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INVERSE SUM INDEG ENERGY OF A GRAPH

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Abstract. Let G be a graph with n vertices and let d_i denote the degree of the vertex v_i. For a given graph, there are more than 100 matrices obtained by using some properties of the graph. Most important and used ones are the adjacency, incidency and Laplacian matrices. Recently, several graph topological indices have been used in defining new graph matrices. Graph energy is the quantity obtained as the sum of the absolute values of all eigenvalues of the adjacency matrix corresponding to the graph. Several types of energy have been defined and applied in different applications by means of such graph matrices in place of the adjacency matrix.

The inverse sum indeg matrix of a graph G is the $n \times n$ matrix whose (i, j)-th entry is equal to $\frac{d_i d_j}{di+dj}$ if the i^{th} and the j^{th} vertices are adjacent and 0 otherwise. The inverse sum indeg energy $ISIE(G)$ of G is similarly defined as the sum of the absolute values of the eigenvalues of the inverse sum indeg matrix. In this paper, we compute the inverse sum indeg characteristic polynomial and the inverse sum indeg energy for standard graphs. Some properties and bounds for $ISIE(G)$ are also obtained.

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1. Introduction

Let G be a simple graph, that is a graph without loops and multiple edges, and let $\{v_1, v_2, \cdots, v_n\}$ be the set of its vertices. If $e = v_i v_j$ is an edge of G, then the vertices v_i and v_j of G are called adjacent where $i, j = 1, 2, \cdots, n$. This is denoted by the notation $v_i \sim v_j$. In such a case, the edge e is said to be incident to v_i and v_j. For $v_i \in V(G)$, the degree of the vertex v_i is denoted by d_i and is defined as the total number of the edges incident to v_i.

We can model a real life example by a graph. Such a graph is constructed by modelling the items by vertices and the links between the items by edges. Topological graph indices are very important tools in graph theory. Such an index gives us a mathematical value which can be commented by looking at other values of it as the quantity that helps to comment about the properties of given example. As a popular example, we can give molecular graphs. In such a graph, atoms in the molecule are represented by vertices and the chemical bonds between the atoms are represented by the edges of the graph. There is a large number of topological indices with several applications. The inverse sum indeg index is one of those indices and is given by

$$ISI(G) = \sum_{v_i \sim v_j} \frac{d_i d_j}{d_i + d_j},$$

[10]. This topological index is a significant predictor of total surface area of octane isomers.

Motivated by the work on classical energy which is defined by means of the adjacency matrix, several other graph matrices have been defined and studied, see e.g. [4, 5, 6, 7]. Here we are introducing the inverse sum indeg matrix $ISI(G)$ which is symmetric. The inverse sum indeg matrix $ISI(G) = (I_{ij})_{n \times n}$ is defined as

$$I_{ij} = \begin{cases} \frac{d_i d_j}{d_i + d_j} & \text{if } v_i \sim v_j, \\ 0 & otherwise. \end{cases}$$

2. The inverse sum indeg energy of graph

Let G be a simple, finite, undirected graph. The energy $E(G)$ is defined as the sum of the absolute values of the eigenvalues of its adjacency matrix. For more details on energy of a graph, see [2, 3].

Let G be a simple graph of order n with vertex set $V = \{v_1, v_2, \cdots, v_n\}$ and edge set E. Let $ISI(G)$ be the inverse sum indeg matrix, see [9]. The characteristic polynomial of $ISI(G)$ is denoted by $\phi_{ISI}(G, \lambda) = det(\lambda I - ISI(G))$. Since the inverse sum indeg matrix is real and symmetric, its eigenvalues are real numbers and we label them in non-increasing order $\lambda_1 > \lambda_2 > \cdots > \lambda_n$. The inverse sum indeg energy is given by

$$ISIE(G) = \sum_{i=1}^{n} |\lambda_i|. \tag{1}$$

This paper is organized as follows: In Section 3, we first obtain some basic properties of the inverse sum indeg energy of a graph. In Section 4, the inverse sum indeg energy of some standard graphs are calculated. In Section 5, we find the inverse sum indeg energy of some specific graphs with one edge deleted. In Section 6, we find the inverse sum indeg energy of some complements of graphs. The energy of cubic graphs of order 10 are discussed in Section 7.

3. Some basic properties of inverse sum indeg energy of a graph

Let d_i and d_j be the degrees of two adjacent vertices. We need to define a number P as

$$P = \sum_{i<j} \left(\frac{d_i d_j}{d_i + d_j} \right)^2.$$

Proposition 3.1. *The first three coefficients of $\phi_{ISI}(G, \lambda)$ are given as follows:*
(i) $a_0 = 1$,

(ii) $a_1 = 0$,
(iii) $a_2 = -P$.

Proof. (i) From the definition, we have $\Phi_{ISI}(G, \lambda) = det[\lambda I - ISI(G)]$, so we get $a_0 = 1$.

(ii) The sum of the determinants of all 1×1 principal submatrices of $ISI(G)$ is equal to the trace of $ISI(G)$. Therefore $a_1 = (-1)^1$ ·the trace of $[ISI(G)] = 0$.

