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ON SKEW-GENERALISED ENERGY OF DIGRAPHS

D. D. Somashekara*, H. E. Ravi †

*,† Department of Studies in Mathematics,
University of Mysore, Manasagangotri, Mysuru-570006, INDIA
email : *dsomashekara@yahoo.com, †ravihemath64@gmail.com

Abstract

In this paper we introduce the concept of skew-generalised energy of directed graphs. We then obtain upper and lower bounds for skew-generalised energy of digraphs. Then we compute the skew-generalised energy of some graphs such as star digraph, complete bipartite digraph, the $(S_m \wedge P_2)$ digraph, $(n, 2n-3)$ strong vertex graceful digraph and a crown digraph.

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1 Introduction

In 2010, Bo Zhou and Nenad Trinajstic [4] have introduced the sum-connectivity energy of a graph. In the same year, Burcu Bozkurt, Dilek Güngör, Gutman and Sinan Çevik [3], have introduced the Randić energy of a graph. Recently in [7] we gave a common generalization of the sum-connectivity energy and the Randić energy. We called it as the generalised energy and introduced as follows. Let a and b be two nonnegative real numbers with $a + b \neq 0$. The generalised adjacency matrix of G is the $n \times n$ matrix $A_g = (a_{ij})$, where

$$a_{ij} = \begin{cases} 0, & \text{if } i = j, \\ \frac{1}{\sqrt{a(d_i+d_j)+b(d_i d_j)}}, & \text{if the vertices } v_i \text{ and } v_j \text{ are adjacent,} \\ 0, & \text{if the vertices } v_i \text{ and } v_j \text{ are not adjacent.} \end{cases}$$

The generalised energy of G is defined as the sum of absolute values of the eigenvalues of the generalised adjacency matrix of G. Note that $a = 1$, $b = 0$ gives the sum-connectivity energy and $a = 0$, $b = 1$ gives the Randić energy of the graph.

In 2010, Adiga, Balakrishnan and Wasin So [1] have introduced the skew energy of a digraph as follows. Let D be a digraph of order n with vertex set $V(D) = \{v_1, v_2, \ldots, v_n\}$ and arc set $\Gamma(D) \subset V(D) \times V(D)$ where $(v_i, v_i) \notin \Gamma(D)$ for all i and $(v_i, v_j) \in \Gamma(D)$ implies $(v_j, v_i) \notin \Gamma(D)$. The skew-adjacency matrix of D is the $n \times n$ matrix $S(D) = (s_{ij})$ where $s_{ij} = 1$ whenever $(v_i, v_j) \in \Gamma(D)$, $s_{ij} = -1$ whenever $(v_j, v_i) \in \Gamma(D)$ and $s_{ij} = 0$ otherwise. Hence $S(D)$ is a skew symmetric matrix of order n and all its eigenvalues are of the form $i\lambda$ where $i = \sqrt{-1}$ and λ is a real number. The skew energy of G is the sum of the absolute values of eigenvalues of $S(D)$.

Motivated by these works, we introduce the concept of skew-generalised energy of a digraph as follows. Let a and b be two nonnegative real numbers with $a+b \neq 0$ and D be a digraph of order n with vertex set $V(D) = \{v_1, v_2, \ldots, v_n\}$ and arc set $\Gamma(D) \subset V(D) \times V(D)$ where $(v_i, v_i) \notin \Gamma(D)$ for all i and $(v_i, v_j) \in \Gamma(D)$ implies $(v_j, v_i) \notin \Gamma(D)$. Then the skew-generalised adjacency matrix of D is the $n \times n$ matrix $A_{sg} = (a_{ij})$ where

$$a_{ij} = \begin{cases} \frac{1}{\sqrt{a(d_i+d_j)+b(d_i d_j)}}, & \text{if } (v_i, v_j) \in \Gamma(D), \\ -\frac{1}{\sqrt{a(d_i+d_j)+b(d_i d_j)}}, & \text{if } (v_j, v_i) \in \Gamma(D), \\ 0, & \text{otherwise.} \end{cases}$$

Then the skew-generalised energy $E_{sg}(D)$ of D is defined as the sum of the absolute values of eigenvalues of A_{sg}.

In section 2 of this paper we obtain the upper and lower bounds for skew-generalised energy of digraphs. In Section 3 we compute the skew-generalised energy of some directed graphs such as complete bipartite digraph, star digraph, the $(S_m \wedge P_2)$ digraph, $(n, 2n-3)$ strong vertex graceful digraph and a crown digraph.

