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# SKEW-ZAGREB ENERGY OF DIRECTED GRAPHS

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Abstract. In molecular graph, vertex symbolizes atom and edge represents bond and degree of a vertex closely related to valence in chemistry. In the year 2010, Adiga et al. were introduced the skew energy for digraphs. Using this new combinatorial technique for digraphs, we introduced and analyzed the skew first Zagreb energy ($SFZE$) and skew second Zagreb energy ($SSZE$) of some digraphs.

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## 1. Introduction

Graph Theory is a branch of mathematics that has a wealth of applications to other science and engineering disciplines, specifically Chemistry. The primary application of graphs to Chemistry is related to understanding of structure and symmetry at the molecular level. By projecting a molecule to the plane and examining it as a graph, a lot can be learned about the underlying molecular structure of a given compound.

Topological graph indices have been used in a lot of areas to study required properties of different objects such as atoms and molecules. Such indices have been described and studied by many mathematicians and chemists since most graphs are generated from molecules by replacing each atom with a vertex and each chemical bond with an edge. These indices are also topological graph invariants measuring several chemical, physical, biological, pharmacological, pharmaceutical, etc. properties of graphs corresponding to real life situations. They are grouped mainly into three classes according to their definition: by vertex degrees, by distances or by matrices.

A topological index is a numerical parameter of a graph which characterizes some of the topological properties of the graph. The concepts of hyper-Zagreb index, first multiple Zagreb index, second multiple Zagreb index and relatedly the Zagreb polynomials were established in chemical graph theory by means of the vertex degrees. We can compute the characteristic polynomial, spectra, nullity, etc. by the matrix representation of the molecular graph which have the crucial role in molecular orbital theory. In general, the topological index can be represented as (See [2]):

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$$TI(G) = \sum_{i \sim j} F(d_i, d_j)$$

where $F = F(x, y)$ is well defined function. The first and second Zagreb indices of a graph G are defined by

$$F(x, y) = x + y$$

and

$$F(x, y) = xy.$$

In 2013, Shirdel et al. [9] introduced a new degree based Zagreb index named "hyper-Zagreb index" by

$$F(x, y) = (x + y)^2.$$

The adjacency matrix of each topological index is: $A = (a_{ij})$ of order $n \times n$ is given by (See [6])

$$a_{ij} = \begin{cases} F(d_i, d_j), & \text{if } i \sim j \\ 0, & \text{otherwise} \end{cases}$$

In 2018, Rad et al. [5] were introduced the concept of Zagreb energy of graph $G$ by taking into account, the first Zagreb matrix $Z_1(G) = (t_{ij})_{n \times n}$ is defined as:

$$t_{ij} = \begin{cases} d_i + d_j, & \text{if } v_i \sim v_j \\ 0, & \text{otherwise} \end{cases}$$

In the similar way, the seond Zagreb adjacency matrix $Z_2(G)$ is defined as $Z_2(G) = (m_{ij})_{n \times n}$, where

$$m_{ij} = \begin{cases} d_i d_j, & \text{if } v_i \sim v_j \\ 0, & \text{otherwise} \end{cases}$$

Let $D^\sigma$ be a directed graph of order $n$ with vertex set $V(D^\sigma)$ and the arc set is given by $\Gamma(D^\sigma) \subset V(D^\sigma) \times V(D^\sigma)$. The skew adjacency matrix of $D^\sigma$ is the $n \times n$ matrix $S(D^\sigma) = [s_{ij}]$, where $s_{ij} = 1$ whenever $(v_i, v_j) \in \Gamma(D^\sigma)$, $s_{ij} = -1$ whenever $(v_j, v_i) \in \Gamma(D^\sigma)$ and $s_{ij} = 0$, otherwise.