(iii) Similarly, the sum of all 2×2 determinants give

$$\begin{aligned} (-1)^2 a_2 &= \sum_{1 \le i < j \le n} \begin{vmatrix} a_{ii} & a_{ij} \\ a_{ji} & a_{jj} \end{vmatrix} \\ &= \sum_{1 \le i < j \le n} a_{ii}a_{jj} - a_{ji}a_{ij} \\ &= \sum_{1 \le i < j \le n} a_{ii}a_{jj} - \sum_{1 \le i < j \le n} a_{ji}a_{ij} \\ &= -P. \end{aligned}$$

□

Proposition 3.2. *If $\lambda_1, \lambda_2, \cdots, \lambda_n$ are the inverse sum indeg eigenvalues of $ISI(G)$, then*

$$\sum_{i=1}^{n} {\lambda_i}^2 = 2P.$$

Proof. We know that

$$\begin{aligned} \sum_{i=1}^{n} \lambda_i^2 &= \sum_{i=1}^{n}\sum_{j=1}^{n} a_{ij}a_{ji} \\ &= 2\sum_{i<j}(a_{ij})^2 + \sum_{i=1}^{n}(a_{ii})^2 \\ &= 2\sum_{i<j}(a_{ij})^2 \\ &= 2P. \end{aligned}$$

□

Theorem 3.3. *Let G be a graph with n vertices. Then*

$$ISIE(G) \le \sqrt{2nP}.$$

Proof. Let $\lambda_1, \lambda_2, \cdots, \lambda_n$ be the eigenvalues of $ISI(G)$. By the Cauchy-Schwartz inequality, we have

$$\left(\sum_{i=1}^{n} a_i b_i\right)^2 \le \left(\sum_{i=1}^{n} {a_i}^2\right)\left(\sum_{i=1}^{n} {b_i}^2\right).$$

Let $a_i = 1$, $b_i =\mid \lambda_i \mid$. Then

$$\left(\sum_{i=1}^{n} |\lambda_i|\right)^2 \le \left(\sum_{i=1}^{n} 1\right)\left(\sum_{i=1}^{n} |\lambda_i|^2\right)$$

implying that

$$[ISIE]^2 \leq n \cdot 2P$$

and hence we get

$$[ISIE] \leq \sqrt{2nP}$$

as an upper bound. □

Theorem 3.4. *Let G be a graph with n vertices. If $R = det\ ISI(G)$, then*

$$ISIE(G) \geq \sqrt{2P + n(n-1)R^{\frac{2}{n}}}.$$

Proof. By definition, we write

$$\begin{aligned}(ISIE(G))^2 &= \left(\sum_{i=1}^{n} \mid \lambda_i \mid\right)^2 \\ &= \sum_{i=1}^{n} \mid \lambda_i \mid \sum_{j=1}^{n} \mid \lambda_j \mid \\ &= \sum_{i=1}^{n} \mid \lambda_i \mid^2 + \sum_{i \neq j} \mid \lambda_i \mid\mid \lambda_j \mid .\end{aligned}$$

Using the arithmetic-geometric mean inequality, we have

$$\frac{1}{n(n-1)} \sum_{i \neq j} \mid \lambda_i \mid\mid \lambda_j \mid \quad \geq \quad \left(\prod_{i \neq j} \mid \lambda_i \mid\mid \lambda_j \mid\right)^{\frac{1}{n(n-1)}}.$$

Therefore,

$$\begin{aligned}[ISI(G)]^2 &\geq \sum_{i=1}^{n} \mid \lambda_i \mid^2 + n(n-1) \left(\prod_{i \neq j} \mid \lambda_i \mid\mid \lambda_j \mid\right)^{\frac{1}{n(n-1)}} \\ &\geq \sum_{i=1}^{n} \mid \lambda_i \mid^2 + n(n-1) \left(\prod_{i=1}^{n} \mid \lambda_i \mid^{2(n-1)}\right)^{\frac{1}{n(n-1)}} \\ &= \sum_{i=1}^{n} \mid \lambda_i \mid^2 + n(n-1)R^{\frac{2}{n}} \\ &= 2P + n(n-1)R^{\frac{2}{n}}.\end{aligned}$$

Thus,

$$ISIE(G) \geq \sqrt{2P + n(n-1)R^{\frac{2}{n}}}.$$

□

Let λ_n and λ_1 be the minimum and maximum values of all $\lambda_i' s$, respectively. Then the following results can easily be proven by means of the above results:

Theorem 3.5. *For a graph G of order n,*

$$ISIE(G) \geq \sqrt{2Pn - \frac{n^2}{4}(\lambda_1 - \lambda_n)^2}.$$

Theorem 3.6. *For a graph G of order n with non-zero eigenvalues, we have*

$$ISIE(G) \geq \frac{2\sqrt{2nP\lambda_1\lambda_n}}{(\lambda_1+\lambda_n)^2}.$$

Theorem 3.7. *Let G be a graph of order n. Let $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_n$ be the eigenvalues of G in increasing order. Then*

$$ISIE(G) \geq \frac{|\lambda_1||\lambda_n|n + 2P}{|\lambda_1| + |\lambda_n|}.$$

4. Inverse sum indeg energy of some standard graphs

In this section, we investigate the inverse sum indeg energy of some frequently used graph classes:

Theorem 4.1. *The inverse sum indeg energy of a complete graph K_n is*

$$ISI(K_n) = (n-1)^2.$$

Proof. Let K_n be the complete graph with vertex set $V = \{v_1, v_2, \cdots, v_n\}$. The inverse sum indeg matrix of this graph is

$$ISI(K_n) = \ \tfrac{n-1}{2}\,(J-I)$$

where J is a matrix with all entries equal to 1. In this case, the characteristic equation is

$$\left(\lambda + \frac{n-1}{2}\right)^{n-1}\left(\lambda - \frac{(n-1)^2}{2}\right) = 0$$

and the spectrum is $Spec_{ISI}(K_n) = \begin{pmatrix} \frac{n-1}{2} & \frac{(n-1)^2}{2} \\ n-1 & 1 \end{pmatrix}$. Therefore, $ISI(K_n) = (n-1)^2$. □

Theorem 4.2. *The inverse sum indeg energy of the star graph $K_{1,n-1}$ is*

$$ISIE(K_{1,n-1}) = 2\frac{(n-1)\sqrt{n-1}}{n}.$$

Proof. Let $K_{1,n-1}$ be the star graph with vertex set $V = \{v_0, v_1, \cdots, v_{n-1}\}$. The inverse sum indeg matrix is

$$ISI(K_{1,n-1}) = \frac{n-1}{n}\begin{pmatrix} 0_{1\times 1} & J_{1\times(n-1)} \\ J_{(n-1)\times 1} & 0_{(n-1)\times(n-1)} \end{pmatrix}.$$

Then the characteristic equation would be

$$\lambda^{n-2}\left(\lambda - \frac{(n-1)\sqrt{n-1}}{n}\right)\left(\lambda + \frac{(n-1)\sqrt{n-1}}{n}\right) = 0$$

and the spectrum would be $Spec_{ISI}(K_{1,n-1}) = \begin{pmatrix} \frac{(n-1)\sqrt{n-1}}{n} & 0 & -\frac{(n-1)\sqrt{n-1}}{n} \\ 1 & n-2 & 1 \end{pmatrix}$.

Therefore $ISIE(K_{1,n-1}) = 2\frac{(n-1)\sqrt{n-1}}{n}$. □

Theorem 4.3. *The inverse sum indeg energy of a crown graph S_n^0 is*

$$ISIE(S_n^0) = 2(n-1)^2.$$

Proof. Let S_n^0 be a crown graph of order $2n$ with vertex set $\{u_1, u_2, \cdots, u_n, v_1, v_2, \cdots, v_n\}$. The inverse sum indeg matrix is

$$ISIE(S_n^0) = \frac{n-1}{2}\begin{pmatrix} 0_{n\times n} & J_{n\times n} \\ J_{n\times n} & 0_{n\times n} \end{pmatrix}.$$

Hence the characteristic equation will be obtained as

$$\left(\lambda - \frac{n-1}{2}\right)^{n-1}\left(\lambda + \frac{n-1}{2}\right)^{n-1}\left(\lambda + \frac{(n-1)^2}{2}\right)\left(\lambda - \frac{(n-1)^2}{2}\right) = 0$$

giving the spectrum as

$$Spec_{ISI}(S_n^0) = \begin{pmatrix} \frac{(n-1)^2}{2} & -\frac{(n-1)^2}{2} & \frac{n-1}{2} & -\frac{n-1}{2} \\ 1 & 1 & n-1 & n-1 \end{pmatrix}.$$

Therefore

$$ISIE(S_n^0) = 2(n-1)^2.$$

□

Theorem 4.4. *The inverse sum indeg energy of a cocktail party graph $K_{n\times 2}$ is*

$$ISIE(K_{n\times 2}) = 4(n-1)^2.$$

Proof. Let $K_{n\times 2}$ be a cocktail party graph of order $2n$ with vertex set $\{u_1, u_2, \cdots, u_n, v_1, v_2, \cdots, v_n\}$. The inverse sum indeg matrix is

$$ISIE(K_{n\times 2}) = (n-1)\begin{pmatrix} (J-I)_{n\times n} & (J-I)_{n\times n} \\ (J-I)_{n\times n} & (J-I)_{n\times n} \end{pmatrix}.$$

Then the characteristic equation is obtained as

$$\lambda^n(\lambda + 2n - 2)^{n-1}(\lambda + (2(n-1)^2)) = 0.$$

Hence, the spectrum is

$$Spec_{ISI}(K_{n\times 2}) = \begin{pmatrix} -2(n-1)^2 & 0 & -(2n-2) \\ 1 & n & n-1 \end{pmatrix}.$$

Therefore we find

$$ISIE(K_{n\times 2}) = 4(n-1)^2.$$

□

Definition 4.1. *The friendship graph denoted by F_n^3 is the graph obtained by taking n copies of the cycle graph C_3 with a vertex in common.*

Note that $|V(F_n^3)| = 2n + 1$.

