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CHARACTERISTIC POLYNOMIALS OF SUBDIVISION GRAPHS

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Abstract. Energy of a graph, firstly defined by E. Hückel is an important sub area of graph theory with numerous applications in Chemistry and Physics together with all areas they are used as fundamental methods. Schrödinger equation is a second order differential equation which include the energy of the corresponding system. Subdivision of a graph is a method of obtaining a derived graph from a given one which helps to calculate some properties of a complex molecular graph by calculating the same for some easier molecular graph. Unlike many other areas in graph theory, to obtain a general result in spectral graph theory is indeed quite difficult and mostly impossible. In this article, as a result of this idea, the spectral polynomials and their reccurence relations of the subdivision graphs of some well-known graphs are studied. Also the energy of some subdivision graphs are obtained by using the definition of energy. A new relation for the characteristic polynomial of a cyclic graph in terms of the triangular numbers is also obtained.

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1. Introduction

In this paper, we let $G = (V, E)$ be a connected undirected graph with no loops nor multiple edges. We call two vertices u and v of G adjacent if there is an edge e of G connecting u and v. Let G have n vertices $v_1, v_2, \cdots, v_n$. By means of adjacency, we can form an $n \times n$ matrix $A = (a_{ij})$ as follows:

$$a_{ij} = \begin{cases} 1, & if\ v_i\ and\ v_j\ are\ adjacent \\ 0, & otherwise. \end{cases}$$

This matrix is called the adjacency matrix of the graph G and it is the most useful and famous matrix amongst hundreds of graph matrices. As well-known, the eigenvalues $\lambda_1, \lambda_2, \cdots, \lambda_n$ of a square $n \times n$ matrix A, called the eigenvalues of the graph G, are the roots of the equation $|\lambda I_n - A| = 0$. The polynomial on the left hand side of this equation is called the characteristic (or spectral) polynomial of A (or of the graph G), and will be denoted by $Pol(G)$. The set of all eigenvalues of the adjacency matrix A is called the spectrum of the graph G, denoted by $S(G)$.

The sum of absolute values of the eigenvalues of G which was defined for the first time by E. Hückel in [12] is called the energy of G, denoted by $E(G)$, which is an important aspect for the subfield of graph theory called spectral graph theory, and also for the molecular calculations, see [1], [4], [8], [9], [13], [14], [20].

There are several methods of studying some certain property of graphs. Some of them are vertex degrees, distances, matrices, graph operations, clique number, matching number, derived graphs, etc. In this paper, we give some relations for the characteristic polynomials of the subdivision graph which is one of the derived graphs, of some well-known graphs such as cyclic, path, star, complete and complete bipartite graphs, and also study the energy of these subdivision graphs. The results in this paper are obtained by means of the results obtained by the authors in [6, 7].

The rest of the paper is organized as follows: In Section 2, the spectra and the characteristic polynomials of some graphs are recalled. To achieve this aim, in Subsection 2.1, the characteristic polynomials of some well-known graph classes are recalled. In Subsection 2.2, the subdivision graph of a given graph is introduced and studied. In Subsection 2.3, recurrence relations for the characteristic polynomials of subdivision graphs are studied. In Subsection 2.4, these results are generalized to r-subdivision graphs which were defined and studied by the third author and others in [16, 17]. In the final Subsection, using the results obtained in the previous subsections, the energy of the subdivision graphs is studied.

2. Characteristic polynomials of subdivision graphs

In this section, our main aim is to obtain the characteristic polynomials of the subdivision graphs. To do this, first we recall the characteristic polynomials of some of the well known graphs. These are path, cycle, star, complete and complete bipartite graphs. Several results on these polynomials are obtained because of the importance of the notion of energy of a graph. Also we give the interesting exact formula for the characteristic polynomial of the cycle graph C_n by means of the coefficients of Lucas (or Cardan) polynomials.