Thus, the corresponding skew adjacency matrix $\mathcal{F} = (f_{ij})$ of order $n \times n$ can be associated to each topological indices as:

$$f_{ij}^\sigma = \begin{cases} F(d_i, d_j), & \text{if } v_i \sim v_j \\ -F(d_i, d_j), & \text{if } v_j \sim v_i \\ 0, & \text{otherwise} \end{cases} \tag{1}$$

We now define the skew-first Zagreb matrix $Z_1(D^\sigma) = (t_{ij}^\sigma)_{n \times n}$ of a digraph as follows:

$$t_{ij}^\sigma = \begin{cases} d_i + d_j, & \text{if } (v_i, v_j) \in \Gamma(D^\sigma) \\ -(d_i + d_j), & \text{if } (v_j, v_i) \in \Gamma(D^\sigma) \\ 0, & \text{otherwise} \end{cases} \tag{2}$$

In a similar way, the skew-second Zagreb matrix $Z_2(D^\sigma) = (m^\sigma_{ij})_{n\times n}$ of a digraph is defined as:

$$m^\sigma_{ij} = \begin{cases} d_i.d_j, & \text{if } (v_i, v_j) \in \Gamma(D^\sigma) \\ -d_i.d_j, & \text{if } (v_j, v_i) \in \Gamma(D^\sigma) \\ 0, & \text{otherwise} \end{cases} \tag{3}$$

**Lemma 1.1.** *Let $D^\sigma$ be a directed graph containg $n$ vertices and let $Z = (f^\sigma_{ij})_{n\times n}$ be the skew-Zagreb matrix defined in equations* (1.2) *and* (1.3). *Then,*

(i) $Tr(Z) = 0$, *where Tr denotes the trace of the respective matrix.*
(ii) $Tr(Z^2) = -2\sum_{i,j=1}^{n} F(d_i, d_j)^2 = -2HM$, *where $F(d_i, d_j) = d_i + d_j$ and $F(d_i, d_j) = d_i.d_j$ for first and second Zagreb indecies respectively and HM represents hyper-Zagreb index.*
(iii) $Tr(Z^i) = 0$, *for all odd $i$.*

*Proof.*

(i) The diagonal entries of matrix $Z$ are equal to zero. Hence, the result follows.
(ii) The diagonal entries of $Z^2$ are

$$(Z^2)_{ii} = \sum_{j=1}^{n} f_{ij} f_{ji} = -\sum_{j=1}^{n} f^2_{ij}$$

Therefore,

$$Tr(Z^2) = -\sum_{i=1}^{n}\sum_{j=1}^{n} f^2_{ij} = -2\sum_{i,j=1}^{n} F(d_i, d_j)^2 = -2HM$$

(iii) Since for all odd $i$, $Z^i$ is a skew symmetric matrix. Then

$$Tr(Z^i) = 0$$

□

**Proposition 1.2.** [8] *Let $\mathcal{C} = \begin{pmatrix} 0 & Y \\ Y^T & 0 \end{pmatrix}$ and $\mathcal{D} = \begin{pmatrix} 0 & Y \\ -Y^T & 0 \end{pmatrix}$ be two real matrices, then Spec($\mathcal{D}$)= iSpec($\mathcal{C}$).*

**Proposition 1.3.** [3] *Let $\mathcal{C} \in M_{m\times m}$ and $\mathcal{D} \in M_{n\times n}$ and Let $\lambda$ and $\mu$ be the eigenvalues of $\mathcal{C}$ and $\mathcal{D}$ respectively. Then $\lambda\mu$ is an eigenvalue of $\mathcal{C} \otimes \mathcal{D}$.*

**Definition 1.1.** [7] *The shadow graph $S(H)$ of a connected graph $H$ is constructed by taking two copies of $H$, say $H'$ and $H''$. Join each vertex $h'$ in $H'$ to the neighbors of the corresponding vertex $h''$ in $H''$ and we note that*

$$E(H) = 2E(S(H))$$

In [4], the authors defined skew-symmetric division degree and skew-inverse sum indegree energies of some digraphs. In this paper, we introduced and analyzed the skew first Zagreb energy ($SFZE$) and skew second Zagreb energy ($SSZE$) of some digraphs. The orientations of all arcs taken from low labels to high labels.

