - S)$. Since G is $AI-$changing, we have $AI(G) \neq AI(G - v)$ for every $v \in V(G)$ and by Corollary 3.11, $v \in V^-(G)$. Thus we obtain $AI(G - v) < AI(G)$. For any graph G, there is at least one AI-set satisfying the accessibility integrity of G. Therefore we discuss the following cases:

Case 1) If $|V(G)| = 3$, then $G \cong K_3$ or P_3. Since G is AI-changing, $G \cong K_3$. Consider $V(K_3) = \{v_1, v_2, v_3\}$. AI-sets are $\{v_1\}, \{v_2\}, \{v_3\}, \{v_1, v_2\}, \{v_1, v_3\}, \{v_2, v_3\}$ and $\{v_1, v_2, v_3\}$. The sets that include only one vertex, for example, satisfy the condition (1).

Case 2) If $|V(G)| \geq 4$, then for an arbitrary vertex v of G such that v does not belong to some AI-set S, there exists a complete bipartite graph $K_{n,n}$ satisfying condition (1). Since $K_{n,n}$ has two AI-sets, namely $S_1 = \{v_1, v_2, \cdots, v_n\}$ and $S_2 = \{u_1, u_2, \cdots, u_n\}$ for any v belonging to S_1, then $N(v) \cap S_1 = \phi$. Hence $|N(v) \cap S_1| = 0$ and this completes the proof. Now by Corollary 3.11, we have $\delta(G) \geq 2$. Then $|N(v)| \geq 2$ for any $v \in V(G)$. Also as $V(G) = S$, by Corollary 3.11, it is clear that $|N(v) \cap S| \geq 2$. Therefore v does not belong to S. Then condition (2) holds. □

Lemma 3.13. *The converse of Proposition 3.12 is not true. For example, let* $G = P_4$ *with* $V(P_4) = \{v_1, v_2, v_3, v_4\}$. *Then the AI-sets are* $S_1 = \{v_1, v_3\}, S_2 = \{v_2, v_4\}$, *and* $S_3 = \{v_2, v_3\}$, *so all vertices of* G *belong to* S_1, S_2 *and* S_3, *and if we take the vertex* $v = v_4$ *and the set* S_2, *then* $N(v_4) \cap S_2 = \phi$. *Finally as* $v_2 \notin S_1$, *we get* $|N(v_2) \cap S_1| = 2$. *Then all conditions are satisfied, but* G *is not an AI-changing graph.*

Proposition 3.12 implies the following corollary:

Corollary 3.14. *Let* G *be an AI-changing graph. Then a* (p, q) *graph* G *has at least two distinct AI-sets.*

Proof. Suppose that G is an AI-changing graph. Then $AI(G - v) \neq AI(G)$ for every $v \in V(G)$. Let us assume that there exists only one AI-set S. Then S satisfies the conditions of Proposition 3.12. Hence $|S| = |V(G)| = p$. Since all vertices of G are contained in S, for any $v \in V(G)$, we have $N(v) \cap S \neq \phi$, so $|N(v) \cap S| \neq 0$, which is a contradiction to the condition (1) of Proposition 3.12. □

Corollary 3.15. *Let* $v \in V^+(G)$. *Then by Eqn. (1), it is not necessary that* v *belongs to any AI-set of* G.

(2) If S *is an AI-set of* G *containing* v, *and if* $u \in pn(v, S)$, *then* u *may belong to* V^-, V^+ *or* V^0. *For example, the graphs* G_1 *and* G_2 *in Fig. 2 are such graphs.*