2.1. **Characteristic polynomials of some graphs.**

The characteristic polynomials, eigenvalues and the spectrum of some graph types including path, cycle, star, complete and complete bipartite graphs are known in literature, see [4, 6, 7]. The spectrum of path and cycle graphs show difference with the other graph types as they can be stated as algebraic numbers in terms of roots of unity. In this section, we will recall the spectrum of these graph types by means of the characteristic polynomial. We shall give exact formulae for the characteristic polynomials and also the recurrence relations for these polynomials.

Let G be a graph. Let A denote the adjacency matrix of G and let λ_1, λ_2, $\cdots$, λ_n be the eigenvalues of A. We now recall the characteristic polynomials of path graph P_n, star graph S_n, complete graph K_n and complete bipartite graph $K_{m,n}$:

$$
Pol(G) = \begin{cases}
\sum_{k=0}^{\frac{n}{2}} (-1)^k \binom{n-k}{k} \lambda^{n-2k}, & if\ G = P_n\ and\ n\ is\ even, \\
\sum_{k=0}^{\frac{n-1}{2}} (-1)^{k+1} \binom{n-k}{k} \lambda^{n-2k}, & if\ G = P_n\ and\ n\ is\ odd, \\
(-\lambda)^{n-2} \left(\lambda^2 - n + 1\right), & if\ G = S_n, \\
(-1)^n (\lambda + 1)^{n-1} (\lambda - n + 1), & if\ G = K_n, \\
(-1)^{m+n} \lambda^{m+n-2} \left(\lambda^2 - mn\right), & if\ G = K_{m,n}.
\end{cases}
$$

Secondly in this paper, we study the characteristic polynomial $Pol(C_n)$ of a cycle graph C_n. The roots of this polynomial which are the eigenvalues of the corresponding adjacency matrix are well known in literature. They can be stated as trigonometric algebraic numbers. In [7], the authors obtained relations between the spectra of C_n and C_{2n}. Now we give the exact formula for the characteristic polynomial $Pol(C_n)$ in terms of the coefficients of the Lucas (or Cardan) polynomials:

Theorem 2.1. *The characteristic polynomial of C_n can be given by the following formula:*

- *if n is even,*

$$Pol(C_n) = \left[\sum_{k=0}^{\frac{n-2}{2}} (-1)^k T(n,k) \lambda^{n-2k} - 2 \right] + 2(-1)^{\frac{n}{2}},$$

- *if n is odd,*

$$Pol(C_n) = \sum_{k=0}^{\frac{n-1}{2}} (-1)^{k+1} T(n,k) \lambda^{n-2k} + 2,$$

where $T(n,k)$s are the coefficients of the Lucas (or Cardan) polynomials (OEIS A034807) given by

$$T(n,0) = 1$$

and

$$T(n,k) = \binom{n-k}{k} + \binom{n-k-1}{k-1} = \frac{n(n-k-1)!}{(n-2k)!k!}, \quad k \leq \frac{n}{2}.$$

Proof. By using combinatorial calculations, it is easy to find the first few characteristic polynomials as $Pol(C_3) = -\lambda^3 + 3\lambda + 2$, $Pol(C_4) = \lambda^4 - 4\lambda^2$, $Pol(C_5) = -\lambda^5 + 5\lambda^3 - 5\lambda + 2$. In [5], a recurrence relation for these polynomials is given by

$$Pol(C_n) = -\lambda Pol(C_{n-1}) - Pol(C_{n-2}) + 2(-1)^{n+1} Pol(C_1).$$

Recursively, one can calculate all other characteristic polynomials of the cycle graphs C_n. By OEIS A034807 giving the coefficients of $Pol(C_n)$ in terms of the coefficients of the Lucas (or Cardan) polynomials, the result follows. □

2.2. **Characteristic polynomials of the subdivision graphs.** The subdivision graph $S(G)$ of a graph G is obtained from G by inserting a new vertex of degree 2 on each edge of G.

