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Integrity of Quasi-Total Graphs

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Abstract: A communication network can be considered to be highly vulnerable to disruption if the failure of few members can result in no member's being able to communicate with very many others. This idea suggests the concept of the integrity of a graph. The integrity $I(G)$ of a graph G is defined as $I(G) = \min_{S \subset V(G)} \{|S| + m(G-S)\}$ where $m(G-S)$ denotes the order of the largest component of $G-S$. In this paper, we obtain the integrity of quasi-total graph of some standard graph families and combinations of these graphs. Further, we establish some relations connecting the integrity of some graph families and integrity of their quasi-total graphs.

Keywords: Vulnerability, Connectivity, Integrity, quasi-total graph.
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1 INTRODUCTION

In communication networks, we require greater degrees of stability or less vulnerability. The vulnerability measures resistance of the network to the disruption in operation after the failure of certain stations or communication links. The stability of a communication network is of prime importance to network designers. As the network starts losing links or nodes, ultimately there is a loss in its efficiency. Thus, communication networks must be assembled to be as stable as possible, not only with respect to the initial interruption, but

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also with respect to the possible reconstruction of the network. A communication network can be represented as an undirected graph. Tree, mesh, hypercube and star graph are most popular communication networks. If we model a network through a graph, then there are many graph theoretical parameters to describe the stability of communication networks. Most notably, the vertex-connectivity and edge-connectivity have been frequently used. The best known measure of reliability of a graph is its vertex-connectivity $\kappa(G)$ defined to be the minimum number of vertices whose removal results in a disconnected or trivial graph. The difficulty with these parameters is that they do not consider what remains after the graph is disconnected. Consequently, a number of other parameters have recently been introduced in order to survive with this difficulty. The connectivity of two different graphs may be same, but the orders of their largest components need not be same. Then they may differ with respect to stability. Well, how can we measure this property? The idea behind the answer is the concept of integrity, which is different from connectivity.

The concept of integrity was introduced as a measure of graph vulnerability. The integrity of a graph G is

$$I(G) = \min_{S \subset V(G)} \{|S| + m(G-S)\},$$

where $m(G-S)$ denotes the order of a largest component of $G-S$. This concept was introduced by Barefoot et al., in [3], who discovered many of the early results on the subject [1–4, 8–10].

The order of a graph G, that is the number of vertices, will generally be denoted by n. As usual $V(G)$ and $E(G)$ will respectively denote the sets of vertices and edges of G, and S will denote a proper subset of V. As noted earlier, $m(G-S)$ equals to the order of the largest component among all components of $G-S$. In this note, we consider nontrivial finite undirected graphs with no loops nor multiple edges, and we denote by P_n, C_n, K_n, $K_{a,b}$, $K_{1,n}$ and $W_{1,n}$ the path, cycle, complete graph, complete bipartite graph, star graph and wheel graph, respectively. The symbol $\lceil x \rceil$ denotes the smallest integer that is greater than or equal to x, $\lfloor x \rfloor$ denotes the greatest integer that is smaller than or equal to x and the absolute value of x is denoted by $|x|$. The basic graph theoretic terminologies and notations can be found in [7, 11, 13].

The following Theorem 1.1 is useful to prove our results.

Theorem 1.1. *[1] The integrity of*

(a) complete graph K_n *is* $I(K_n) = n$,

(b) the null graph is $I(\overline{K_n}) = 1$,

(c) the star graph is $I(K_{1,n}) = 2$,

(d) the path graph is $I(P_n) = \lceil 2\sqrt{n+1} \rceil - 2$,

(e) the cycle graph is $I(C_n) = \lceil 2\sqrt{n} \rceil - 1$,

(f) the complete bipartite graph is $I(K_{a,b}) = 1 + \min\{a, b\}$.

In the next section, we formulize integrity of quasi-total graph of some basic graph families and we establish some relations connecting the integrity of some graph families and integrity of their quasi-total graphs. Finally, we find integrity of quasi-total graph of graphs that are obtained from cartesian product as well as corona of some graphs.

2 Integrity of quasi-total graph of some basic graph families

In this section, firstly we define quasi-total graph of a graph. Then we obtain the integrity of quasi-total graph of some basic graph families.

Definition 1. *[14] The quasi-total graph $P(G)$ of a graph G is the graph whose vertex set is $V(G) \cup E(G)$ and two vertices in $P(G)$ are adjacent if and only if they correspond to two non-adjacent vertices of G or to two adjacent edges of G or to a vertex and an edge incident with it, in G (see Fig. 1).*

The details of quasi-total graph can be found in [5, 6, 12, 14]. In figures, the dark circles denote the vertices of G while light circles denote the edges of G.

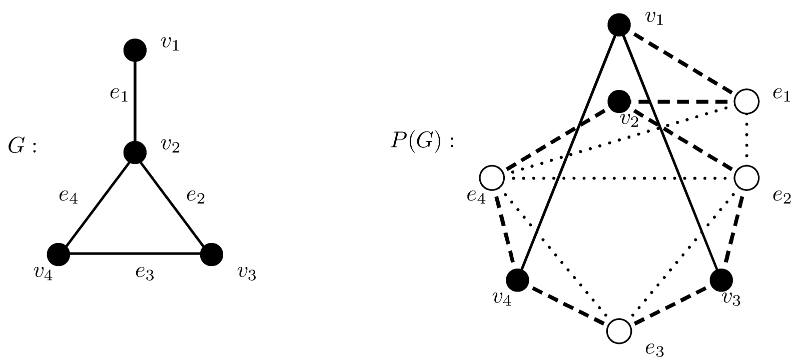

Figure 1: The graph G and its quasi-total graph

Example 1. *Let $P(P_6)$ be the quasi-total graph of the path graph P_6. Then $I(P(P_6)) = 8$.*

Illustration. Let $S \subset V(P(P_6))$. There are three cases to choose the set S (as shown in Fig. 2).