Theorem 4.5. *The inverse sum indeg energy of a friendship graph F_n^3 is*

$$ISIE(F_n^3) = 2n - 1 + \frac{\sqrt{32n^3 + n^2 + 2n + 1}}{n+1}.$$

Proof. Let F_n^3 be the friendship graph with $2n+1$ vertices. The inverse sum indeg matrix is

$$\begin{bmatrix} 0 & \frac{2n}{\sqrt{n+1}} & \frac{2n}{\sqrt{n+1}} & \frac{2n}{\sqrt{n+1}} & \frac{2n}{\sqrt{n+1}} & \cdots & \frac{2n}{\sqrt{n+1}} & \frac{2n}{\sqrt{n+1}} \\ \frac{2n}{\sqrt{n+1}} & 0 & 1 & 0 & 0 & \dots & 0 & 0 \\ \frac{2n}{\sqrt{n+1}} & 1 & 0 & 0 & 0 & \dots & 0 & 0 \\ \frac{2n}{\sqrt{n+1}} & 0 & 0 & 0 & 1 & \dots & 0 & 0 \\ \frac{2n}{\sqrt{n+1}} & 0 & 0 & 1 & 0 & \dots & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ \frac{2n}{\sqrt{n+1}} & 0 & 0 & 0 & 0 & \dots & 0 & 1 \\ \frac{2n}{\sqrt{n+1}} & 0 & 0 & 0 & 0 & \dots & 1 & 0 \end{bmatrix}.$$

Then the characteristic equation is

$$\left(\lambda^2 - \lambda - \frac{8n^3}{(n+1)^2}\right)(\lambda - 1)^{n-1}(\lambda + 1)^n = 0.$$

Hence the spectrum is

$$Spec_{ISI}(F_n^3) = \begin{pmatrix} -1 & 1 & \frac{n+1+\sqrt{32n^3+n^2+2n+1}}{2(n+1)} & \frac{n+1+\sqrt{32n^3+n^2+2n+1}}{2(n+1)} \\ n & n-1 & 1 & 1 \end{pmatrix}.$$

Therefore, $ISIE(F_n^3) = 2n - 1 + \frac{\sqrt{32n^3+n^2+2n+1}}{n+1}$. □

Theorem 4.6. *The inverse sum indeg energy of a cycle graph C_n is*

$$ISIE(C_n) = \begin{cases} 2cosec\frac{\pi}{2n} & \text{if } n \equiv 1 \pmod 2, \\ 4cosec\frac{\pi}{n} & \text{if } n \equiv 2 \pmod 4, \\ 4cot\frac{\pi}{n} & \text{if } n \equiv 0 \pmod 4. \end{cases}$$

Proof. The inverse sum indeg matrix is

$$\begin{bmatrix} 0 & 1 & 0 & 0 & 0 & \dots & 0 & 1 \\ 1 & 0 & 1 & 0 & 0 & \dots & 0 & 0 \\ 0 & 1 & 0 & 1 & 0 & \dots & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 1 & 0 & 0 & 0 & 0 & \dots & 1 & 0 \end{bmatrix}.$$

This is a circulant matrix of order n which is similar to the adjacency matrix of the cycle graph. Thus

$$ISIE(C_n) = \begin{cases} 2cosec\frac{\pi}{2n} & \text{if } n \equiv 1 \pmod 2, \\ 4cosec\frac{\pi}{n} & \text{if } n \equiv 2 \pmod 4, \\ 4cot\frac{\pi}{n} & \text{if } n \equiv 0 \pmod 4. \end{cases}$$

□

Definition 4.2. *The double star graph $S_{n,m}$ is the graph constructed from $K_{1,n-1}$ and $K_{1,m-1}$ by joining their centers v_0 and u_0.*

Note that $V(S_{n,m}) = V(K_{1,n-1}) \cup V(K_{1,m-1})$ and $E(S_{n,m}) = \{v_0u_0; v_0v_i; u_0u_j : 1 \le i \le n-1, 1 \le i \le n-1\}$ Therefore, double star graph is a bipartite graph.