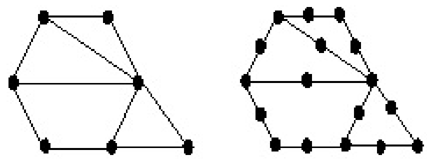

Figure 2.1 A graph G and its subdivision graph $S(G)$

In [2, 3, 10, 11, 15, 16, 17, 18, 19, 21, 22], several properties of the subdivision graphs and their topological indices were studied. Directly from the definition, first we form the subdivision graphs of some well-known graphs like P_n, C_n, S_n, K_n, $K_{m,n}$. Then we obtain the characteristic polynomials of these subdivision graphs. From the definition of subdivision graphs, it is clear that $S(P_n) = P_{2n-1}$ and $S(C_n) = C_{2n}$.

Theorem 2.2. *The characteristic polynomial of the subdivision graph of P_n is*

$$Pol(S(P_n)) = \sum_{k=0}^{n-1}(-1)^{k+1}\binom{2n-k-1}{k}\lambda^{2n-2k-1}.$$

Proof. From the definition of subdivision graph, it is clear that the subdivision graph of P_n is again a path graph P_{2n-1}. By using the well known results

$$Pol(S(P_n)) = Pol(P_{2n-1}) = \sum_{k=0}^{n-1}(-1)^{k+1}\binom{2n-k-1}{k}\lambda^{2n-2k-1}.$$

□

Theorem 2.3. *The characteristic polynomial of the subdivision graph of C_n is*

$$Pol(S(C_n)) = \left[\sum_{k=0}^{n-1}(-1)^k T(2n,k)\lambda^{2n-2k} - 2\right] + 2(-1)^n,$$

where $T(2n,k) = \binom{2n-k}{k} + \binom{2n-k-1}{k-1}$, $\quad k \leq n$ *and* $T(2n,0) = 1$.

Proof. The subdivision graph of C_n is again a cycle graph C_{2n}. By using the well known results and the Theorem 2.1, we can obtain the characteristic polynomial of $S(C_n)$. □

Theorem 2.4. *The characteristic polynomial of the subdivision graph of S_n is*

$$Pol(S(S_n)) \quad = \quad -\lambda\left(\lambda^2-1\right)^{n-2}\left(\lambda^2-n\right).$$

Proof. The adjacency matrix of $S(S_n)$ is

$$A = \begin{bmatrix} 0 & 0 & \dots & 0 & 1 & 0 & 0 & \dots & 0 & 0 \\ 0 & 0 & \dots & 0 & 0 & 1 & 0 & \dots & 0 & 0 \\ 0 & 0 & \dots & 0 & 0 & 0 & 1 & \dots & 0 & 0 \\ \vdots & \vdots & \dots & \vdots & \vdots & \vdots & \vdots & \dots & \vdots & \vdots \\ 0 & 0 & \dots & \dots & 0 & 0 & 0 & \dots & 1 & 0 \\ 1 & 0 & \dots & \dots & 0 & 0 & 0 & \dots & 0 & 1 \\ 0 & 1 & \dots & \dots & 0 & 0 & 0 & \dots & 0 & 1 \\ 0 & 0 & 1 & \dots & 0 & 0 & 0 & \dots & 0 & 1 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \dots & \vdots & \vdots \\ 0 & 0 & 0 & \dots & 1 & 0 & 0 & \vdots & 0 & 1 \\ 0 & 0 & 0 & \dots & 0 & 1 & 1 & \dots & 1 & 0 \end{bmatrix}_{(2n-1)\times(2n-1)}.$$

In this adjacency matrix, we can easily see that its elements have some special rule in rows and columns. By dividing this adjacency matrix into parts, we rewrite it as follows:

$$A = \begin{bmatrix} 0_{(n-1)\times(n-1)} & I_{n-1} & 0_{(n-1)\times 1} \\ I_{n-1} & 0_{(n-1)\times(n-1)} & 1_{(n-1)\times 1} \\ 0_{(n-1)\times 1} & 1_{1\times(n-1)} & 0_{1\times 1} \end{bmatrix}_{(2n-1)\times(2n-1)}$$

where $0_{(n-1)\times(n-1)}$, $0_{(n-1)\times 1}$ and $0_{1\times 1}$ denote the zero matrices, and the same notations for the element 1. Therefore the characteristic polynomial of $S(S_n)$ is given by

$$Pol(S(S_n)) = \begin{vmatrix} \lambda I_{n-1} & -I_{n-1} & 0_{(n-1)\times 1} \\ -I_{n-1} & \lambda I_{n-1} & -1_{(n-1)\times 1} \\ 0_{1\times(n-1)} & -1_{1\times(n-1)} & \lambda \end{vmatrix}_{(2n-1)\times(2n-1)}.$$