Case 1. In Fig. 2 (a), if we choose the set $S = \{a_1, a_2, a_3, a_4\}$, then we have $m(P(P_6) - S) = 6$. So $I(P(P_6)) = 4 + 6 = 10$.

Case 2. In Fig. 2 (b), if we choose the set $S = \{a_1, a_2, a_3, a_4, a_5, a_6\}$, then we have $m(P(P_6) - S) = 3$. So $I(P(P_6)) = 6 + 3 = 9$.

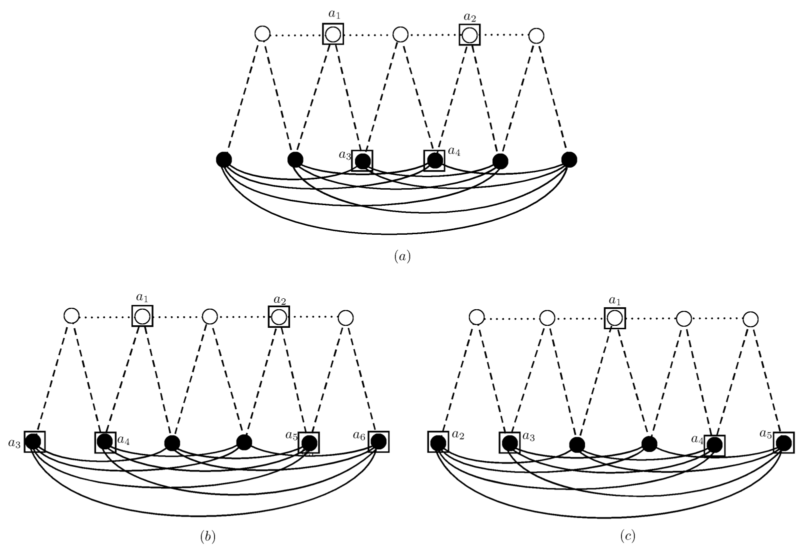

Figure 2: Quasi-total graph of P_6.

Case 3. In Fig. 2 (c), if we choose the set $S = \{a_1, a_2, a_3, a_4, a_5\}$, then we have $m(P(P_6)-S) = 3$. So $I(P(P_6)) = 5 + 3 = 8$.

Theorem 2.1. *Let $P(P_n)$ be the quasi-total graph of a path graph $P_n, (n \geq 3)$ with n vertices. Then*

$$I(P(P_n)) = n + \left\lceil \frac{n-1}{3} \right\rceil.$$

Proof. Let $S_I = V_I \cup E_I$ and $|S_I| = n + \lceil \frac{n-1}{3} \rceil - 2$, where $|V_I| = n-2, |E_I| = \lceil \frac{n-1}{3} \rceil$. The set $V_I \subset V(P_n)$ contains both terminal vertices of P_n such that $V(P_n) \setminus V_I$ contains two adjacent vertices of P_n say u and v, while the set $E_I \subset E(P_n)$ is chosen in such a way that it must contain the edge uv and two edges which are incident with the open neighbours of u and v in $V(P_n) \setminus \{u, v\}$, also note that, the distance between two members of E_I must be greater than or equal to two. Then clearly, S_I will form a subset of $V(P(P_n))$ such that

$$|S_I| + m(P(P_n) - S_I) = \min_{S \subset V(P(P_n))} \{|S| + m(P(P_n) - S)\},$$

where $m(P(P_n) - S_I) = \begin{cases} 3 & if\ n = 5, 6, \\ 2 & otherwise. \end{cases}$

Thus,

$$\begin{aligned} I(P(P_n)) &= |S_I| + m(P(P_n) - S_I) \\ &= n + \left\lceil \frac{n-1}{3} \right\rceil - 2 + 2 \\ &= n + \left\lceil \frac{n-1}{3} \right\rceil. \end{aligned}$$

□

Corollary 2.2. *Let P_n and P_m be two path graphs of order n and m, respectively. Then for $n, m \geq 3$, $I(P_n) = I(P(P_m))$ if and only if $\lceil 2\sqrt{n+1} \rceil = m + \lceil \frac{m-1}{3} \rceil + 2$.*

Example 2. *Let $P(C_5)$ be the quasi-total graph of a cycle graph C_5. Then $I(P(C_5)) = 8$.*

Illustration. Let $S \subset V(P(C_5))$. There are three cases to choose the set S (as shown in Fig. 3).
Case 1. In Fig. 3 (a), if we choose the set $S = \{a_1, a_2, a_3, a_4, a_5\}$, then we have $m(P(C_5) - S) = 5$. So $I(P(C_5)) = 5 + 5 = 10$.
Case 2. In Fig. 3 (b), if we choose the set $S = \{a_1, a_2, a_3, a_4, a_5, a_6\}$, then we have $m(P(C_5) - S) = 2$. So $I(P(C_5)) = 6 + 2 = 8$.
Case 3. In Fig. 3 (c), if we choose the set $S = \{a_1, a_2, a_3, a_4, a_5\}$, then we have $m(P(C_5) - S) = 4$. So $I(P(C_5)) = 5 + 4 = 9$.