## 2. MAIN RESULTS

**Theorem 2.1.** *Let $V(K_n^\sigma) = \{v_1, ..., v_n\}$ and $\Gamma(K_n^\sigma) = \{(v_i, v_j) \mid 1 \leq i, j \leq n\}$ be vertex set and arc set of complete digraph $K_n^\sigma$ respectively. Then, skew-first Zagreb energy of $K_n^\sigma$ is*

$$SFZE(K_n^\sigma) = 2(n-1)\sum_{k=0}^{n-1} cot(2k+1)\frac{\pi}{2n}$$

*Proof.* The skew-first Zagreb matrix of $K_n^\sigma$ is given by

$$A = \begin{bmatrix} 0 & 2(n-1) & \cdots & 2(n-1) \\ -2(n-1) & 0 & \cdots & 2(n-1) \\ \vdots & \vdots & \ddots & \vdots \\ -2(n-1) & -2(n-1) & \cdots & 2(n-1) \\ -2(n-1) & -2(n-1) & \cdots & 0 \end{bmatrix}$$

Then the characteristic polynomial is

$$\frac{(-1)^n[(\lambda-2(n-1))^n + (\lambda+2(n-1))^n]}{2} = 0.$$

The spectrum is

$$\lambda = 2(n-1)icot(2k+1)\tfrac{\pi}{2n} \ ; \ k = 0, 1, 2..., (n-1)$$

Hence,

$$SFZE(K_n^\sigma) = 2(n-1)\sum_{k=0}^{n-1} cot(2k+1)\frac{\pi}{2n}$$

□

**Theorem 2.2.** *Let $V(K_n^\sigma) = \{v_1, ..., v_n\}$ and $\Gamma(K_n^\sigma) = \{(v_i, v_j) \mid 1 \leq i, j \leq n\}$ be vertex set and arc set of complete digraph $K_n^\sigma$ respectively. Then*

$$SSZE(K_n^\sigma) = (n-1)^2\sum_{k=0}^{n-1} cot(2k+1)\frac{\pi}{2n}.$$

*Proof.* The skew-second Zagreb matrix of $K_n^\sigma$ is given by

$$A = \begin{bmatrix} 0 & (n-1)^2 & \cdots & (n-1)^2 \\ -(n-1)^2 & 0 & \cdots & (n-1)^2 \\ \vdots & \vdots & \ddots & \vdots \\ -(n-1)^2 & -(n-1)^2 & \cdots & (n-1)^2 \\ -(n-1)^2 & -(n-1)^2 & \cdots & 0 \end{bmatrix}$$

Thus, the spectrum is

$$\lambda = (n-1)^2 i cot(2k+1)\tfrac{\pi}{2n} \ ; \ k = 0, 1, 2..., (n-1)$$

Hence,

$$SSZE(K_n^\sigma) = (n-1)^2 \sum_{k=0}^{n-1} cot(2k+1)\frac{\pi}{2n}.$$

□

**Theorem 2.3.** *Let $V(S_n^\sigma) = \{v_1, ..., v_n\}$ and $\Gamma(S_n^\sigma) = \{(v_1, v_j) \mid 2 \le j \le n\}$ be vertex set and arc set of star digraph $S_n^\sigma$ respectively. Then,*

$$SFZE(S_n^\sigma) = 2n\sqrt{n-1}.$$

*Proof.* The skew-first Zagreb matrix of $S_n^\sigma$ is given by

$$A = \begin{bmatrix} 0 & n & \cdots & n \\ -n & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ -n & 0 & \cdots & 0 \end{bmatrix}$$

Thus, the spectrum is

$$Spec(S_n^\sigma) = \begin{pmatrix} 0 & in\sqrt{n-1} & -in\sqrt{n-1} \\ n-2 & 1 & 1 \end{pmatrix}$$

Hence, the skew-first Zagreb energy of $S_n^\sigma$ is

$$SFZE(S_n^\sigma) = 2n\sqrt{n-1}.$$

□

**Theorem 2.4.** *Let $V(S_n^\sigma) = \{v_1, ..., v_n\}$ and $\Gamma(S_n^\sigma) = \{(v_1, v_j) \mid 2 \le j \le n\}$ be vertex set and arc set of star digraph $S_n$ respectively. Then*

$$SSZE(S_n^\sigma) = 2(n-1)^{\frac{3}{2}}.$$

*Proof.* The skew-second Zagreb matrix of $S_n^\sigma$ is given by

$$A = \begin{bmatrix} 0 & n-1 & \cdots & n-1 \\ -(n-1) & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ -(n-1) & 0 & \cdots & 0 \end{bmatrix}$$