2 Upper and lower bounds for skew-generalised energy

Theorem 2.1. *Let D be a simple digraph of order n and a, b be as defined above. Then*

$$E_{sg}(D) \leq \sqrt{2n \sum_{j \sim k} \frac{1}{a(d_j + d_k) + b(d_j d_k)}}. \tag{1}$$

Proof. On using the identities,

$$\sum_{j=1}^{n} (i\lambda_j)^2 = tr(A_{sg}^2) = -\sum_{j=1}^{n} \sum_{k=1}^{n} a_{jk}^2 = -2 \sum_{j \sim k} \frac{1}{a(d_j + d_k) + b(d_j d_k)} \tag{2}$$

and Cauchy-Schwartz inequality

$$\left(\sum_{j=1}^{n} a_j b_j \right)^2 \leq \left(\sum_{j=1}^{n} a_j^2 \right) \cdot \left(\sum_{j=1}^{n} b_j^2 \right),$$

we obtain (1).

□

Theorem 2.2. *Let D be a simple digraph of order n with and a, b be as defined above. Then*

$$E_{sg}(D) \geq \sqrt{2\sum_{j\sim k} \frac{1}{a(d_j+d_k)+b(d_jd_k)} + n(n-1)p^{\frac{2}{n}}}, \text{ where } p = |detA_{sg}| = \prod_{j=1}^{n} |\lambda_j|. \tag{3}$$

Proof. On using again (2) and a special case of the arithmetic-geometric mean inequality, we obtain (3). □

3 Skew- generalised energies of some families of graphs

We begin with some basic definitions and notations.

Definition 3.1. *[5] A graph G is said to be complete if every pair of its distinct vertices are adjacent. A complete graph on n vertices is denoted by K_n.*

Definition 3.2. *[5] A bigraph or bipartite graph G is a graph whose vertex set $V(G)$ can be partitioned into two subsets V_1 and V_2 such that every line of G joins V_1 with V_2. (V_1, V_2) is a bipartition of G. If G contains every line joining V_1 and V_2, then G is a complete bigraph. If V_1 and V_2 have m and n points, we write $G = K_{m,n}$. A star is a complete bigraph $K_{1,n}$.*

Definition 3.3. *[2] The Crown graph S_n^0 for an integer $n \geq 3$ is the graph with vertex set $\{u_1, u_2, \ldots, u_n, v_1, v_2, \ldots, v_n\}$ and edge set $\{u_iv_j; 1 \leq i, j \leq n, i \neq j\}$. S_n^0 is therefore S_n^0 coincides with complete bipartite graph $K_{n,n}$ with the horizontal edges removed.*

Definition 3.4. *[6] The conjunction $(S_m \wedge P_2)$ of $S_m = \overline{K}_m + K_1$ and P_2 is the graph having the vertex set $V(S_m) \times V(P_2)$ and edge set $\{(v_i, v_j)(v_k, v_l)|v_iv_k \in E(S_m)$ and $v_jv_l \in E(P_2)$ and $1 \leq i, k \leq m+1, 1 \leq j, l \leq 2\}$.*

Definition 3.5. *[8] A graph G is said to be strong vertex graceful if there exists a bijective mapping $f : V(G) \to \{1, 2, \ldots, n\}$ such that for the induced labeling $f^+ : E(G) \to \mathbb{N}$ defined by $f^+(e) = f(u) + f(v)$, where $e = uv$, the set $f^+(E(G))$ consists of consecutive integers.*

Now we compute skew- generalised energies of some directed graphs such as complete bipartite digraph, star digraph, the $(S_m \wedge P_2)$ digraph, $(n, 2n-3)$ strong vertex graceful digraph and a crown digraph.

Theorem 3.6. *Let the vertex set $V(D)$ and arc set $\Gamma(D)$ of $K_{m,n}$ complete bipartite digraph be respectively given by*

$$V(D) = \{u_1, u_2, \ldots, u_m, v_1, v_2, \ldots, v_n\} \text{ and}$$
$$\Gamma(D) = \{(u_i, v_j) \,|\, 1 \leq i \leq m, 1 \leq j \leq n\}.$$

Then the skew-generalised energy of the complete bipartite digraph is $2\sqrt{\frac{mn}{a(m+n)+b(mn)}}$.