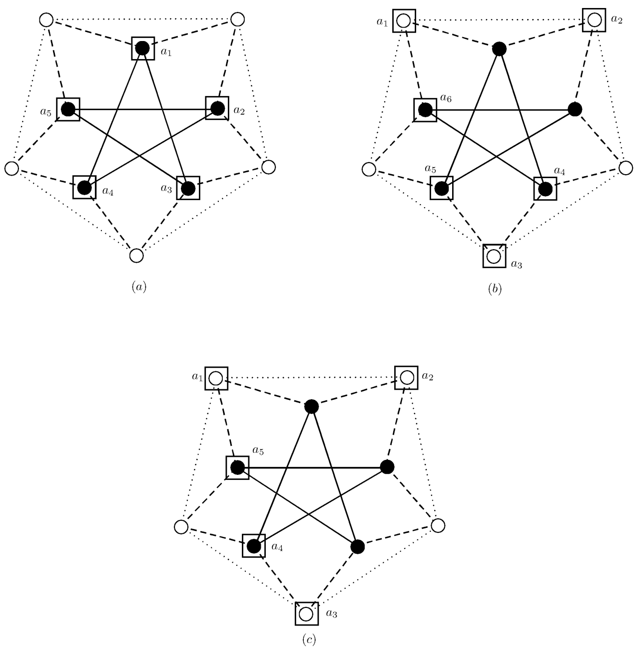

Figure 3: Quasi-total graph of C_5.

Theorem 2.3. *Let $P(C_n)$ be the quasi-total graph of a cycle C_n with n vertices. Then*

$$I(P(C_n)) = \begin{cases} 4 & \text{if } n = 3, \\ n + \lceil 2\sqrt{n+2} \rceil - 3 & \text{otherwise.} \end{cases}$$

Proof. If $n = 3$, then by removing three vertices (corresponding to the edges of C_3) from $P(C_3)$, we get $P(C_3) - S = 3K_1$ which yields $I(P(C_3)) = 4$.

If $n \geq 4$, then let $S_I = V_I \cup E_I$ and $|S_I| = n + \lceil 2\sqrt{n+2} \rceil - 5$, where $|V_I| = n - 2$, $|E_I| = \lceil 2\sqrt{n+2} \rceil - 3$. The set $V_I \subset V(C_n)$ such that $V(C_n) \setminus V_I$ contains two adjacent vertices of C_n say u and v, while the set $E_I \subset E(C_n)$ is chosen in such a way that, it must contain the edge uv and the distance between two members of $E_I \setminus \{uv\}$ must be greater than or equal to two. Then clearly, S_I will form a subset of $V(P(C_n))$ such that

$$|S_I| + m(P(C_n) - S_I) = \min_{S \subset V(P(C_n))} \{|S| + m(P(C_n) - S)\},$$

where $m(P(C_n) - S_I) = 2$. Thus,

$$\begin{aligned} I(P(C_n)) &= |S_I| + m(P(C_n) - S_I) \\ &= n + \left\lceil 2\sqrt{n+2} \right\rceil - 5 + 2 \\ &= n + \left\lceil 2\sqrt{n+2} \right\rceil - 3. \end{aligned}$$

□

Corollary 2.4. *Let $P(C_n), n \geq 4$, be the quasi-total graph of a cycle C_n with n vertices. Then $I(P(C_n)) = n + I(C_{n+2}) - 2$.*

Corollary 2.5. *Let C_n and C_m be two cycle graphs of orders n and m, respectively. Then $I(C_n) = I(P(C_m))$ if and only if $\lceil 2\sqrt{n} \rceil = m + \left\lceil \sqrt{m+2} \right\rceil - 4$.*

Example 3. *Let $P(K_4)$ be the quasi-total graph of a complete graph K_4. Then $I(P(K_4)) = 7$.*

Illustration. Let $S \subset V(P(K_4))$. There are three cases to choose the set S (as shown in Fig. 4).
Case 1. In Fig. 4 (a), if we choose the set $S = \{a_1, a_2, a_3, a_4, a_5, a_6\}$, then we have $m(P(K_4) - S) = 1$. So $I(P(K_4)) = 6 + 1 = 7$.
Case 2. In Fig. 4 (b), if we choose the set $S = \{a_1, a_2, a_3, a_4, a_5, a_6\}$, then we have $m(P(K_4) - S) = 2$. So $I(P(K_4)) = 6 + 2 = 8$.
Case 3. In Fig. 4 (c), if we choose the set $S = \{a_1, a_2, a_3, a_4\}$, then we have $m(P(K_4) - S) = 6$. So $I(P(K_4)) = 4 + 6 = 10$.