Then the corresponding characteristic polynomial is

$$\lambda^{n-2}[\lambda^2 + (n-1)^3] = 0$$

and the spectrum is

$$Spec(S_n^\sigma) = \begin{pmatrix} 0 & i(n-1)^{\frac{3}{2}} & -i(n-1)^{\frac{3}{2}} \\ n-2 & 1 & 1 \end{pmatrix}$$

Hence,

$$SSZE(S_n^\sigma) = 2(n-1)^{\frac{3}{2}}.$$

□

**Theorem 2.5.** *Let* $V(H_{2n}^\sigma) = \{u_1, ..., u_n, v_1, ..., v_n\}$ *and* $\Gamma(H_{2n}^\sigma) = \{(u_i, v_j), (u_i, u_j), (v_i, v_j) \mid 1 \leq i, j \leq n\}$ *be vertex set and arc set of hyperoctahedral graph digraph* $H_{2n}^\sigma$ *respectively. Then,*

$$SFZE(H_{2n}^\sigma) = 8(n-1)\sum_{k=0}^{n-1} cot(2k+1)\frac{\pi}{2n}.$$

*Proof.* The skew-first Zagreb matrix of $H_n^\sigma$ is given by

$$A = \begin{bmatrix} 0 & 4(n-1) & \cdots & 4(n-1) & 0 & 4(n-1) & \cdots & 4(n-1) \\ -4(n-1) & 0 & \cdots & 4(n-1) & -4(n-1) & 0 & \cdots & 4(n-1) \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots \\ -4(n-1) & -4(n-1) & \cdots & 0 & -4(n-1) & -4(n-1) & \cdots & 0 \\ 0 & 4(n-1) & \cdots & 4(n-1) & 0 & 4(n-1) & \cdots & 4(n-1) \\ -4(n-1) & 0 & \cdots & 4(n-1) & -4(n-1) & 0 & \cdots & 4(n-1) \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots \\ -4(n-1) & -4(n-1) & \cdots & 0 & -4(n-1) & -4(n-1) & \cdots & 0 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} \otimes \begin{bmatrix} 0 & 4(n-1) & \cdots & 4(n-1) & 4(n-1) \\ -4(n-1) & 0 & \cdots & 4(n-1) & 4(n-1) \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ -4(n-1) & -4(n-1) & \cdots & -4(n-1) & 0 \end{bmatrix}$$

The spectrum is

$\lambda = 8(n-1)icot(2k+1)\frac{\pi}{2n}$ ; $k = 0, 1, 2..., (n-1)$ and 0 ($n$ times).

Hence,

$$SFZE(H_{2n}^\sigma) = 8(n-1)\sum_{k=0}^{n-1} cot(2k+1)\frac{\pi}{2n}.$$

□

**Theorem 2.6.** *Let* $V(H_{2n}^\sigma) = \{u_1, ..., u_n, v_1, ..., v_n\}$ *and* $\Gamma(H_{2n}^\sigma) = \{(u_i, v_j), (u_i, u_j), (v_i, v_j) \mid 1 \leq i, j \leq n\}$ *be vertex set and arc set of hyperoctahedral graph digraph* $H_{2n}^\sigma$ *respectively. Then,*

$$SSZE(H_{2n}^\sigma) = 8(n-1)^2\sum_{k=0}^{n-1} cot(2k+1)\frac{\pi}{2n}.$$

*Proof.* The skew-second Zagreb matrix of $H_{2n}^{\sigma}$ is given by

$$A=\begin{bmatrix} 0 & 4(n-1)^2 & \cdots & 4(n-1)^2 & 0 & 4(n-1)^2 & \cdots & 4(n-1)^2\\ -4(n-1)^2 & 0 & \cdots & 4(n-1)^2 & -4(n-1)^2 & 0 & \cdots & 4(n-1)^2\\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots\\ -4(n-1)^2 & -4(n-1)^2 & \cdots & 0 & -4(n-1)^2 & -4(n-1)^2 & \cdots & 0\\ 0 & 4(n-1)^2 & \cdots & 4(n-1)^2 & 0 & 4(n-1)^2 & \cdots & 4(n-1)^2\\ -4(n-1)^2 & 0 & \cdots & 4(n-1)^2 & -4(n-1)^2 & 0 & \cdots & 4(n-1)^2\\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots\\ -4(n-1)^2 & -4(n-1)^2 & \cdots & 0 & -4(n-1)^2 & -4(n-1)^2 & \cdots & 0 \end{bmatrix}$$

$$=\begin{bmatrix} 1 & 1\\ 1 & 1\end{bmatrix}\otimes\begin{bmatrix} 0 & 4(n-1)^2 & \cdots & 4(n-1)^2 & 4(n-1)^2\\ -4(n-1)^2 & 0 & \cdots & 4(n-1)^2 & 4(n-1)^2\\ \vdots & \vdots & \ddots & \vdots & \vdots\\ -4(n-1)^2 & -4(n-1)^2 & \cdots & -4(n-1)^2 & 0 \end{bmatrix}$$