Theorem 4.7. *The inverse sum indeg energy of double star graph $S_{n,n}$ is*

$$ISIE(S_{n,n}) = \frac{n\sqrt{n^2+18n-15}}{n+1}.$$

Proof. The inverse sum indeg matrix is

$$ISI(S_{n,n}) = \begin{bmatrix} 0 & \frac{n}{n+1} & \frac{n}{n+1} & \cdots & \frac{n}{n+1} & \frac{n}{2} & 0 & 0 & \cdots & 0 \\ \frac{n}{n+1} & 0 & 0 & \cdots & 0 & 0 & 0 & 0 & \cdots & 0 \\ \frac{n}{n+1} & 0 & 0 & \cdots & 0 & 0 & 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ \frac{n}{n+1} & 0 & 0 & \cdots & 0 & 0 & 0 & 0 & \cdots & 0 \\ \frac{n}{2} & 0 & 0 & \cdots & 0 & 0 & \frac{n}{n+1} & \frac{n}{n+1} & \cdots & \frac{n}{n+1} \\ 0 & 0 & 0 & \cdots & 0 & \frac{n}{n+1} & 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & 0 & \frac{n}{n+1} & 0 & 0 & \cdots & 0 \\ 0 & 0 & 0 & \cdots & 0 & \frac{n}{n+1} & 0 & 0 & \cdots & 0 \end{bmatrix}.$$

Hence the characteristic equation is

$$\lambda^{2n-4}\left(\lambda^2 + \frac{n}{2}\lambda - \frac{n^2(n-1)}{(n+1)^2}\right)\left(\lambda^2 - \frac{n}{2}\lambda - \frac{n^2(n-1)}{(n+1)^2}\right) = 0.$$

Hence the spectrum is

$$Spec_{ISI}(S_{n,n}) = \begin{pmatrix} 0 & \frac{n}{4}+A & \frac{n}{4}-A & \frac{-n}{4}+A & \frac{-n}{4}-A \\ 2n-4 & 1 & 1 & 1 & 1 \end{pmatrix}$$

where $A = \frac{n\sqrt{n^2+18n-15}}{4(n+1)}$. Therefore, $ISIE(S_{n,n}) = \frac{n\sqrt{n^2+18n-15}}{n+1}$. □

Theorem 4.8. *For a complete bipartite graph $K_{m,n}$, the inverse sum indeg energy will be*

$$ISIE(K_{m,n}) = 2\frac{(mn)^{\frac{3}{2}}}{m+n}.$$

Proof. $ISI(K_{m,n}) = \frac{mn}{m+n}\begin{pmatrix} 0_{m\times m} & J_{m\times n} \\ J_{n\times m} & 0_{n\times n} \end{pmatrix}$. Hence the characteristic equation is

$$\lambda^{m+n-2}\left(\lambda + \frac{(mn)^{\frac{3}{2}}}{m+n}\right)\left(\lambda - \frac{(mn)^{\frac{3}{2}}}{m+n}\right) = 0.$$

Hence the spectrum is

$$Spec_{ISI}(K_{m,n}) = \begin{pmatrix} 0 & \frac{-(mn)^{\frac{3}{2}}}{m+n} & \frac{(mn)^{\frac{3}{2}}}{m+n} \\ m+n-2 & 1 & 1 \end{pmatrix}.$$

Therefore, $ISIE(K_{m,n}) = 2\frac{(mn)^{\frac{3}{2}}}{m+n}$. □

5. Inverse sum indeg energy of graphs with one edge deleted

Edge and vertex deletion or addition are very useful tools for graphs when we deal with the large graphs. We can calculate some required property of a large graph in terms of the same property of smaller graphs which are obtained by deleting edges or vertices. In this section, we obtain the inverse sum indeg energy for certain graphs having one edge deleted.

Theorem 5.1. *Let e be an edge of the complete graph K_n. The inverse sum indeg energy of $K_n - e$ is*

$$ISIE(K_n - e) = \frac{n-1}{2}\left(n-3+\frac{\sqrt{4n^4-4n^3-75n^2+222n-175}}{2n-3}\right).$$

Proof. Note that $ISI(K_n - e) = \begin{pmatrix} 0_{2\times 2} & \frac{(n-2)(n-1)}{2n-3}J_{2\times(n-2)} \\ \frac{(n-2)(n-1)}{2n-3}J_{(n-2)\times 2} & \frac{n-1}{2}(J-I)_{(n-2)} \end{pmatrix}$.
Hence the characteristic equation is

$$\lambda(\lambda+\frac{n-1}{2})^{n-3}\left(\lambda^2-\left(\frac{n^2}{2}-2n+1.5\right)\lambda-\frac{2(n-1)^2(n-2)^3}{(2n-3)^2}\right)=0$$

and then the spectrum is

$$Spec_{ISI}(K_n - e) = \begin{pmatrix} \frac{-(n-1)}{2} & \frac{n-1}{4}(n-3+B) & \frac{n-1}{4}(n-3-B) & 0 \\ n-3 & 1 & 1 & 1 \end{pmatrix}$$

where $B = \frac{\sqrt{4n^4-4n^3-75n^2+222n-175}}{2n-3}$. Therefore

$$ISIE(K_n - e) = \frac{n-1}{2}\left(n-3+\frac{\sqrt{4n^4-4n^3-75n^2+222n-175}}{2n-3}\right).$$