Proof. The skew-generalised matrix of complete bipartite digraph is given by

$$A_{sg} = \begin{pmatrix} 0 & 0 & \cdots & 0 & \gamma & \gamma & \cdots & \gamma \\ 0 & 0 & \cdots & 0 & \gamma & \gamma & \cdots & \gamma \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 0 & \gamma & \gamma & \cdots & \gamma \\ -\gamma & -\gamma & \cdots & -\gamma & 0 & 0 & \cdots & 0 \\ -\gamma & -\gamma & \cdots & -\gamma & 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots \\ -\gamma & -\gamma & \cdots & -\gamma & 0 & 0 & \cdots & 0 \end{pmatrix},$$

where $\gamma = \frac{1}{\sqrt{a(m+n)+b(mn)}}$. Then its characteristic polynomial is

$$\begin{aligned} |\lambda I - A_{sg}| &= \begin{vmatrix} \lambda & 0 & \cdots & 0 & -\gamma & -\gamma & \cdots & -\gamma \\ 0 & \lambda & \cdots & 0 & -\gamma & -\gamma & \cdots & -\gamma \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \lambda & -\gamma & -\gamma & \cdots & -\gamma \\ \gamma & \gamma & \cdots & \gamma & \lambda & 0 & \cdots & 0 \\ \gamma & \gamma & \cdots & \gamma & 0 & \lambda & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots \\ \gamma & \gamma & \cdots & \gamma & 0 & 0 & \cdots & \lambda \end{vmatrix} \\ &= \begin{vmatrix} \lambda I_m & -\frac{1}{\sqrt{a(m+n)+b(mn)}} J^T \\ \frac{1}{\sqrt{a(m+n)+b(mn)}} J & \lambda I_n \end{vmatrix}, \end{aligned}$$

where J is an $n \times m$ matrix with all the entries are equal to 1. Hence the characteristic equation is given by

$$\begin{vmatrix} \lambda I_m & -\frac{1}{\sqrt{a(m+n)+b(mn)}} J^T \\ \frac{1}{\sqrt{a(m+n)+b(mn)}} J & \lambda I_n \end{vmatrix} = 0.$$

This can be written as

$$|\lambda I_m| \left| \lambda I_n - \left(\frac{1}{\sqrt{a(m+n)+b(mn)}} J \right) \frac{I_m}{\lambda} \left(-\frac{1}{\sqrt{a(m+n)+b(mn)}} J^T \right) \right| = 0.$$

On simplification, we obtain

$$\frac{\lambda^{m-n}}{a(m+n)+b(mn)} \left| (a(m+n)+b(mn))\lambda^2 I_n + JJ^T \right| = 0,$$

which can be written as

$$\frac{\lambda^{m-n}}{a(m+n)+b(mn)} P_{JJ^T}(-(a(m+n)+b(mn))\lambda^2) = 0,$$

where $P_{JJ^T}(\lambda)$ is the characteristic polynomial of the matrix ${}_mJ_n$. Thus, we have

$$\frac{\lambda^{m-n}}{a(m+n)+b(mn)}((a(m+n)+b(mn))\lambda^2+mn)((a(m+n)+b(mn))\lambda^2)^{n-1}=0,$$

which is same as

$$\lambda^{m+n-2}(\lambda^2+\frac{mn}{a(m+n)+b(mn)})=0.$$

Hence,

$$Spec\,(D)=\begin{pmatrix} 0 & i\sqrt{\frac{mn}{a(m+n)+b(mn)}} & -i\sqrt{\frac{mn}{a(m+n)+b(mn)}} \\ m+n-2 & 1 & 1 \end{pmatrix},$$

from which the theorem follows. □

Theorem 3.7. *Let the vertex set $V(D)$ and arc set $\Gamma(D)$ of S_n star digraph be respectively given by*

$$V(D)=\{v_1,v_2,\dots,v_n\} \text{ and } \Gamma(D)=\{(v_1,v_j)\,|\,2\le j\le n\}.$$

Then the skew-generalised energy of D is $2\sqrt{\frac{n-1}{an+b(n-1)}}$.