Theorem 2.6. *Let $P(K_n)$ be the quasi-total graph of a complete graph K_n with n vertices. Then*

$$I(P(K_n)) = \frac{n(n-1)}{2} + 1.$$

Proof. The number of edges in K_n is $\frac{n(n-1)}{2}$. Therefore, the number of vertices in $P(K_n)$ is $n + \frac{n(n-1)}{2}$. If we consider $S = E(K_n)$, then $|S| = \frac{n(n-1)}{2}$ and $m(P(K_n) - S) = 1$ as $P(K_n) - S = \overline{K_n}$. Thus, $I(P(K_n)) = \frac{n(n-1)}{2} + 1$. □

Corollary 2.7. *Let K_n and K_m be two complete graphs of orders n and m, respectively. Then $I(K_n) = I(P(K_m))$ if and only if $2(n-1) = m(m-1)$.*

Example 4. *Let $P(K_{1,4})$ be the quasi-total graph of a star graph $K_{1,4}$. Then $I(P(K_{1,4})) = 7$.*

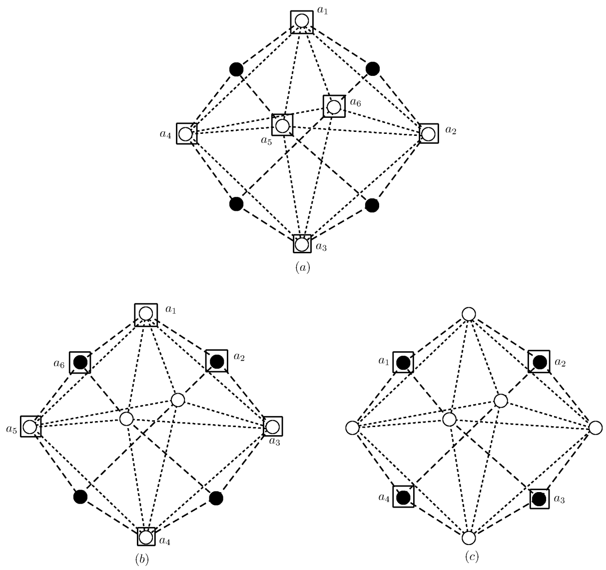

Figure 4: Quasi-total graph of K_4.

Illustration. Let $S \subset V(P(K_{1,4}))$. There are three cases to choose the set S (as shown in Fig. 5).

Case 1. In Fig. 5 (a), if we choose the set $S = \{a_1, a_2, a_3, a_4, a_5\}$, then we have $m(P(K_{1,4}) - S) = 3$. So $I(P(K_{1,4})) = 5 + 3 = 8$.

Case 2. In Fig. 5 (b), if we choose the set $S = \{a_1, a_2, a_3, a_4\}$, then we have $m(P(K_{1,4}) - S) = 4$. So $I(P(K_{1,4})) = 4 + 4 = 8$.

Case 3. In Fig. 5 (c), if we choose the set $S = \{a_1, a_2, a_3, a_4\}$, then we have $m(P(K_{1,4}) - S) = 3$. So $I(P(K_{1,4})) = 4 + 3 = 7$.

Theorem 2.8. *Let $P(K_{1,n})$ be the quasi-total graph of a star graph $K_{1,n}$ with $n+1$ vertices. Then*

$$I(P(K_{1,n})) = n + \left\lceil \frac{n+1}{2} \right\rceil.$$

Proof. The number of edges in $K_{1,n}$ is n. Therefore, the number of vertices in $P(K_{1,n})$ is $2n+1$. If we consider $S = V_I \cup E_I$ and $S \subset V(P(K_{1,n}))$ with $|S| = n$, where $|V_I| = n - \lceil \frac{n+1}{2} \rceil$, $|E_I| = \lceil \frac{n+1}{2} \rceil$. The set $V_I \subset V(K_{1,n})$ and $E_I \subset E(K_{1,n})$ such that all members of E_I are not adjacent to any member of V_I in $P(K_{1,n})$. Clearly, any graph $P(K_{1,n})$ contains two complete graphs K_n and K_{n+1}.

If the members of S are removed from $P(K_{1,n})$, then $P(K_{1,n}) - S$ has two components $K_{\lceil \frac{n+1}{2} \rceil}$ and $K_{n - \lceil \frac{n+1}{2} \rceil + 1}$. Since $m(G-S)$ is the order of the largest component of $G - S$. Therefore, $m(P(K_{1,n}) - S) = \lceil \frac{n+1}{2} \rceil$ as $\lceil \frac{n+1}{2} \rceil \geq n - \lceil \frac{n+1}{2} \rceil + 1$. Thus, $I(P(K_{1,n})) = n + \lceil \frac{n+1}{2} \rceil$. □

Corollary 2.9. *The integrity of a star graph never equals to the integrity of the quasi-total graph of a star graph except for $n = 1$.*

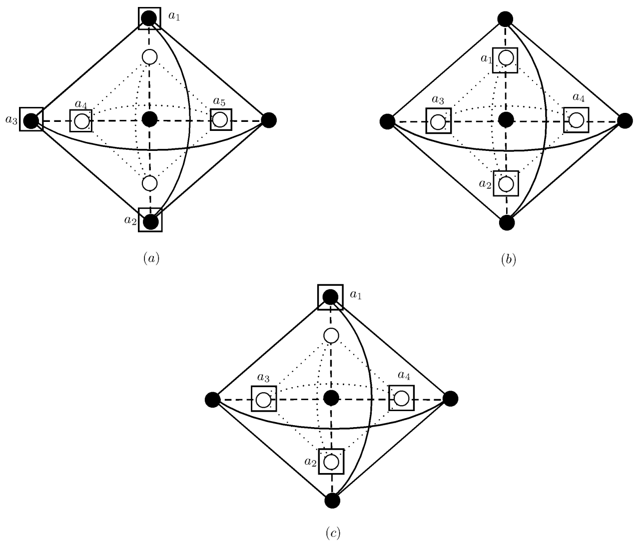

Figure 5: Quasi-total graph of $K_{1,4}$.