□

Theorem 5.2. *Let e be an edge of complete bipartite graph $K_{n,n}$. The inverse sum indeg energy of $K_{n,n} - e$ is*

$$ISIE(K_{n,n} - e) = 2\sqrt{\frac{2n^3+3n^2-4n-1}{2n(2n-1)}}.$$

Proof. Note that we have $ISI(K_{n,n} - e) = \begin{pmatrix} 0_{n\times n} & A \\ A & 0_{n\times n} \end{pmatrix}$ where

$$A = \begin{pmatrix} \frac{1}{\sqrt{2n}}J_{(n-1)\times(n-1)} & \frac{1}{\sqrt{2n-1}}J_{(n-1)\times 1} \\ \frac{1}{\sqrt{2n-1}}J_{1\times(n-1)} & 0_{1\times 1} \end{pmatrix}.$$

Hence the characteristic equation is

$$\lambda^{2n-4}\left(\lambda^2+\frac{n-1}{\sqrt{2n}}\lambda-\frac{n-1}{2n-1}\right)\left(\lambda^2-\frac{n-1}{\sqrt{2n}}\lambda-\frac{n-1}{2n-1}\right)=0.$$

Hence, the spectrum is

$$Spec_{ISI}(K_{n,n} - e) = \begin{pmatrix} \frac{1}{2}(-C+D) & \frac{1}{2}(-C-D) & \frac{1}{2}(C+D) & \frac{1}{2}(C-D) & 0 \\ 1 & 1 & 1 & 1 & 2n-4 \end{pmatrix}$$

where $C = \frac{n-1}{\sqrt{2\sqrt{2n}}}$ and $D = \sqrt{\frac{2n^3+3n^2-4n-1}{2n(2n-1)}}$. Therefore, $ISIE(K_{n,n} - e) = 2\sqrt{\frac{2n^3+3n^2-4n-1}{(2n-1)2n}}$. □

6. Inverse sum indeg energy of the complements

In this section, we study the inverse sum indeg energy of some special complement graphs.

Definition 6.1. [8] *Let G be a graph and $P_k = \{V_1, V_2, \cdots, V_k\}$ be a partition of its vertex set V. Then the k-complement of G is obtained as follows: For all V_i and V_j in P_k for $i \neq j$, remove the edges between V_i and V_j and add the edges between the vertices of V_i and V_j which are not in G and the obtained graph is denoted by $\overline{G_k}$.*

Definition 6.2. [8] *Let G be a graph and $P_k = \{V_1, V_2, \cdots, V_k\}$ be a partition of its vertex set V. Then the $k(i)$-complement of G is obtained as follows: For each set V_r in P_k, remove the edges of G joining the vertices within V_r and add the edges of $\overline{G}$ (complement of G) joining the vertices of V_r, and the obtained graph is denoted by $\overline{G_{k(i)}}$.*

Theorem 6.1. *The inverse sum indeg energy of complement of a complete graph $\overline{K_n}$ is*

$$ISIE(\overline{K_n}) = 0.$$

Proof. Let K_n be the complete graph with vertex set $V = \{v_1, v_2, \cdots, v_n\}$. The inverse sum indeg matrix is

$$ISI(\overline{K_n}) = 0_{n\times n}.$$

The characteristic equation is $\lambda^n = 0$. Therefore, $ISIE(\overline{K_n}) = 0$. □

Theorem 6.2. *The inverse sum indeg energy of the complement $\overline{K_{1,n-1}}$ of the star graph is*

$$ISIE(\overline{K_{1,n-1}}) = (n-2)^2.$$

Proof. Let $\overline{K_{1,n-1}}$ be the complement of the star graph with vertex set $V = \{v_0, v_1, \cdots, v_{n-1}\}$. The inverse sum indeg matrix is

$$ISI(\overline{K_{1,n-1}}) = \begin{pmatrix} 0_{(n-1)\times 1} & 0_{1\times(n-1)} \\ 0 & \frac{n-2}{2}(J-I)_{(n-1)} \end{pmatrix}.$$

The characteristic equation is

$$\lambda\left(\lambda - \frac{n-2}{2}\right)^{n-2}\left(\lambda - \frac{(n-2)^2}{2}\right) = 0.$$

Hence the spectrum is $Spec_{ISI}\overline{K_{1,n-1}} = \begin{pmatrix} \frac{(n-2)^2}{2} & 0 & \frac{n-2}{2} \\ 1 & 1 & n-2 \end{pmatrix}$. Therefore,

$$ISIE(\overline{K_{1,n-1}}) = (n-2)^2.$$

□

Theorem 6.3. *The inverse sum indeg energy of the complement of the cocktail party graph $\overline{K_{n\times 2}}$ of order $2n$ with vertex set $\{u_1, u_2, \cdots, u_n, v_1, v_2, \cdots, v_n\}$ is*

$$ISIE(\overline{K_{n\times 2}}) = n.$$

Proof. Let $\overline{K_{n\times 2}}$ be the cocktail party graph of order $2n$ with vertex set $\{u_1, u_2,$
$\cdots, u_n, v_1, v_2, \cdots, v_n\}$. The inverse sum indeg matrix is

$$ISI(\overline{K_{n\times 2}}) = \begin{pmatrix} 0_{n\times n} & \frac{1}{2}I_{n\times n} \\ \frac{1}{2}I_{n\times n} & 0_{n\times n} \end{pmatrix}.$$