The spectrum is

$\lambda = 8(n-1)^2 i cot(2k+1)\frac{\pi}{2n}$ ; $k = 0, 1, 2..., (n-1)$ and 0 ($n$ times).

Hence,

$$SFZE(H_{2n}^{\sigma}) = 8(n-1)^2\sum_{k=0}^{n-1} cot(2k+1)\frac{\pi}{2n}.$$

□

**Theorem 2.7.** *Let $V(C_{2n}^{\sigma}) = \{u_1, ..., u_n, v_1, ..., v_n\}$ and $\Gamma(C_{2n}^{\sigma}) = \{(u_i, v_j) \mid 1 \leq i, j \leq n\}$ be vertex set and arc set of crown digraph $C_{2n}^{\sigma}$ respectively. Then,*

$$SFZE(C_{2n}^{\sigma}) = 4(n-1)\sum_{k=0}^{n-1} cot(2k+1)\frac{\pi}{2n}.$$

*Proof.* The skew-first Zagreb matrix of $C_{2n}^{\sigma}$ is given by

$$A=\begin{bmatrix} 0 & 0 & \cdots & 0 & 0 & 2(n-1) & \cdots & 2(n-1)\\ 0 & 0 & \cdots & 0 & -2(n-1) & 0 & \cdots & 2(n-1)\\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots\\ 0 & 0 & \cdots & 0 & -2(n-1) & -2(n-1) & \cdots & 0\\ 0 & 2(n-1) & \cdots & 2(n-1) & 0 & 0 & \cdots & 0\\ -2(n-1) & 0 & \cdots & 2(n-1) & 0 & 0 & \cdots & 0\\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots\\ -2(n-1) & -2(n-1) & \cdots & 0 & 0 & 0 & \cdots & 0 \end{bmatrix}$$

$$= \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \otimes \begin{bmatrix} 0 & 2(n-1) & \cdots & 2(n-1) & 2(n-1) \\ -2(n-1) & 0 & \cdots & 2(n-1) & 2(n-1) \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ -2(n-1) & -2(n-1) & \cdots & -2(n-1) & 0 \end{bmatrix}$$

The spectrum is

$$\lambda = \pm 2(n-1)icot(2k+1)\tfrac{\pi}{2n} \ ; \ k = 0, 1, 2..., (n-1).$$

Hence,

$$SFZE(C_{2n}^{\sigma}) = 4(n-1)\sum_{k=0}^{n-1} cot(2k+1)\frac{\pi}{2n}.$$

□

**Theorem 2.8.** *Let* $V(C_{2n}^{\sigma}) = \{u_1, ..., u_n, v_1, ..., v_n\}$ *and* $\Gamma(C_{2n}^{\sigma}) = \{(u_i, v_j) \mid 1 \leq i, j \leq n\}$ *be vertex set and arc set of crown digraph* $C_{2n}^{\sigma}$ *respectively. Then,*

$$SSZE(C_{2n}^{\sigma}) = 2(n-1)^2 \sum_{k=0}^{n-1} cot(2k+1)\frac{\pi}{2n}.$$

*Proof.* The skew-second Zagreb matrix of $C_{2n}^{\sigma}$ is given by

$$A = \begin{bmatrix} 0 & 0 & \cdots & 0 & 0 & (n-1)^2 & \cdots & (n-1)^2 \\ 0 & 0 & \cdots & 0 & -(n-1)^2 & 0 & \cdots & (n-1)^2 \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 0 & -(n-1)^2 & -(n-1)^2 & \cdots & 0 \\ 0 & (n-1)^2 & \cdots & (n-1)^2 & 0 & 0 & \cdots & 0 \\ -(n-1)^2 & 0 & \cdots & (n-1)^2 & 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots \\ -(n-1)^2 & -(n-1)^2 & \cdots & 0 & 0 & 0 & \cdots & 0 \end{bmatrix}$$

$$= \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \otimes \begin{bmatrix} 0 & (n-1)^2 & \cdots & (n-1)^2 & (n-1)^2 \\ -(n-1)^2 & 0 & \cdots & (n-1)^2 & (n-1)^2 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ -(n-1)^2 & -(n-1)^2 & \cdots & -(n-1)^2 & 0 \end{bmatrix}$$