Theorem 2.10. *Let $P(K_{a,b})$ be the quasi-total graph of a complete bipartite graph $K_{a,b}$ with $a+b$ vertices. Then*

$$I(P(K_{a,b})) = ab + \max\{a, b\}.$$

Proof. The number of edges in $K_{a,b}$ is ab. Therefore, the number of vertices in $P(K_{a,b})$ is $a+b+ab$. If we consider $S = E(K_{a,b})$ and $S \subset V(P(K_{a,b}))$ with $|S| = ab$. If the members of S are removed from $P(K_{a,b})$, then $P(K_{a,b}) - S = K_a \cup K_b$. Since $m(G-S)$ is the order of the largest component of $G-S$. Therefore, $m(P(K_{a,b}) - S) = \max\{a, b\}$. Thus, $I(P(K_{a,b})) = ab + \max\{a, b\}$. □

Corollary 2.11. *Let $K_{a,b}$ and $K_{c,d}$ be two complete bipartite graphs of orders $a+b$ and $c+d$, respectively. Then $I(K_{a,b}) = I(P(K_{c,d}))$ if and only if $\min\{a, b\} = cd-1+\max\{c, d\}$.*

Example 5. *Let $P(W_{1,5})$ be the quasi-total graph of a wheel graph $W_{1,5}$. Then $I(P(W_{1,5})) =$* 14.

Illustration: Let $S \subset V(P(W_{1,5}))$. There are three cases to choose the set S (as shown in Fig. 7).
Case 1. In Fig. 7 (a), if we choose the set $S = \{a_1, a_2, \cdots, a_9\}$, then we have $m(P(W_{1,5})-S) = 7$. So $I(P(W_{1,5})) = 9 + 7 = 16$.
Case 2. In Fig. 7 (b), if we choose the set $S = \{a_1, a_2, \cdots, a_{10}\}$, then we have $m(P(W_{1,5}) - S) = 5$. So $I(P(W_{1,5})) = 10 + 5 = 15$.
Case 3. In Fig. 7 (c), if we choose the set $S = \{a_1, a_2, \cdots, a_{12}\}$, then we have $m(P(W_{1,5}) - S) = 2$. So $I(P(W_{1,5})) = 12 + 2 = 14$.

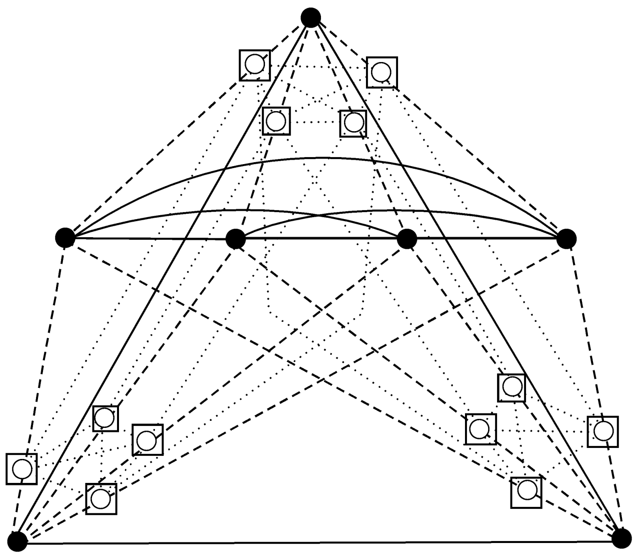

Figure 6: Quasi-total graph of $K_{3,4}$.

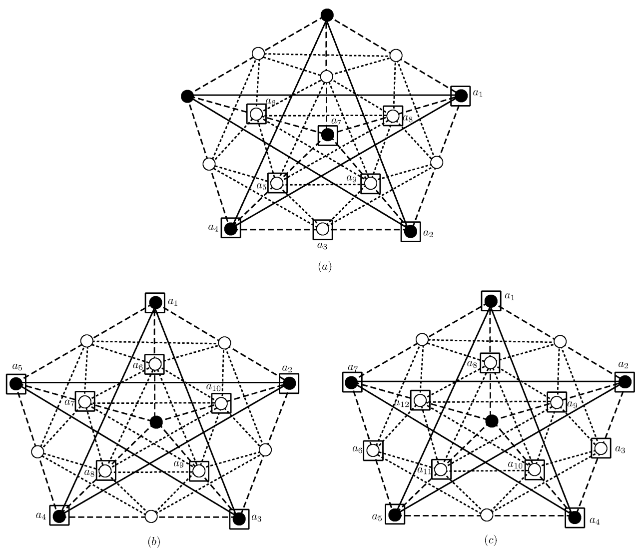

Figure 7: Quasi-total graph of $W_{1,5}$.