Proof. The proof follows on replacing m by 1 and n by $n-1$ in theorem 3.6. □

Theorem 3.8. *Let the vertex set $V(D)$ and arc set $\Gamma(D)$ of S_n^0 crown digraph be respectively given by*

$$V(D)=\{u_1,u_2,\dots,u_n,v_1,v_2,\dots,v_n\} \text{ and}$$
$$\Gamma(D)=\{(u_i,v_j)\,|\,1\le i\le n, 1\le j\le n, i\ne j\}.$$

Then the skew-generalised energy of the crown digraph is $\frac{4(n-1)}{\sqrt{2a(n-1)+b(n-1)^2}}$.

Proof. The skew-generalised matrix of crown digraph is given by

$$A_{sg}=\begin{pmatrix} 0 & 0 & \cdots & 0 & 0 & \gamma & \cdots & \gamma \\ 0 & 0 & \cdots & 0 & \gamma & 0 & \cdots & \gamma \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 0 & \gamma & \gamma & \cdots & 0 \\ 0 & -\gamma & \cdots & -\gamma & 0 & 0 & \cdots & 0 \\ -\gamma & 0 & \cdots & -\gamma & 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots \\ -\gamma & -\gamma & \cdots & 0 & 0 & 0 & \cdots & 0 \end{pmatrix}.$$

where $\gamma=\frac{1}{\sqrt{2a(n-1)+b(n-1)^2}}$. Then its characteristic polynomial is

$$|\lambda I-A_{sg}|=\begin{vmatrix} \lambda I_n & -\frac{1}{\sqrt{2a(n-1)+b(n-1)^2}}K^T \\ \frac{1}{\sqrt{2a(n-1)+b(n-1)^2}}K & \lambda I_n \end{vmatrix},$$

where K is an $n \times n$ matrix. Hence the characteristic equation is given by

$$\begin{vmatrix} \lambda I_n & -\frac{1}{\sqrt{2a(n-1)+b(n-1)^2}}K^T \\ \frac{1}{\sqrt{2a(n-1)+b(n-1)^2}}K & \lambda I_n. \end{vmatrix} = 0.$$

Now, on using the technique similar to the one used in the proof of the theorem 3.6, theorem 3.8 follows. □

Theorem 3.9. *Let the vertex set $V(D)$ and arc set $\Gamma(D)$ of $(S_m \wedge P_2)$ digraph be respectively given by*

$$V(D) = \{v_1, v_2, \dots, v_{2m+2}\} \text{ and}$$
$$\Gamma(D) = \{(v_1, v_j), (v_{m+2}, v_k) \,|\, 2 \leq k \leq m+1, m+3 \leq j \leq 2m+2\}.$$

Then the skew-generalised energy of D is $4\sqrt{\frac{n-1}{an+b(n-1)}}$.

Proof. The skew-generalised matrix of $(S_m \wedge P_2)$ digraph is given by

$$A_{sg} = \begin{pmatrix} 0 & 0 & \cdots & 0 & 0 & \gamma & \cdots & \gamma \\ 0 & 0 & \cdots & 0 & -\gamma & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 0 & -\gamma & 0 & \cdots & 0 \\ 0 & \gamma & \cdots & \gamma & 0 & 0 & \cdots & 0 \\ -\gamma & 0 & \cdots & 0 & 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots \\ -\gamma & 0 & \cdots & 0 & 0 & 0 & \cdots & 0 \end{pmatrix}_{2n \times 2n},$$

where $m+1 = n$ and $\gamma = \frac{1}{\sqrt{an+b(n-1)}}$. Then the characteristic equation is given by

$$\left(\frac{1}{\sqrt{an+b(n-1)}}\right)^{2n} \begin{vmatrix} \Lambda & 0 & \cdots & 0 & 0 & -1 & \cdots & -1 \\ 0 & \Lambda & \cdots & 0 & 1 & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \Lambda & 1 & 0 & \cdots & 0 \\ 0 & -1 & \cdots & -1 & \Lambda & 0 & \cdots & 0 \\ 1 & 0 & \cdots & 0 & 0 & \Lambda & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & 0 & \cdots & 0 & 0 & 0 & \cdots & \Lambda \end{vmatrix}_{2n \times 2n} = 0,$$

$$\text{where } \Lambda = \sqrt{an+b(n-1)}\lambda.$$

Using the usual properties of the determinants we obtain

$$Spec\,(D) = \begin{pmatrix} 0 & i\sqrt{\frac{n-1}{an+b(n-1)}} & -i\sqrt{\frac{n-1}{an+b(n-1)}} \\ 2n-4 & 2 & 2 \end{pmatrix},$$

from which the theorem 3.9 follows. □

Theorem 3.10. *Let the vertex set $V(D)$ and arc set $\Gamma(D)$ of $(n, 2n-3)$ strong vertex graceful digraph $D = K_2 + \overline{K}_{n-2}$ be respectively given by*

$$V(D) = \{v_1, v_2, \ldots, v_n\} \text{ and}$$
$$\Gamma(D) = \{(v_1, v_j) \,|\, 2 \le j \le n\} \cup \{(v_j, v_n) \,|\, 2 \le j \le n-1\}.$$