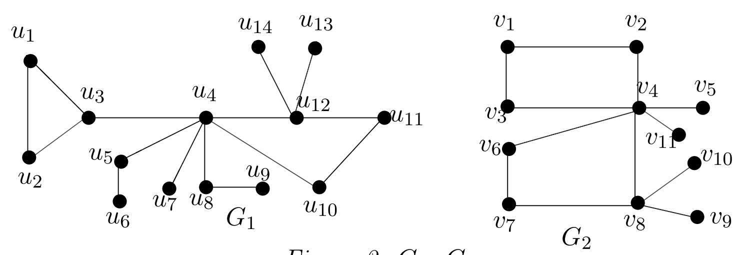

Figure 2: G_1, G_2

We have $S = \{u_3, u_4, u_{12}\}$ *is an AI-set of* G_1, *and* $u_3, u_4, u_{12}, u_5, u_8 \in V^+(G_1)$. *Also* $u_3, u_4, u_{12} \in S$, *while* $u_5, u_8 \notin S$. *For the vertex* $u_4 \in S$, *we have* $pn(u_4, S) = \{u_5, u_7, u_8, u_{10}\}$. *Also* $u_7, u_{10} \in V^0(G_1)$ *and* $u_5, u_8 \in V^+(G_1)$. *Then* $S_1 = \{v_4, v_8\}, S_2 = \{v_1, v_4, v_8\}, S_3 = \{v_1, v_4, v_6, v_8\}$ *and* $S_4 = \{v_1, v_4, v_7, v_8\}$ *are AI-sets of* G_2. S_1 *contains* v_4 *and* $pn(v_4, S_1) = \{v_2, v_3, v_5, v_6, v_{11}\}$ *and* $v_2, v_3, v_6 \in V^-(G_2)$ *and* $v_5, v_{11} \in V^0(G_2)$.

Proposition 3.16. *If* $uv \in E(G)$, *where* $u \in V^+(G)$ *and* $v \in V^-(G)$, *then* u *is a support or* $deg u = \Delta(G)$.

Also we have

Proposition 3.17. $v \in V^+(G)$ *if the following conditions are satisfied:*

i) v *is not an isolated vertex of* G.

ii) *v is a support vertex or $deg(v) = \Delta(G)$.*

iii) *Let S be an AI-set of G containing v. Then $|pn(v, S)| \geq 2$.*

Remark 3.4. *Let G be a graph. Then*

i) *If $e = uv$ is an edge such that $u, v \in V^+$, then $deg(u)$ or $deg(v)$ is equal to $\Delta(G)$.*

ii) *If $v \in V^+$, then v is a cut vertex.*

iii) *If G is tree such that $G \neq P_p$, then every pendant vertex is AI-poor.*

Observation *In a tree T with p points, $p \geq 4$, if AI-set is unique, then for every $v \in AI$-set, $v \in V^+$. But, if G is not a tree and if AI-set is unique, then for every $v \in AI$-set, $v \in V^0$. For example, the graphs shown in Fig. 3 are such graphs.*

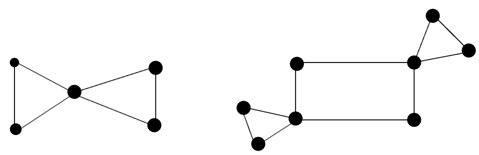

Figure 3

Lemma 3.18. *If G is disconnected, and if every component is complete, then G is an AI-stellar graph.*

Remark 3.5. *An AI-stellar graph G is possibly disconnected. For example, have a look at the graph G in Fig. 4.*

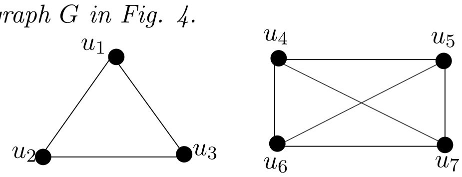

Figure 4: $G = K_3 \cup K_4$

The AI-sets of G are $\{u_1, u_4\}, \{u_2, u_5\}, \{u_3, u_6\}, \{u_1, u_7\}, \cdots$ and $AI(G) = 5$. Thus G is an AI-stellar graph.

We now define just AI-stellar graphs and study some properties of just AI-stellar graph of a graph G.

Definition 3.4. *A graph G is said to be just AI-stellar graph if for each $v \in V(G)$, there exists a unique AI-set of G containing v.*

Some examples of just AI-stellar graphs are $\overline{K_p}$, $K_{n,n}$ and P_9.