Theorem 2.12. *Let $P(W_{1,n})$ be the quasi-total graph of a wheel graph $W_{1,n}$ with $n+1$ vertices. Then*

$$I(P(W_{1,n})) = \begin{cases} 7 & if\ n = 3, \\ 2n + \lceil 2\sqrt{n} \rceil - 1 & otherwise. \end{cases}$$

Proof. The number of edges in $W_{1,n}$ is $2n$. Therefore, the number of vertices in $P(W_{1,n})$

is $3n+1$. If $n=3$, then $W_{1,3}=K_4$. Therefore, from Theorem 2.6, we can have $I(P(W_{1,3}))=7$.

Suppose $n \geq 4$. Consider $S = V_I \cup E_I$ and $S \subset V(P(W_{1,n}))$ with $|S| = 2n$, where $|V_I| = n$, $|E_I| = n$. The set $V_I = V(W_{1,n})$ and $E_I \subset E(W_{1,n})$ such that all members of E_I are incident with a vertex of degree n in $W_{1,n}$. If the members of S are removed from $P(W_{1,n})$, then $P(W_{1,n}) - S = C_n$. Thus,

$$\begin{aligned} I(P(W_{1,n})) &= 2n + I(C_n) \\ &= 2n + \lceil 2\sqrt{n} \rceil - 1. \end{aligned}$$

□

Corollary 2.13. *Let $P(W_{1,n}), n \geq 4$, be the quasi-total graph of a wheel graph $W_{1,n}$ with $n+1$ vertices. Then $I(P(W_{1,n})) = 2n + I(C_n)$.*

Corollary 2.14. *Let $W_{1,n}$ and $W_{1,m}$ be two wheel graphs of orders $n+1$ and $m+1$, respectively. Then $I(W_{1,n}) = I(P(W_{1,m}))$ if and only if $\lceil 2\sqrt{n} \rceil = 2m + \lceil 2\sqrt{m} \rceil - 1$.*

Definition 2. *The product $G \times H$ of graphs G (which has n_1 vertices, m_1 edges) and H (which has n_2 vertices, m_2 edges) has the vertex set $V(G \times H) = V(G) \times V(H)$ and $(a,x)(b,y)$ is an edge of $G \times H$ if and only if [$a = b$ and $xy \in E(H)$] or [$x = y$ and $ab \in E(G)$], [11]. It follows from the definition of product that $G \times H$ has $n_1 n_2$ vertices and $n_1 m_2 + n_2 m_1$ edges.*

Theorem 2.15. *Let $P(K_2 \times P_n)$ be the quasi-total graph of $K_2 \times P_n$ with $2n$ vertices. Then*

$$I(P(K_2 \times P_n)) = 3n - 2 + 2\left\lfloor \frac{n}{4} \right\rfloor + \left\lceil \frac{n - \lfloor \frac{n}{4} \rfloor}{\lfloor \frac{n}{4} \rfloor + 1} \right\rceil.$$

Proof. If we consider $S = V_I \cup E_I$ and $S \subset V(P(K_2 \times P_n))$ with $|S| = 3n-2+2\left\lfloor \frac{n}{4} \right\rfloor$, where $V_I \subset V(P(K_2 \times P_n)), |V_I| = 2(n-1)$ and $E_I \subset E(P(K_2 \times P_n)), |E_I| = n + 2\left\lfloor \frac{n}{4} \right\rfloor$. If the members of V_I and n members of E_I are removed from $P(K_2 \times P_n)$, then we remain with $2P_n$ and $m(P(K_2 \times P_n) - S) = \left\lceil \frac{n - \lfloor \frac{n}{4} \rfloor}{\lfloor \frac{n}{4} \rfloor + 1} \right\rceil$. Thus, $I(P(K_2 \times P_n)) = 3n-2+2\left\lfloor \frac{n}{4} \right\rfloor + \left\lceil \frac{n - \lfloor \frac{n}{4} \rfloor}{\lfloor \frac{n}{4} \rfloor + 1} \right\rceil$. An example is shown in Fig. 8. □

Theorem 2.16. *Let $P(K_2 \times C_n)$ be the quasi-total graph of $K_2 \times C_n$ with $2n$ vertices. Then*

$$I(P(K_2 \times C_n)) = 3n + 2\left\lfloor \frac{n}{4} \right\rfloor + \left\lceil \frac{n - \lfloor \frac{n}{4} \rfloor}{\lfloor \frac{n}{4} \rfloor + 1} \right\rceil.$$

Proof. If we consider $S = V_I \cup E_I$ and $S \subset V(P(K_2 \times C_n))$ with $|S| = 3n-2+2\left\lfloor \frac{n}{4} \right\rfloor$, where $V_I \subset V(P(K_2 \times C_n)), |V_I| = 2(n-1)$ and $E_I \subset E(P(K_2 \times C_n)), |E_I| = n+2+2\left\lfloor \frac{n}{4} \right\rfloor$. If the members of V_I and $n+2$ members of E_I are removed from $P(K_2 \times P_n)$, then we remain with $2P_n$ and $m(P(K_2 \times C_n) - S) = \left\lceil \frac{n - \lfloor \frac{n}{4} \rfloor}{\lfloor \frac{n}{4} \rfloor + 1} \right\rceil$. Thus, $I(P(K_2 \times C_n)) = 3n+2\left\lfloor \frac{n}{4} \right\rfloor + \left\lceil \frac{n - \lfloor \frac{n}{4} \rfloor}{\lfloor \frac{n}{4} \rfloor + 1} \right\rceil$. □