The spectrum is

$$\lambda = \pm \ i(n-1)^2 cot(2k+1)\tfrac{\pi}{2n} \ ; \ k = 0, 1, 2..., (n-1).$$

Hence,

$$SSZE(C_{2n}^{\sigma}) = 2(n-1)^2 \sum_{k=0}^{n-1} cot(2k+1)\frac{\pi}{2n}.$$

□

**Theorem 2.9.** *Let* $V(K^{\sigma}_{m,n}) = \{u_1, ..., u_m, v_1, ..., v_n\}$ *and* $\Gamma(K^{\sigma}_{m,n}) = \{(u_i, v_j) \mid 1 \le i \le m, 1 \le j \le n\}$ *be vertex set and arc set of complete bipartite digraph* $K^{\sigma}_{m,n}$ *respectively. Then,*

$$SFSE(K^{\sigma}_{m,n}) = 2\sqrt{mn}(m+n).$$

*Proof.* The skew-first Zagreb matrix of $K^{\sigma}_{m,n}$ is given by

$$A = \begin{bmatrix} 0 & 0 & \cdots & m+n & m+n \\ 0 & 0 & \cdots & m+n & m+n \\ 0 & 0 & \cdots & m+n & m+n \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ -(m+n) & -(m+n) & \cdots & 0 & 0 \\ -(m+n) & -(m+n) & \cdots & 0 & 0 \\ -(m+n) & -(m+n) & \cdots & 0 & 0 \end{bmatrix}$$

Thus, the spectrum is

$$spec(K^{\sigma}_{m,n}) = \begin{pmatrix} 0 & i\sqrt{mn}(m+n) & -i\sqrt{mn}(m+n) \\ m+n-2 & 1 & 1 \end{pmatrix}$$

Hence,

$$SFSE(K^{\sigma}_{m,n}) = 2\sqrt{mn}(m+n).$$

□

**Theorem 2.10.** *Let* $V(K^{\sigma}_{m,n}) = \{u_1, ..., u_m, v_1, ..., v_n\}$ *and* $\Gamma(K^{\sigma}_{m,n}) = \{(u_i, v_j) \mid 1 \le i \le m, 1 \le j \le n\}$ *be vertex set and arc set of complete bipartite digraph* $K^{\sigma}_{m,n}$ *respectively. Then,*

$$SFSE(K^{\sigma}_{m,n}) = 2mn.$$

*Proof.* The skew-second Zagreb matrix of $K^{\sigma}_{m,n}$ is given by

$$A = \begin{bmatrix} 0 & 0 & \cdots & mn & mn \\ 0 & 0 & \cdots & mn & mn \\ 0 & 0 & \cdots & mn & mn \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ -mn & -mn & \cdots & 0 & 0 \\ -mn & -mn & \cdots & 0 & 0 \\ -mn & -mn & \cdots & 0 & 0 \end{bmatrix}$$

Characteristic polynomial is given by

$$\lambda^{m+n-2}(\lambda^2 + (mn)^3) = 0$$

and the spectrum is

$$spec(K^{\sigma}_{m,n}) = \begin{pmatrix} i(mn)^{\frac{3}{2}} & -i(mn)^{\frac{3}{2}} & 0 \\ 1 & 1 & m+n-2 \end{pmatrix}$$

Hence,

$$SFSE(K^{\sigma}_{m,n}) = 2(mn)^{\frac{3}{2}}.$$

□

## 3. Skew Zagreb energies of complements of Digraphs

**Theorem 3.1.** *Let $V(\overline{K^{\sigma}_n}) = \{v_1, ..., v_n\}$ and $\Gamma(\overline{K^{\sigma}_n}) = \{(v_i, v_j) \mid 1 \leq i, j \leq n\}$ be vertex set and arc set of $\overline{K^{\sigma}_n}$ respectively. Then,*

$$SFZE(\overline{K^{\sigma}_n}) = 0.$$

*Proof.* It follows from the fact that the complement of complete digraph is a null graph. □