Hence the characteristic equation is

$$(\lambda + \frac{1}{2})^n(\lambda - \frac{1}{2})^n = 0.$$

Hence the spectrum is

$$Spec_{ISI}(K_{n\times 2}) = \begin{pmatrix} \frac{1}{2} & -\frac{1}{2} \\ n & n \end{pmatrix}.$$

Therefore $ISIE(\overline{K_{n\times 2}}) = n.$ □

Theorem 6.4. *The inverse sum indeg energy of the* $2(i)$*-complement of a double star graph* $S_{n,n}$ *is*

$$ISIE(\overline{S_{n,n_{2(i)}}}) = \frac{n-1}{3}\left(6n - 12 + \sqrt{16n-17} + \sqrt{36n^2 - 92n + 65}\right).$$

Proof. The inverse sum indeg matrix for $2(i)$-complement of a double star graph is

$$ISI(\overline{S_{n,n_{2(i)}}}) = \begin{pmatrix} 0_{n\times n} & A \\ A & 0_{n\times n} \end{pmatrix}$$

where

$$A = \begin{pmatrix} 0_{1\times 1} & \frac{2(n-1)}{3}I_{1\times(n-1)} \\ \frac{2(n-1)}{3}I_{(n-1)\times 1} & (n-1)(J-I)_{(n-1)\times(n-1)} \end{pmatrix}.$$

Hence the characteristic equation is

$$(\lambda + n - 1)^{2n-4}\left(\lambda^2 + (n-1)\lambda - \frac{4}{9}(n-1)^3\right)\left(\lambda^2 - (2n^2 - 5n + 3)\lambda - \frac{4}{9}(n-1)^3\right) = 0$$

Then the spectrum is

$$\begin{pmatrix} -n+1 & \frac{n-1}{6}(-3+M) & \frac{n-1}{6}(-3-M) & \frac{n-1}{6}(6n-9+N) & \frac{n-1}{6}(6n-9-N) \\ 2n-4 & 1 & 1 & 1 & 1 \end{pmatrix}$$

where $M = \sqrt{16n-7}$ and $N = \sqrt{36n^2 - 92n + 65}$. Therefore,

$$ISIE(\overline{S_{n,n_{2(i)}}}) = \frac{n-1}{3}\left(6n - 12 + \sqrt{16n-17} + \sqrt{36n^2 - 92n + 65}\right).$$

□

Theorem 6.5. *The inverse sum indeg energy of the* 2*-complement of the cocktail party graph* $K_{n\times 2}$ *is*

$$ISIE(\overline{K_{n\times 2(2)}}) = 2n(n-1).$$

Proof. Consider the 2-complement $\overline{K_{n\times 2(2)}}$ of the cocktail party graph. The corresponding matrix is

$$ISI(\overline{K_{n\times 2(2)}}) = (n-1)\begin{pmatrix} (J-I)_{n\times n} & (J-I)_{n\times n} \\ (J-I)_{n\times n} & (J-I)_{n\times n} \end{pmatrix}.$$

Hence the characteristic polynomial is

$$\lambda^{n-1}\left(\lambda+\frac{2}{\sqrt{2n}}\right)^{n-1}\left(\lambda-\frac{n-2}{\sqrt{2n}}\right)\left(\lambda-\frac{n}{\sqrt{2n}}\right)=0$$

implying that the inverse sum indeg spectrum is

$$Spec(\overline{K_{n\times 2(2)}})=\begin{pmatrix} 0 & \frac{n^2}{2} & \frac{n(n-2)}{2} & -n \\ n-1 & 1 & 1 & n-1 \end{pmatrix}.$$

As the result, the inverse sum indeg energy is

$$ISIE(\overline{K_{n\times 2(2)}})=2n(n-1).$$

□

7. Inverse sum indeg energy of cubic graphs of order 10

There are 21 cubic graphs of order 10. They are represented in the following figure:

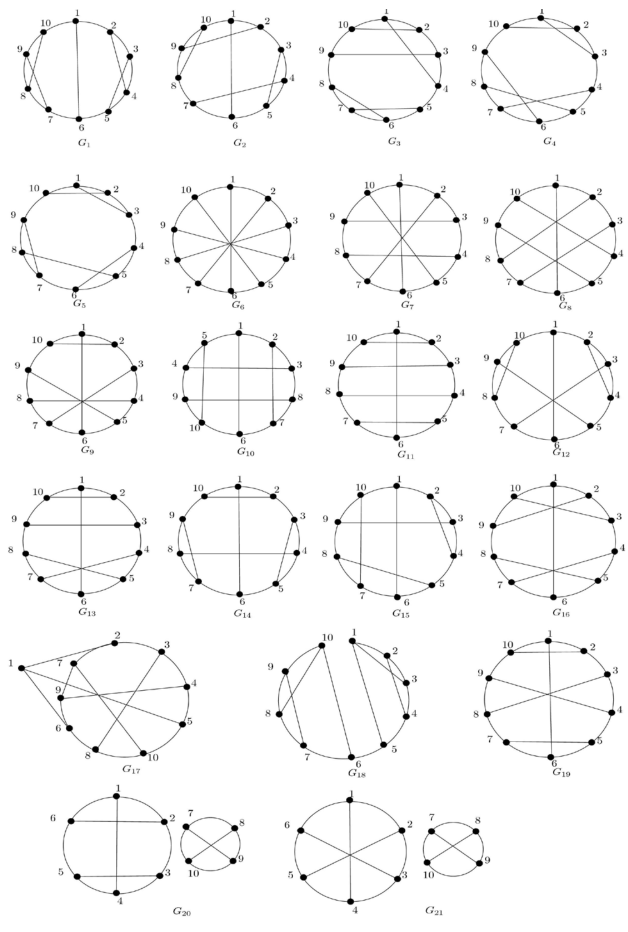

Figure 1. Cubic graphs of order 10

The eigenvalues and the inverse sum indeg energy of cubic graphs of order 10 are given in the following table:

graph	eigenvalues	ISI Energy
G_1	-3, -3, -2.3423, -1.5, -1.5, 1.5, 1.5, 0, 3.8423, 4.5	22.6846
G_2	-3.6843, -3, -2.4079, -1.5, -0.5525, 0, 1.5, 1.9247, 3.22, 4.5	22.2894
G_3	-3.7202, -2.7029, -2.3025, -1.5, -0.6676, -0.2228, 1.0882, 1.8705, 3.6572, 4.5	22.2319
G_4	-4.2204, -3, -1.5, -0.794, -0.621, 0, 0, 2.0144, 3.6213, 4.5	20.2711
G_5	-3, -3, -2.5981, -1.5, -0.6213, 0, 0, 2.5981, 3.6213, 4.5	21.4388
G_6	-4.5, -2.4271, -2.4271, -0.9271 , -0.9271, 0.9271, 0.9271, 2.4271, 2.4271, 4.5	22.4168
G_7	-0.9271, -3.9251, -3.4542, -2.4271, -0.5729, 0.9271, 1.5, 1.9542, 2.4271, 4.5	22.6148
G_8	-3.8423, -3, -3, -1.5, 0, 1.5, 1.5, 1.5, 2.3423, 4.5	22.6846
G_9	-3.8942, -3, -2.2981, -1.7740, -0.5209, 0.7736, 1.5, 1.8946, 2.8191, 4.5	22.9745
G_{10}	-3.9271, -3.9271, -0.9271, -0.9271, -0.5729, -0.5729, 1.5, 4.5, 2.4271, 2.4271	21.7084
G_{11}	-3.9271, -2.7912, -2.4271, -0.5729, -0.9271, -0.3812, 0.9271, 2.4271, 3.1724, 4.5	22.0532
G_{12}	-3, -3, -3, -1.5, -1.5, 1.5, 1.5, 1.5, 3, 4.5	24
G_{13}	-4.0004, -3.3705, -1.9411, -0.8324, -0.639, 0, 1.2029, 1.9641, 3.1165, 4.5	21.5668
G_{14}	-3, -3, -2.2981, -2.2981, -0.5209, -0.5209, 1.5, 2.8191, 2.8191, 4.5	23.2762
G_{15}	-4.0639, -2.7029, -2.7029, -0.6676, -0.6676, -0.2909, 1.8705, 1.8705, 2.8548, 4.5	22.1916
G_{16}	-4.5, -3, -1.5, -1.5, 0, 0, 1.5, 1.5, 3, 4.5	21
G_{17}	-3, -3, -3, -3, 1.5, 1.5, 1.5, 1.5, 1.5, 4.5	24
G_{18}	-3.7339, -3, -1.5, -1.5, -0.4338, 0, 0, 1.5, 4.1677, 4.5	20.3354
G_{19}	-3.7093, -2.4271, -2.4271, -2.1939, -0.9271, 0.9271, 0.9271, 2.4271, 2.9031, 4.5	23.3689
G_{20}	-3, -3, -1.5, -1.5, -1.5, 0, 1.5, 4.5, 4.5	21
G_{21}	-4.5, -1.5, -1.5, -1.5, 0, 0, 0, 0, 4.5, 4.5	18

Theorem 7.1. *If two connected cubic graphs of order* 10 *have the same inverse sum indeg energy, then their adjacency matrices have the same permanent.*

Proof. By the above table, we have $ISIE(G_{12}) = ISIE(G_{17}) = 24$ and $ISIE(G_1) = ISIE(G_8) = 22.6846$. We are using the Ryser's method to find the permanent.

$$A(G_{12}) = \begin{bmatrix} 0 & 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 \\ 1 & 0 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 1 & 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 & 1 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 1 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 1 & 0 & 1 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 0 \end{bmatrix}.$$

We have $per(A(G_{12})) = 60$. Similarly we have $per(A(G_{17})) = 60$, $per(A(G_1)) = 72$, $per(A(G_8)) = 72$. □

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