By adding negatives of the first $n-1$ columns to the last column, we get

$$Pol(S(S_n)) = \begin{vmatrix} \lambda I_{n-1} & -I_{n-1} & -\lambda_{(n-1)\times 1} \\ -I_{n-1} & \lambda I_{n-1} & 0_{(n-1)\times 1} \\ 0_{1\times(n-1)} & -1_{1\times(n-1)} & \lambda \end{vmatrix}_{(2n-1)\times(2n-1)}.$$

By adding the last row to the first $n-1$ rows, we get

$$Pol(S(S_n)) = \begin{vmatrix} \lambda I_{n-1} & -I_{n-1} + (-1)_{(n-1)\times(n-1)} & 0_{(n-1)\times 1} \\ -I_{n-1} & \lambda I_{n-1} & 0_{(n-1)\times 1} \\ 0_{1\times(n-1)} & -1_{1\times(n-1)} & \lambda \end{vmatrix}_{(2n-1)\times(2n-1)}.$$

Then by using the elementary column and row operations, we change the columns and rows with each other and then expanding this determinant according to the first column, we get

$$Pol(S(S_n)) = -\lambda \begin{vmatrix} \lambda I_{n-1} & -I_{n-1} + (-1)_{(n-1)\times(n-1)} \\ -I_{n-1} & \lambda I_{n-1} \end{vmatrix}_{(2n-2)\times(2n-2)}.$$

By using the elementary row operations $\frac{1}{\lambda}R_1 + R_n \longrightarrow R_n$, $\frac{1}{\lambda}R_2 + R_{n+1} \longrightarrow R_{n+1}$, $\frac{1}{\lambda}R_3 + R_{n+2} \longrightarrow +R_{n+2}$, $\dots \frac{1}{\lambda}R_{n-1} + R_{2n-2} \longrightarrow R_{2n-2}$, we get

$$\begin{aligned} Pol(S(S_n)) &= -\lambda \begin{vmatrix} \lambda I_{n-1} & -I_{n-1} + (-1)_{(n-1)\times(n-1)} \\ 0_{(n-1)\times(n-1)} & (\frac{\lambda^2-1}{\lambda})I_{n-1} + (\frac{-1}{\lambda})_{(n-1)\times(n-1)} \end{vmatrix}_{(2n-2)\times(2n-2)} \\ &= -\lambda \left|\lambda I_{n-1}\right| \left|(\frac{\lambda^2-1}{\lambda})I_{n-1} + (\frac{-1}{\lambda})_{(n-1)\times(n-1)}\right|_{(n-1)\times(n-1)} \\ &= -\lambda^n \left|(\frac{\lambda^2-1}{\lambda})I_{n-1} + (\frac{-1}{\lambda})_{(n-1)\times(n-1)}\right|_{(n-1)\times(n-1)}. \end{aligned}$$

In the last determinant, by using elementary column operations, $C_2 + C_3 + \cdots + C_{n-1} \to C_1$, we get