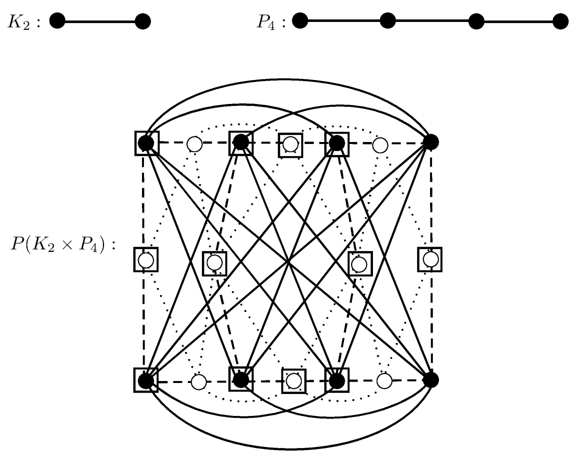

Figure 8: Quasi-total graph of $K_2 \times P_4$.

Theorem 2.17. *Let $P(K_2 \times K_{1,n})$ be the quasi-total graph of $K_2 \times K_{1,n}$ with $2(n+1)$ vertices. Then*

$$I(P(K_2 \times K_{1,n})) = 4n - 2.$$

Proof. If we consider $S = V_I \cup E_I$ and $S \subset V(P(K_2 \times K_{1,n}))$ with $|S| = 3n-2$, where $V_I \subset V(P(K_2 \times K_{1,n})), |V_I| = 2(n-1)$ and $E_I \subset E(P(K_2 \times K_{1,n})), |E_I| = n$. If the members of S are removed from $P(K_2 \times K_{1,n})$, then $P(K_2 \times K_{1,n}) - S$ has two components of $K_2 \cdot K_n$. Thus, $I(P(K_2 \times K_{1,n})) = 3n-2+n = 4n-2$. An example is shown in Figure 9. □

Definition 3. *The corona $G_1 \circ G_2$ of two graphs G_1 and G_2 is the graph obtained by taking one copy of G_1 (which has n_1 vertices, m_1 edges) and n_1 copies of G_2 (which has n_2 vertices, m_2 edges) and then joining the i^{th} vertex of G_1 to every vertex in the i^{th} copy of G_2, [11] . It follows from the definition of corona that $G_1 \circ G_2$ has $n_1(1+n_2)$ vertices and $m_1 + n_1 m_2 + n_1 n_2$ edges.*

Theorem 2.18. *Let $P(K_2 \circ P_n)$ be the quasi-total graph of $K_2 \circ P_n$ with $2(n+1)$ vertices. Then*

$$I(P(K_2 \circ P_n)) = \begin{cases} 2n + \lceil \frac{n-1}{3} \rceil + 2 & if\ 4 \le n \le 6, \\ 2\left(n + \lceil \frac{n-1}{3} \rceil\right) + 1 & otherwise, \end{cases}$$
$$= \begin{cases} n + 2 + I(P(P_n)) & if\ 4 \le n \le 6, \\ 2I(P(P_n)) + 1 & otherwise. \end{cases}$$

Proof. If $4 \le n \le 6$, then consider $S = V_I \cup E_I$ and $S \subset V(P(K_2 \circ P_n))$ with $|S| = 2n + \lceil \frac{n-1}{3} \rceil - 1$, where $V_I \subset V(P(K_2 \circ P_n)), |V_I| = 2(n-1)$ and $E_I \subset E(P(K_2 \circ P_n)), |E_I| = \lfloor \frac{n-1}{3} \rfloor + 1$. If the members of S are taken as given in Theorem 2.1, and removed from $P(K_2 \circ P_n)$, then $m(P(K_2 \circ P_n) - S) = 3$. The set S considered above

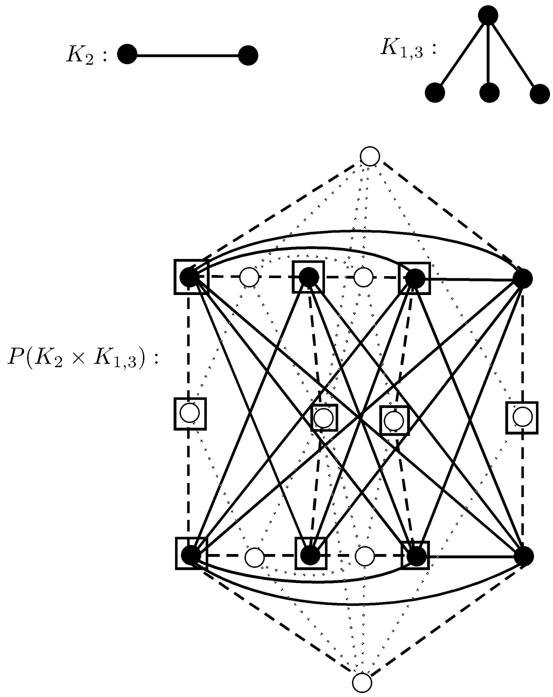

Figure 9: Quasi-total graph of $K_2 \times K_{1,3}$.

is the smallest subset of $V(P(K_2 \circ P_n))$ which yeilds the integrity of $P(K_2 \circ P_n)$. Thus, $I(P(K_2 \circ P_n)) = 2n + \lceil \frac{n-1}{3} \rceil + 2$.

