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NOT NECESSARILY CONTINUOUS LOCALLY BOUNDED FINALLY PRECONTINUOUS FINITE-DIMENSIONAL IRREDUCIBLE REPRESENTATIONS OF CONNECTED LOCALLY COMPACT GROUPS

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Abstract. We obtain an explicit form of every (not necessarily continuous) locally bounded finally precontinuous finite-dimensional representation of a connected locally compact group in terms of representations of the radical and a Levi subgroup of the quotient Lie group of the locally compact group by a normal subgroup whose image under the representation is contained in a ball of radius less than one centered at the identity operator in the representation space.

§ 1. Introduction

An explicit form of continuous irreducible finite-dimensional representations of connected Lie groups in terms of representations of the radical and a Levi subgroup is well known (see, e.g., [1], Ch. 8, § 7).

An explicit form of not necessarily continuous irreducible finite-dimensional representations of connected Lie groups in terms of representations of the radical and a Levi subgroup was obtained in [2].

In the present note, using the clear passage to a class of "good" not necessarily continuous finite-dimensional representations, we show that a

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similar explicit form holds for the not necessarily continuous irreducible locally bounded finally precontinuous finite-dimensional representations of connected locally compact groups in terms of representations of the radical and a Levi subgroup of the quotient Lie group of the locally compact group by a normal subgroup whose image under the representation is contained in a ball of radius less than one centered at the identity operator in the representation space.

§ 2. Preliminaries

To clarify the matter, we begin with recalling a lemma used in [3] in the proof of a generalization of the well-known Lie theorem concerning continuous finite-dimensional representations of any solvable Lie group to the case of not necessary continuous representations (Lemma 2.1 of [3]).

Lemma. *Let G be a group, let N be a normal subgroup of G, let π be an irreducible representation of G in a finite-dimensional vector space E, and let there be a one-dimensional subspace $L \subset E$ invariant with respect to the restriction σ of the representation π to N. If there is a divisible subset $X \subset G/N$ generating G/N, then all operators of the representation σ are multiples of the identity operator on the space E.*

This lemma is used in the proof of one of the main results of [2].

Theorem 1. *Let G be a connected Lie group, let R be the radical of G, let L be a Levi subgroup of G, and let π be a* (*not necessarily continuous*) *irreducible locally bounded finite-dimensional representation of G in a space E. Then there is a* (*not necessarily continuous*) *character χ of R satisfying the condition*

$$\chi(grg^{-1}) = \chi(r) \quad \text{for every} \quad g \in G,\ r \in R, \tag{1}$$

and an (*automatically continuous*) *irreducible representation ρ of L on E such that*

$$\pi(g) = \pi(lr) = \chi(r)\rho(l) \quad \text{for every} \quad g = lr,\ l \in L,\ r \in R. \tag{2}$$

§ 3. Main theorem

Recall that a (not necessarily continuous) finite-dimensional representation is said to be *locally bounded* is there is a neighborhood of the identity element whose image under the representation is bounded and *finally precontinuous* if there is a normal subgroup whose image under the representation is contained in a ball centered at the identity operator (in the space of the operators in the representation space) whose radius is less than one and the quotient group by this normal subgroup is a Lie group.

Lemma. *Let G be a connected locally compact group and let π be a finally precontinuous representation of G on the space E. Let N be a normal subgroup entering the definition of final precontinuity. Then $\pi(N) = 1_E$, where 1_E stands for the identity operator on E.*

Obviously, the lemma implies that π defines a representation of G/N, which is locally bounded if π is.

Proof. Let $g \in N$. By assumption, $\|\pi(g^n) - 1_E\| \le q < 1$ for all $n \in \mathbb{Z}$. Applying an invariant mean on the group $\mathbb{Z}$ of integers to $\{\pi(g^n), n \in \mathbb{Z}\}$, we obtain a linear operator B on E such that $\|B - 1_E\| \le q < 1$, and hence B is invertible, and $\pi(g)B = B$ since the mean is invariant, which implies that $\pi(g) = 1_E$. Since $g \in N$ is arbitrary, this completes the proof.

Theorem 2. *Let G be a connected locally compact group, let π be a locally bounded finally precontinuous finite-dimensional* (*not necessarily continuous*) *representation of G on a space E. Let N be a normal subgroup of G satisfying the definition of final precontinuity. Let ρ be the* (*not necessarily continuous*) *locally bounded representation of G/N defined by π. Let R be the radical of G/N and let L be a Levi subgroup of G/N. Then there is a* (*not necessarily continuous*) *character χ of R satisfying the condition*

$$\chi(grg^{-1}) = \chi(r) \quad \textit{for every} \quad g \in G,\ r \in R, \tag{3}$$

and an (*automatically continuous*) *irreducible representation θ of L on E such that, for $g \in G$, $gN = lr$, $l \in L$, $r \in R$, we have*

$$\pi(g) = \rho(lr) = \chi(r)\theta(l) \quad \text{for every} \quad g \in G,\ gN = lr,\ l \in L,\ r \in R. \tag{4}$$

Proof. Immediate.

§ 4. Concluding remarks

Thus, we have obtained a formula giving a general form of every (not necessarily continuous) locally bounded finally precontinuous finite-dimensional representation of a connected locally compact group. This enables us to prove the following corollary.

Corollary. *A connected locally compact group G has a (not necessarily continuous) finite-dimensional finally precontinuous irreducible unitary representation of dimension exceeding one if and only if the Levi subgroup L of some quotient Lie group of G contains a nontrivial compact normal subgroup.*

Proof. Let us apply Theorem 2. A unitary representation of G is bounded, and therefore every finally precontinuous unitary finite-dimensional representation of G has the form (4), where the unitary character χ of R satisfies (3) and θ is a unitary (automatically continuous) representation of the semisimple group L. The representation θ of L can be simultaneously irreducible, nontrivial, and unitary if and only if the list of simple factors of the group L contains a nontrivial simple compact group, because noncompact simple Lie groups have no nontrivial finite-dimensional unitary representations.

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