**Theorem 3.2.** *Let $V(\overline{K^{\sigma}_n}) = \{v_1, ..., v_n\}$ and $\Gamma(\overline{K^{\sigma}_n}) = \{(v_i, v_j) \mid 1 \leq i, j \leq n\}$ be vertex set and arc set of $\overline{K_n}$ respectively. Then,*

$$SSZE(\overline{K^{\sigma}_n}) = 0.$$

**Theorem 3.3.** *Let $V(\overline{S^{\sigma}_n}) = \{v_1, ..., v_n\}$ and $\Gamma(\overline{S^{\sigma}_n}) = \{(v_1, v_j) \mid 2 \leq j \leq n\}$ be vertex set and arc set of $\overline{S^{\sigma}_n}$ respectively. Then,*

$$SFZE(\overline{S^{\sigma}_n}) = (2n-4)\sum_{k=0}^{n-2} cot(2k+1)\frac{\pi}{2n}, \quad n \geq 2.$$

*Proof.* The skew-first Zagreb matrix of $\overline{S^{\sigma}_n}$ is given by

$$A = \begin{bmatrix} 0 & 0 & 0 & \cdots & 0 & 0 \\ 0 & 0 & (2n-4) & \cdots & (2n-4) & (2n-4) \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & -(2n-4) & -(2n-4) & \cdots & 0 & (2n-4) \\ 0 & -(2n-4) & -(2n-4) & \cdots & -(2n-4) & 0 \end{bmatrix}$$

Thus, the spectrum is

$$spec(\overline{S^{\sigma}_n}) = \begin{pmatrix} 0 & (2n-4)icot(\frac{\pi}{2n}) & (2n-4)icot(\frac{3\pi}{2n}) & \cdots & (2n-4)icot(\frac{2n-3}{2n})\pi \\ 1 & 1 & 1 & \cdots & 1 \end{pmatrix}$$

Hence,

$$SFZE(\overline{S^{\sigma}_n}) = (2n-4)\sum_{k=0}^{n-2} cot(2k+1)\frac{\pi}{2n}, \quad n \geq 2.$$

□

Similarly, we have the following:

**Theorem 3.4.** *Let $V(\overline{S_n^\sigma}) = \{v_1, ..., v_n\}$ and $\Gamma(\overline{S_n^\sigma}) = \{(v_1, v_j) \mid 2 \leq j \leq n\}$ be vertex set and arc set of $\overline{S_n^\sigma}$ respectively. Then,*

$$SSZE(\overline{S_n^\sigma}) = (n-2)^2 \sum_{k=0}^{n-2} cot(2k+1)\frac{\pi}{2n}, \quad n \geq 2.$$

*Proof.* The skew-second Zagreb matrix of $\overline{S_n^\sigma}$ is given by

$$A = \begin{bmatrix} 0 & 0 & 0 & \cdots & 0 & 0 & 0 \\ 0 & 0 & (n-2)^2 & \cdots & (n-2)^2 & (n-2)^2 & (n-2)^2 \\ 0 & -(n-2)^2 & 0 & \cdots & (n-2)^2 & (n-2)^2 & (n-2)^2 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ 0 & -(n-2)^2 & -(n-2)^2 & \cdots & -(n-2)^2 & 0 & (n-2)^2 \\ 0 & -(n-2)^2 & -(n-2)^2 & \cdots & -(n-2)^2 & -(n-2)^2 & 0 \end{bmatrix}$$

Thus, the spectrum is

$$spec(\overline{S_n^\sigma}) = \begin{pmatrix} 0 & i(n-2)^2cot(\frac{\pi}{2n}) & i(n-2)^2cot(\frac{3\pi}{2n}) & \cdots & i(n-2)^2cot(\frac{2n-3}{2n})\pi \\ 1 & 1 & 1 & \cdots & 1 \end{pmatrix}$$

Hence,

$$SSZE(\overline{S_n^\sigma}) = (n-2)^2 \sum_{k=0}^{n-2} cot(2k+1)\frac{\pi}{2n}, \quad n \geq 2.$$