Suppose $n \geq 7$. Then consider $S = V_I \cup E_I$ and $S \subset V(P(K_2 \circ P_n))$ with $|S| = 2\left(n + \lceil \frac{n-1}{3} \rceil\right) - 1$, where $V_I \subset V(P(K_2 \circ P_n)), |V_I| = 2n-1$ and $E_I \subset E(P(K_2 \circ P_n)), |E_I| = 2\left\lfloor \frac{n-1}{3} \right\rfloor$. If the members of S are taken as given in Theorem 2.1, and removed from $P(K_2 \circ P_n)$, then $m(P(K_2 \circ P_n) - S) = 2$. The set S considered above is the smallest subset of $V(P(K_2 \circ P_n))$ which yeilds the integrity of $P(K_2 \circ P_n)$. Thus, $I(P(K_2 \circ P_n)) = 2\left(n + \lceil \frac{n-1}{3} \rceil\right) + 1$. □

Theorem 2.19. *Let $P(K_2 \circ C_n)$ be the quasi-total graph of $K_2 \circ C_n, (n \geq 4)$ with $2(n+2)$ vertices. Then*

$$I(P(K_2 \circ C_n)) = 2(n-1) + \left\lceil \frac{n}{3} \right\rceil + \left\lceil 2\sqrt{n+2} \right\rceil = n + 1 + \left\lceil \frac{n}{3} \right\rceil + I(P(C_n)).$$

Proof. If we consider $S = V_I \cup E_I$ and $S \subset V(P(K_2 \circ C_n))$ with $|S| = 2(n-2) + \lceil \frac{n}{3} \rceil + \lceil 2\sqrt{n+2} \rceil$, where $V_I \subset V(P(K_2 \circ C_n)), |V_I| = 2n-1$ and $E_I \subset E(P(K_2 \circ C_n)), |E_I| = \lceil \frac{n}{3} \rceil + \lceil 2\sqrt{n+2} \rceil - 3$. If the members of S are taken as given in Theorem 2.3, and removed from $P(K_2 \circ C_n)$. The set S considered above is the smallest subset of $V(P(K_2 \circ C_n))$ which yeilds the integrity of $P(K_2 \circ C_n)$. Thus, $I(P(K_2 \circ C_n)) = 2(n-1) + \lceil \frac{n}{3} \rceil + \lceil 2\sqrt{n+2} \rceil$. □

Theorem 2.20. *Let $P(K_2 \circ K_{1,n})$ be the quasi-total graph of $K_2 \circ K_{1,n}$ with $2(n+2)$ vertices. Then*

$$I(P(K_2 \circ K_{1,n})) = 3n + 2.$$

Proof. If we consider $S = V_I \cup E_I$ and $S \subset V(P(K_2 \circ K_{1,n}))$ with $|S| = 3n + 2 - \lceil \frac{n+1}{2} \rceil$, where $V_I \subset V(P(K_2 \circ K_{1,n})), |V_I| = 2(n+1) - \lceil \frac{n+1}{2} \rceil$ and $E_I \subset E(P(K_2 \circ K_{1,n})), |E_I| = n$. If the members of S are taken as given in Theorem 2.8, and removed from $P(K_2 \circ K_{1,n})$, then $P(K_2 \circ K_{1,n}) - S$ has largest component as $K_{\lceil \frac{n+1}{2} \rceil}$. Thus, $I(P(K_2 \circ K_{1,n})) = 3n + 2$. □

Remark 2.1. *For any graph G, $I(G) \leq I(P(G))$, and equality holds for $G = K_1$ or $G = K_2$.*

3 Conclusion

In this article, one of the measures of graph vulnerability called integrity is studied. The values of vulnerability helps the researchers to construct such a communication network which remains stable after some of its nodes or communication links are get defected. The transformation graph considered in this paper is taken as a model of network system and it reveals that, how it would become more stable and strong. For this purpose the new nodes are inserted in the network. This construction of new network is done by using the definition of quasi-total graph of a graph. The integrity of this new graph are calculated.

The integrity of any graph and integrity of quasi-total graph of the same kind of graphs are given in the following Table. As one can see from the Table 1, integrities of

Table 1:

G	$I(G)$	$P(G)$	$I(P(G))$
P_2	2	$P(P_2)$	2
P_6	4	$P(P_3)$	4
C_5	4	$P(C_3)$	4
C_{10}	6	$P(C_4)$	6
K_4	4	$P(K_3)$	4
K_7	4	$P(K_4)$	7
$K_{5,5}$	6	$P(K_{2,2})$	6
$W_{1,10}$	7	$P(W_{1,3})$	7
$W_{1,26}$	11	$P(W_{1,4})$	11
$K_2 \times P_4$	10	$P(K_2 \times P_4)$	14
$K_2 \times C_4$	6	$P(K_2 \times C_4)$	16
$K_2 \times K_{1,3}$	4	$P(K_2 \times K_{1,3})$	10
$K_2 \circ P_4$	6	$P(K_2 \circ P_4)$	11
$K_2 \circ C_4$	6	$P(K_2 \circ C_4)$	13
$K_2 \circ K_{1,3}$	5	$P(K_2 \circ K_{1,3})$	14

quasi-total graphs are equal to or greater than the integrity values of graphs that have the same structure. These results can help the researchers to choose a suitable topology for the network.

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