$$\begin{aligned} Pol(S(S_n)) &= -\lambda^n \begin{vmatrix} \lambda - \frac{n}{\lambda} & -\frac{1}{\lambda} & -\frac{1}{\lambda} & \dots & -\frac{1}{\lambda} \\ \lambda - \frac{n}{\lambda} & \lambda - \frac{2}{\lambda} & -\frac{1}{\lambda} & \dots & -\frac{1}{\lambda} \\ \lambda - \frac{n}{\lambda} & -\frac{1}{\lambda} & \lambda - \frac{2}{\lambda} & \dots & -\frac{1}{\lambda} \\ \vdots & \vdots & -\frac{1}{\lambda} & \ddots & \vdots \\ \vdots & \vdots & \vdots & \dots & \vdots \\ \lambda - \frac{n}{\lambda} & -\frac{1}{\lambda} & -\frac{1}{\lambda} & \dots & \lambda - \frac{2}{\lambda} \end{vmatrix}_{(n-1)\times(n-1)} \\ &= -\lambda^n(\lambda - \frac{n}{\lambda}) \begin{vmatrix} 1 & -\frac{1}{\lambda} & -\frac{1}{\lambda} & \dots & -\frac{1}{\lambda} \\ 1 & \lambda - \frac{2}{\lambda} & -\frac{1}{\lambda} & \dots & -\frac{1}{\lambda} \\ 1 & -\frac{1}{\lambda} & \lambda - \frac{2}{\lambda} & \dots & -\frac{1}{\lambda} \\ \vdots & \vdots & -\frac{1}{\lambda} & \ddots & \vdots \\ \vdots & \vdots & \vdots & \dots & \vdots \\ 1 & -\frac{1}{\lambda} & -\frac{1}{\lambda} & \dots & \lambda - \frac{2}{\lambda} \end{vmatrix}_{(n-1)\times(n-1)}. \end{aligned}$$

By adding -1 times first row to the second row, to the third row, $\cdots$, to the $(n-1)$-th row, we get

$$Pol(S(S_n)) = -\lambda^n(\lambda - \frac{n}{\lambda}) \begin{vmatrix} 1 & -\frac{1}{\lambda} & -\frac{1}{\lambda} & \dots & -\frac{1}{\lambda} \\ 0 & \lambda - \frac{1}{\lambda} & 0 & \dots & 0 \\ 0 & 0 & \lambda - \frac{1}{\lambda} & \dots & 0 \\ \vdots & \vdots & 0 & \ddots & \vdots \\ \vdots & \vdots & \vdots & \dots & \vdots \\ 0 & 0 & 0 & \dots & \lambda - \frac{1}{\lambda} \end{vmatrix}_{(n-1)\times(n-1)}.$$

Expanding the determinant according to first column

$$Pol(S(S_n)) = -\lambda^n(\lambda - \frac{n}{\lambda}) \begin{vmatrix} \lambda - \frac{1}{\lambda} & 0 & \dots & 0 \\ 0 & \lambda - \frac{1}{\lambda} & \dots & 0 \\ \vdots & 0 & \ddots & \vdots \\ \vdots & \vdots & \dots & \vdots \\ 0 & 0 & \dots & \lambda - \frac{1}{\lambda} \end{vmatrix}_{(n-2)\times(n-2)}.$$

Finally we obtain the following equation

$$\begin{aligned} Pol(S(S_n)) &= -\lambda^n(\lambda - \tfrac{n}{\lambda})(\lambda - \tfrac{1}{\lambda})^{n-2} \\ &= -\lambda\left(\lambda^2-1\right)^{n-2}\left(\lambda^2-n\right). \end{aligned}$$

□

According to Theorem 2.4, the spectrum of $S(S_n)$ is

$$\left\{0, \mp\sqrt{n}, \mp 1^{(n-2)}\right\}.$$

Using similar calculations we obtain the following results:

Theorem 2.5. *The characteristic polynomial of subdivision graph of K_n is*

$$Pol(S(K_n)) = (-1)^n \lambda^{\frac{n(n-3)}{2}} \left(\lambda^2-(n-2)\right)^{n-1}\left(\lambda^2-2(n-1)\right) \textit{ for } n \geq 1.$$

According to Theorem 2.5, the spectrum of $S(K_n)$ is

$$\left\{0^{\left(\frac{n(n-3)}{2}\right)}, \mp\sqrt{n-2}^{(n-1)}, \mp\sqrt{2(n-1)}\right\}.$$

Proceeding similary for $K_{m,n}$, we obtained following results:

Theorem 2.6. *The characteristic polynomial of subdivision graph of $K_{m,n}$ can be given by the following formula:*

$$Pol(S(K_{m,n})) = (-1)^n \lambda^{m+(m-1)(n-2)}\left(\lambda^2-(m+n)\right)\left(\lambda^2-m\right)^{n-1}\left(\lambda^2-n\right)^{m-1}.$$