□

Similarly, we obtain the following results:

**Theorem 3.5.** *Let $V(\overline{H_{2n}^\sigma}) = \{u_1, ..., u_n, v_1, ..., v_n\}$ and $\Gamma(\overline{H_{2n}^\sigma}) = \{(u_i, v_j) \mid 1 \leq i, j \leq n\}$ be vertex set and arc set of complement of the hyperoctahedral digraph $\overline{H_{2n}^\sigma}$ respectively. Then*

$$SFZE(\overline{H_{2n}^\sigma}) = 4n.$$

**Theorem 3.6.** *Let $V(\overline{H_{2n}^\sigma}) = \{u_1, ..., u_n, v_1, ..., v_n\}$ and $\Gamma(\overline{H_{2n}^\sigma}) = \{(u_i, v_j) \mid 1 \leq i, j \leq n\}$ be vertex set and arc set of complement of the hyperoctahedral digraph $\overline{H_{2n}^\sigma}$ respectively. Then*

$$SSZE(\overline{H_{2n}^\sigma}) = n.$$

**Theorem 3.7.** *Let $V(\overline{K_{m,n}^\sigma}) = \{u_1, ..., u_m, v_1, ..., v_n\}$ and $\Gamma(\overline{K_{m,n}^\sigma}) = \{(u_i, u_j), (v_i, v_j) \mid 1 \leq i \leq m, 1 \leq j \leq n\}$ be vertex set and arc set of complement of complete bipartite digraph $\overline{K_{m,n}^\sigma}$ respectively. Then,*

$$SFZE(\overline{K_{m,n}^\sigma}) = 2(m-1) \sum_{k=0}^{m-1} cot(2k+1)\frac{\pi}{2m} + 2(n-1) \sum_{k=0}^{n-1} cot(2k+1)\frac{\pi}{2n}.$$

**Theorem 3.8.** *Let* $V(\overline{K^{\sigma}_{m,n}}) = \{u_1, ..., u_m, v_1, ..., v_n\}$ *and* $\Gamma(\overline{K^{\sigma}_{m,n}}) = \{(u_i, u_j), (v_i, v_j) \mid 1 \leq i \leq m, 1 \leq j \leq n\}$ *be vertex set and arc set of complete bipartite digraph* $\overline{K^{\sigma}_{m,n}}$ *respectively. Then,*

$$SSZE(\overline{K^{\sigma}_{m,n}}) = (m-1)^2 \sum_{k=0}^{m-1} cot(2k+1)\frac{\pi}{2m} + (n-1)^2 \sum_{k=0}^{n-1} cot(2k+1)\frac{\pi}{2n}.$$

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## References


[1] C. Adiga, R. Balakrishnan and Wasin So, The skew energy of a digraph, *Linear Algebra Appl.*, 432(7) (2010), 1825-1835.

[2] I. Gutman, Degree-Based Topological Indices, *Croat. Chem. Acta*, 86 (4) (2013), 351-361.

[3] R. A. Horn and C. R. Johnson, ***Topics in Matrix Analysis***, Cambridge Univ. Press, Cambridge, 1991.

[4] V. Lokesha and Y. Shanthakumari, Skew-SDD and Skew-ISI energies of some digraphs, Communicated for publication.

[5] K. N. Prakasha, P. Siva Kota Reddy and Ismail Naci Cangul, Symmetric division deg energy of a graph, *Turkish Journal of Analysis and Number Theory*, 5(6) (2017), 202-209.

[6] N. J Rad, A. Jahanbani and I. Gutman, Zagreb Energy and Zagreb Estrada Index of Graphs, *MATCH Commun. Math. Comput. Chem.*, 79 (2018), 371-386.

[7] Samir K. Vaidya and Kalpesh M. Popat, Some New Results on Energy of Graphs, *MATCH Commun. Math. Comput. Chem.*, 77 (2017), 589-594.

[8] B. Shader and Wasin So, Skew spectra of oriented graphs, *Electron. J. Comb.*, 16(2009), #N32

[9] G. H. Shirdel, H. Rezapour and A. M. Sayadi, The hyper-Zagreb Index of graph operations, *Iran. J. Math. Chem.*, 4(2) (2013), 213-220.



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