Remark 3.6. *Every just AI-stellar graph is also an AI-stellar graph, but the converse is not true, for example $G = P_p$, $p \equiv 1 \pmod 5$ and $p \geq 25$.*

Observation *The union of two just AI-stellar graphs is not necessarily just AI-stellar graphs, for example, the graph G in Fig. 5 below.*

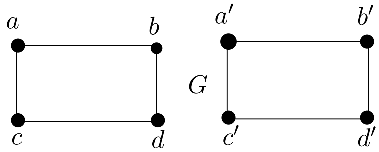

Figure 5: G

The AI-sets of G are $\{a, d, a', d'\}, \{a, d, b', c'\}, \{b, c, a', d'\}$ *and* $\{b, c, b', c'\}$ *and* $AI(G) = 5$. *We note that every element of G belongs to two AI-sets. So G is not just AI-stellar graph.*

Proposition 3.19. *Every just AI-stellar graph* $G \neq \overline{K_p}$ *is connected.*

Proof. Suppose that G is disconnected, then G has at least two components, say G_1 and G_2. Since G is just AI-stellar, G_1 is just AI-stellar. Since $G \neq \overline{K_p}$, so G_1 has at least two AI-sets, namely, S_1 and S_2 such that $V(G_1) = S_1 \cup S_2$. Let S be an $AI-$set of $G - G_1$. Then $S \cup S_1$ and $S \cup S_2$ are two AI-sets of G, a contradiction, since G is a just AI-stellar graph. □

Remark 3.7. *The following are few properties of just AI-stellar graphs:*

(1) *In a just AI-stellar graph G, we have* $V^+(G) = \phi$, *and so every vertex belongs to* V^0 *or* V^-.
(2) *A just AI-stellar graph is not an AI-stable graph.*

Proposition 3.20. *Let G be a just AI-stellar* (p, q) *graph.*

(1) *Then there exists a unique partition of* $V(G)$ *into AI-sets of G.*
(2) *If S is an AI-set of G, then* $|S|$ *is a factor of p.*

Proof. (1) Let V be the vertex set and S be an AI-set of G. Let S_1 be the unique AI-set of G that contains u. If $V \setminus S_1 = \phi$, then the proof is complete. If $V \setminus S_1 \neq \phi$, then for $v \in V \setminus S_1$, there exists a unique AI-set of G, namely S_2 containing v. Since G is just AI-stellar, $S_1 \cap S_2 = \phi$. Now if $V \setminus (S_1 \cup S_2) = \phi$, then the process stops, otherwise there exists $x \in V \setminus (S_1 \cup S_2)$. So there exists a unique AI-set of G, namely S_3, containing x. Since G is just AI-stellar, $S_3 \cap (S_1 \cap S_2) = \phi$, continuing in the same fashion, we get the result.
(2) By (1), $p = n|S|$ where n is the cardinality of partition of $V(G)$ into AI-sets. □

Lemma 3.21. *If G has two disjoint AI-sets* V_1 *and* V_2 *whose union is* $V(G)$, *then*

(1) *G has no isolated vertex,*

(2) $N(V_1) = V_2$ *and* $N(V_2) = V_1$.

Proof. (1) Let $|S| = |V_1|$ or $|V_2|$. Assume that G has an isolated vertex v. Let $v \in V_1$. Clearly $V_2 \cup \{v\}$ is an AI-set of G such that

$$AI(G) = |V_2 \cup \{v\}| + m(G - (V_2 \cup \{v\})) = |S| + 2.$$

Since, $AI(G) \leq |S| + 1$, this means that $m(G - S) = 1$ and $|S| = |V_1|$ or $|V_2|$. Hence, when v is an isolated vertex,

$$AI(G) = |S| + 2 > |S| + 1,$$

a contradiction.