By Theorem 2.6, the spectrum of $S(K_{m,n})$ is

$$\left\{0^{m+(m-1)(n-2)}, \mp\sqrt{m+n}, \mp\sqrt{m}^{(n-1)}, \mp\sqrt{n}^{(m-1)}\right\}.$$

2.3. **Reccurence relations for the characteristic polynomials of subdivision graphs.**

Theorem 2.7. *The recurrence relation for the characteristic polynomial of the subdivision graph of a star graph for $n \geq 4$ is*

$$\left(\lambda^2-n\right) Pol((S(S_{n+1})) = (\lambda^2-1)(\lambda^2-n-1)Pol(S(S_n)).$$

Proof. According to Theorem 2.4, we have $Pol(S(S_n)) = -\lambda\left(\lambda^2-1\right)^{n-2}\left(\lambda^2-n\right)$ and $Pol(S(S_{n+1})) = -\lambda\left(\lambda^2-1\right)^{n-1}\left(\lambda^2-n-1\right)$. Hence

$$\begin{aligned} Pol(S(S_{n+1})) &= -\lambda\left(\lambda^2-1\right)^{n-1}\left(\lambda^2-n-1\right) \\ &= -\lambda\left(\lambda^2-1\right)^{n-2}\left(\lambda^2-1\right)\left(\lambda^2-n-1\right) \\ &= -\lambda\left(\lambda^2-1\right)^{n-2}\left(\lambda^2-1\right)\left(\lambda^2-n-1\right)\frac{\left(\lambda^2-n\right)}{\left(\lambda^2-n\right)} \\ &= \left(\lambda^2-1\right)\frac{\left(\lambda^2-n-1\right)}{\left(\lambda^2-n\right)} Pol(S(S_n)). \end{aligned}$$

Finally we obtain the required result for $n \geq 4$ as follows:

$$\left(\lambda^2-n\right) Pol((S(S_{n+1})) = (\lambda^2-1)(\lambda^2-n-1)Pol(S(S_n)).$$

□

Theorem 2.8. *The recurrence relation for the characteristic polynomial of the subdivision graph for* $n \geq 1$ *of a complete graph is*

$$Pol(S(K_{n+1})) = -\lambda^{n-1}\frac{(\lambda^2-n+1)^n(\lambda^2-2n)}{(\lambda^2-n+2)^{n-1}(\lambda^2-2n+2)}Pol(S(K_n)).$$

Proof. According to Theorem 2.5, we have
$Pol(S(K_{n+1})) = (-1)^{n+1}\lambda^{\frac{(n+1)(n-2)}{2}}\left(\lambda^2-n+1\right)^n\left(\lambda^2-2n\right)$ for $n \geq 0$ and
$Pol(S(K_n)) = (-1)^n\lambda^{\frac{n(n-3)}{2}}\left(\lambda^2-(n-2)\right)^{n-1}\left(\lambda^2-2(n-1)\right)$ for $n \geq 1$.
By using the similar steps with the proof of Theorem 2.7, we obtain the following equation for $n \geq 1$:

$$Pol(S(K_{n+1})) = -\lambda^{n-1}\frac{(\lambda^2-n+1)^n(\lambda^2-2n)}{(\lambda^2-n+2)^{n-1}(\lambda^2-2n+2)}Pol(S(K_n)).$$

□

Finally, proceeding similarly, we obtain the recurrence relation for the characteristic polynomial of $S(K_{m,n})$ as follows:

Theorem 2.9. *The recurrence relation for the characteristic polynomial of the subdivision graph of a complete bipartite graph is*

$$Pol(S(K_{m+1,n})) = \frac{\lambda^{n-1}(\lambda^2-m-n-1)(\lambda^2-m-1)^{n-1}(\lambda^2-n)}{(\lambda^2-m-n)(\lambda^2-m)}Pol(S(K_{m,n})).$$

2.4. **The characteristic polynomials of the $r-$subdivision graphs $S^r(P_n)$ and $S^r(C_n)$.** Let $G=(V,E)$ be a graph and u and v be two vertices of it. The $r-$subdivision graph of G is a graph obtained by inserting r new vertices into every edge, that is, by replacing each edge with a path P_{r+2}. It was defined and some properties of it were studied in [2, 16, 17]. In this paper, the $r-$subdivision graph of G is denoted by $S^r(G)$.