Then the skew-generalised energy of D is $2\sqrt{\frac{a(n+1)+2b(n-1)+2(n-2)(2a(n-1)+b(n-1)^2)}{(a(n+1)+2b(n-1))(2a(n-1)+b(n-1)^2)}}$.

Proof. The skew-generalised matrix is given by

$$A_{sg} = \begin{pmatrix} 0 & \frac{1}{\sqrt{a(n+1)+2b(n-1)}} & \cdots & \frac{1}{\sqrt{2a(n-1)+b(n-1)^2}} \\ -\frac{1}{\sqrt{a(n+1)+2b(n-1)}} & 0 & \cdots & \frac{1}{\sqrt{a(n+1)+2b(n-1)}} \\ -\frac{1}{\sqrt{a(n+1)+2b(n-1)}} & 0 & \cdots & \frac{1}{\sqrt{a(n+1)+2b(n-1)}} \\ \vdots & \vdots & \ddots & \vdots \\ -\frac{1}{\sqrt{2a(n-1)+b(n-1)^2}} & -\frac{1}{\sqrt{a(n+1)+2b(n-1)}} & \cdots & 0 \end{pmatrix}.$$

Its characteristics polynomial is

$$|\lambda I - A_{sg}| = \left(\frac{1}{\sqrt{a(n+1)+2b(n-1)}}\right)^n \begin{vmatrix} \mu & -1 & -1 & \cdots & -1 & -\gamma \\ 1 & \mu & 0 & \cdots & 0 & -1 \\ 1 & 0 & \mu & \cdots & 0 & -1 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 1 & 0 & 0 & \cdots & \mu & -1 \\ \gamma & 1 & 1 & \cdots & 1 & \mu \end{vmatrix},$$

where $\mu = \lambda\sqrt{a(n+1)+2b(n-1)}$ and $\gamma = \sqrt{\frac{a(n+1)+2b(n-1)}{2a(n-1)+b(n-1)^2}}$. Using the properties of the determinants, we obtain

$$Spec\,(D) = \begin{pmatrix} 0 & i\omega & -i\omega \\ n-2 & 1 & 1 \end{pmatrix},$$

where $\omega = \sqrt{\frac{a(n+1)+2b(n-1)+2(n-2)(2a(n-1)+b(n-1)^2)}{(a(n+1)+2b(n-1))(2a(n-1)+b(n-1)^2)}}$. From this the theorem 3.10 follows. □

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References

[1] C. Adiga, R. Balakrishnan and Wasin So, The skew energy of a digraph, *Lin. Algebra Appl.*, **432** (2010), 1825-1835.

[2] C. Adiga, Abdelmejid Bayad, Ivan Gutman and Shrikanth Avant Srinivas , The minimum covering energy of a digraph, *Kragujevac j. Sci.*, **34** (2012), 39-56.

[3] Ş. Burcu Bozkurt, A. Dilek Güngör, I. Gutman and A. Sinan Çevik, Randić matrix and Randić energy, *MATCH Commun. Math. Comput. Chem.*, **64** (2010), 239-250.

[4] Bo Zhou and Nenad Trinajstic , on sum-connectivity matrix and sum-connectivity energy of (molecular) graphs *Acta Chim. Slov.*, **57** (2010), 518523.

[5] F. Harary, *Graph Theory*, 10^{th} Reprint, Narosa Publishing House, New Delhi, 2001.

[6] Ş. M. A. Soeud, Huda Jaber, Some Families of Strongly*-graphs(communicated).

[7] D. D. Somashekara, Xiao-Dong Zhang and H. E. Ravi , On generalised energies of some graphs(Communicated) .

[8] D. D. Somashekara and C. R. Veena, On strong vertex graceful graphs, International Math. Forum, 5 (2010), 2751-2757.