(2) Suppose $N(V_1) \subset V_2$ and let $v \in V_2 \setminus N(V_1)$. Then v is an isolated vertex, a contradiction, since G has no isolated vertices. Thus $N(V_1) = V_2$. Likewise, $N(V_2) = V_1$. □

Lemma 3.22. *A (p, q) graph G is just AI-stellar if and only if*

(1) *$|S|$ divides p,*
(2) *G has exactly $p/|S|$ distinct AI-sets,*
(3) *the maximum cardinality of a partition of $V(G)$ into an accessible set is $p/|S|$.*

Proof. Suppose that the graph G is just AI-stellar. Let $S_1, S_2, \cdots, S_n$ be distinct AI-sets of G. Then

$$V(G) = S_1 \cup S_2 \cup \cdots \cup S_n,$$

where $n \geq 2$ and $S_1 \cap S_2 \cap \cdots \cap S_n = \phi$. Then $S_1, S_2, \cdots, S_n$ is a partition of $V(G)$ into AI-sets. Thus $n|S| = p$. Since $|S_1| = |S_2| = \cdots = |S_n|$, this completes the proof.

Conversely, let G be a graph satisfying the conditions (1), (2) and (3), and let S be an AI-set satisfying $AI(G) = |S| + m(G - S)$. From condition (1), $|S|$ divides p, so $p = n|S|$, for an integer n, and by condition (2), G has exactly $p/|S|$ distinct AI-sets. This means that

$$S_1 \neq S_2 \neq \cdots \neq S_{p/|S|}.$$

From condition (3), there exist accessible sets $S_1, S_2, \cdots, S_n$ such that they are pairwise disjoint and $V = S_1 \cup S_2 \cup S_3 \cup \cdots \cup S_n$. Thus

$$p = \sum_{i=1}^{n} |S_i| \leq n|S|.$$

Then $V = S_1 \cup S_2 \cup S_3 \cup \cdots \cup S_n$ and $S_i,\ 1 \leq i \leq n$ are disjoint AI-sets. Therefore G is AI-stellar. Also the condition (2) shows that $S_1, S_2, \cdots, S_n$ are the only AI-sets of G. □

Observation *Let G be just AI-stellar and $G \neq \overline{K_p}$ and $G \neq K_{n,n}$. Then $|S| \leq p/3$.*

Proof. Let G be just AI-stellar and $G \neq \overline{K_p}$ and $G \neq K_{n,n}$. Since G is just AI-stellar, G has at least two AI-sets. Since $G \neq K_{n,n}$ and $G \neq \overline{K_p}$ by the hypothesis, G has at least three AI-sets. Then $p/|S| \geq 3$ this implies that $|S| \leq p/3$. □

Proposition 3.23. *C_p is just AI-stellar if and only if $p = 4, 9, 16$ or $p \equiv 0(mod\ 5)$, for $p \geq 25$.*

Proof. Suppose that C_p is just AI-stellar. Let S be an AI-set of C_p such that

$$AI(C_p) = |S| + m(C_p - S).$$

Since C_p is just AI-stellar, C_p satisfies the conditions in Lemma 3.22, so $p = n|S|$ where n is the cardinality of the partition of $V(G)$ into AI-sets. Therefore, p is not a prime number. Hence $p \neq 2, 3, 5, 7, 11, 13, 17, 19, 23, \cdots$.

Now, we have the following cases:

Case 1: Let $4 \leq p \leq 24$. By Proposition 2.1, since $|S|$ divides p, we have $p = 4, 6, 8, 9, 12, 16, 20, 24$ and $p \neq 14, 15, 18, 21, 22$. Now, if $p = 8, 10, 12, 20, 24$, there is more than one partition of $V(C_p)$ into AI-sets of C_p, which is a contradiction. So $p \neq 8, 10, 12, 20, 24$. Therefore, $p = 4, 9, 16$, hence the result.

Case 2: Let $p \geq 25$. By Proposition 2.1, we have $p = 25, 30, 35, 40, \cdots$ since $|S|$ divides p. This completes the proof.

Conversely, if $p = 4, 9, 16$ or $p \equiv 0 (mod\ 5)$, $p \geq 25$, it can be verified that C_p is just AI-stellar. □

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