Theorem 2.10. *The characteristic polynomial of* $S^r(P_n)$ *is*

$$Pol(S^r(P_n)) = Pol(P_{n(r+1)-r}) \quad with \quad |V|S^r(P_n)|| = n(r+1)-r.$$

Theorem 2.11. *The characteristic polynomial of* $S^r(C_n)$ *is*

$$Pol(S^r(C_n)) = Pol(C_{n(r+1)}) \quad with \quad |V(S^r(C_n))| = n(r+1).$$

2.5. **The energy of the subdivision graphs.** The sum of the absolute values of the eigenvalues of G was studied for the first time by E. Hückel in [12] and is called the energy of G, which is an important aspect for the subfield of graph theory called spectral graph theory, and also for the molecular calculations, see [1], [8].

The energy of some special graph types are well-known in literature, see [5, 6, 13, 4]. We now recall them:

$$E(G) = \begin{cases} 2\sqrt{n-1} & ifG = S_n, \\ 2(n-1) & ifG = K_n, \\ 2(mn) & ifG = K_{m,n} \end{cases}.$$

In this section, we give the energy of the subdivision graphs of star, complete and complete bipartite graphs. Firstly, we obtain the energy of the $S(S_n)$ as follows:

Theorem 2.12. *Let $E(S(S_n))$ denote the energy of $S(S_n)$. Then*

$$E(S(S_n)) = 2(n-2+\sqrt{n}).$$

Proof. By Theorem 2.4, we know the spectrum of $S(S_n)$. From the definition of energy, the sum of the absolute values of the eigenvalues is

$$E(S(S_n)) = 2(n-2+\sqrt{n}).$$

□

Secondly, we obtain the energy of the $S(K_n)$ as follows:

Theorem 2.13. *Let $E(S(K_n))$ denote the energy of $S(K_n)$. Then*

$$E(S(K_n)) = 2((n-1)\sqrt{n-2}+\sqrt{2(n-1)}).$$

Proof. By Theorem 2.5, the sum of the absolute values of the eigenvalues of $Pol(S(K_n))$ is

$$E(S(K_n)) = 2((n-1)\sqrt{n-2}+\sqrt{2(n-1)}).$$

□

Corollary 2.14. *The relation between the energies of $S(K_n)$ and K_n is given by*

$$E(S(K_n)) \quad = \quad E(K_n)\sqrt{n-2}+2\sqrt{E(K_n)}.$$

Theorem 2.15. *The energy of the $S(K_{m,n})$ is given by*

$$E(K_{m,n}) = 2(\sqrt{m+n}+(n-1)\sqrt{m}+(m-1)\sqrt{n}).$$

Proof. By Theorem 2.6, the proof is clear. □

3. Discussion

Spectral graph theory is one of the main branches of graph theory dealing with spectral study of graphs. Linear algebraic methods are used to obtain the spectrum of a given graph, which consists of the eigenvalues of the characteristic polynomial corresponding to the adjacency matrix. The notion of graph energy was defined by Gutman in [9] and employed in determining many properties of molecular graphs. A large number of research has been done on spectral graph theory. In this paper, we gave the spectral study of subdivision graphs of graphs. Also using the recently defined generalization of the subdivision graph called the r-subdivision graph in [16] and [17], we calculated the spectrum of the r-subdivision graphs of several well-known graph classes.

4. Conclusion

In this work, a spectral study of the subdivision and r-subdivision graphs is realized. By means of the formulae and recurrence relations obtained here, it is possible to study the spectra, characteristic polynomials and energy of some large graphs by means of the smaller ones. These calculations can be extended to other derived graphs such as the line, total, central, middle, Mycielskian, etc.

The method introduced here can be applied to the derived graphs such as the line, total, subdivision, middle graphs etc. and also to graph operations.

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