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ON $(3,4)$-REGULAR BIPARTITIONS WITH DESIGNATED SUMMANDS

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ABSTRACT. Andrews, Lewis and Lovejoy defined a new class of partitions with designated summands by taking ordinary partitions and tagging exactly one of each part size. Let $BPD_{3,4}(n)$ denote the number of bipartitions of n with designated summands in which parts are not multiples of 3 or 4. In this paper, we establish many infinite families of congruences modulo powers of 2 for $BPD_{3,4}(n)$. For example, for any $n \geq 0$ and $\alpha, \beta, \gamma \geq 0$,

$$BPD_{3,4}\left(24 \cdot 3^{2\alpha} \cdot 5^{2\beta+2} \cdot 7^{2\gamma} n + b_1 \cdot 3^{2\alpha} \cdot 5^{2\beta+1} \cdot 7^{2\gamma}\right) \equiv 0 \pmod{4},$$

where $b_1 \in \{39, 63, 87, 111\}$.

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Dedicated to Prof. C. Adiga on the occasion of his 62^{nd} birthday.

1. INTRODUCTION

A partition of a positive integer n is a non-increasing sequence of positive integers whose sum is n. We denote the number of partitions of n by $p(n)$ with $p(0) = 1$. Ramanujan proved for every non-negative integer n that

$$p(5n+4) \equiv 0 \pmod{5},$$
$$p(7n+5) \equiv 0 \pmod{7},$$
$$p(11n+6) \equiv 0 \pmod{11}.$$

Recall that for an integer $\ell > 1$, an ℓ-regular partition is a partition in which none of the parts is divisible by ℓ. We denote the number of ℓ-regular partitions of n by $b_\ell(n)$ with $b_\ell(0) = 1$. The generating function for $b_\ell(n)$ is given by

$$\sum_{n=0}^{\infty} b_\ell(n) q^n = \frac{f_\ell}{f_l}.$$

Throughout this paper, we use the standard q-series notation, and f_ℓ is defined as

$$f_\ell := \left(q^\ell; q^\ell\right)_\infty = \prod_{k=1}^{\infty}(1 - q^{k\ell}).$$

Andrews et al. [1] investigated a new class of partitions with designated summands by taking ordinary partitions and tagging exactly one of each part size. The total number of partitions of n with designated summands is denoted by $PD(n)$. For example, there are 10 partitions of 4 with designated summands, namely

$4'$, $3'+1'$, $2'+2$, $2+2'$, $2'+1'+1$, $2'+1+1'$, $1'+1+1+1$, $1+1'+1+1$, $1+1+1'+1$, $1+1+1+1'$.

Andrews et al. [1] derived the following generating function for $PD(n)$,

$$\sum_{n=0}^{\infty} PD(n)q^n = \frac{f_6}{f_1 f_2 f_3}. \tag{1}$$

Chen et al. [3] established Ramanujan-type identity for the partition function $PD(3n+2)$ which implies the congruence of Andrews et al. [1] and they also gave a combinatorial interpretation of the congruence for $PD(3n+2)$ by introducing a rank for partitions with designated summands. Recently, Xia [11] extended the work on designated summands by deriving congruence properties of $PD(n)$ by employing the generating functions of $PD(3n)$ and $PD(3n+1)$ due to Chen et al. [3].

Mahadeva Naika and Gireesh [6] obtained many congruences for $PD_3(n)$, the number of partitions of n with designated summands whose parts are not divisible by 3 and the generating function is given by

$$\sum_{n=0}^{\infty} PD_3(n)q^n = \frac{f_6^2 f_9}{f_1 f_2 f_{18}}. \tag{2}$$

Mahadeva Naika and Shivaprasada Nayaka [7] derived many congruences for $PD_{2,3}(n)$, the number of partitions of n with designated summands in which parts are not multiples of 2 and 3. The generating function for $PD_{2,3}(n)$ is given by

$$\sum_{n=0}^{\infty} PD_{2,3}(n)q^n = \frac{f_4 f_6^2 f_9 f_{36}}{f_1 f_{12}^2 f_{18}^2}. \tag{3}$$

In [9], Mahadeva Naika and Shivashankar established many congruences for $PD_2(n)$, the number of bipartitions of n with designated summands and the generating function is given by

$$\sum_{n=0}^{\infty} PD_2(n)q^n = \frac{f_6^2}{f_1^2 f_2^2 f_3^2}. \tag{4}$$

In [8], the authors studied $PBD_3(n)$, the number of 3-regular bipartitions of n with designated summands and the generating function is given by

$$\sum_{n=0}^{\infty} PBD_3(n)q^n = \frac{f_6^4 f_9^2}{f_1^2 f_2^2 f_{18}^2}. \tag{5}$$

By the motivation of the above work, in this paper, we define $BPD_{3,4}(n)$, the number of $(3,4)$-regular bipartitions of n with designated summands and the generating function is given by

$$\sum_{n=0}^{\infty} BPD_{3,4}(n) = \frac{f_4^2 f_6^4 f_8^2 f_9^2 f_{72}^2}{f_1^2 f_2^2 f_{18}^2 f_{24}^4 f_{36}^2}. \tag{6}$$

Also, we establish many infinite families of congruences modulo powers of 2 for $BPD_{3,4}(n)$. For example, for any $n \geq 0$ and $\beta \geq 0$,

$$BPD_{3,4}\left(24 \cdot 5^{2\beta+2}n + b_2 \cdot 5^{2\beta+1}\right) \equiv 0 \pmod{16},$$

where $b_2 \in \{11, 59, 83, 107\}$.

2. Preliminary results

In this section, we record some identities which are useful in proving our main results.

Lemma 2.1. *The following* 2*-dissection holds:*

$$f_1^4 = \frac{f_4^{10}}{f_2^2 f_8^4} - 4q\frac{f_2^2 f_8^4}{f_4^2}. \tag{7}$$

The identity (7) is a consequence of dissection formulas of Ramanujan, collected in Berndt's Notebooks [2, p. 40, Entry 25].

Lemma 2.2. *The following* 2*-dissections hold:*

$$\frac{f_9}{f_1} = \frac{f_{12}^3 f_{18}}{f_2^2 f_6 f_{36}} + q\frac{f_4^2 f_6 f_{36}}{f_2^3 f_{12}} \tag{8}$$

and

$$\frac{f_1}{f_9} = \frac{f_2 f_{12}^3}{f_4 f_6 f_{18}^2} - q\frac{f_4 f_6 f_{36}^2}{f_{12} f_{18}^3}. \tag{9}$$

The identity (8) was obtained by Xia and Yao [12]. Replacing q by $-q$ in (8) and using the relation $(-q;-q)_\infty = \dfrac{f_2^3}{f_1 f_4}$, we obtain (9).

Lemma 2.3. *The following* 2*-dissection holds:*

$$\frac{f_3^3}{f_1} = \frac{f_4^3 f_6^2}{f_2^2 f_{12}} + q\frac{f_{12}^3}{f_4}. \tag{10}$$

The identity (10) was obtained by Hirschhorn et al. [4].

Lemma 2.4. *The following* 3*-dissection holds:*

$$f_1^3 = \frac{f_6 f_9^6}{f_3 f_{18}^3} - 3q f_9^3 + 4q^3\frac{f_3^2 f_{18}^6}{f_6^2 f_9^3}. \tag{11}$$

For a proof, see [2, p.345].

Lemma 2.5. *The following* 3*-dissection holds:*

$$f_1 f_2 = \frac{f_6 f_9^4}{f_3 f_{18}^2} - q f_9 f_{18} - 2q^2\frac{f_3 f_{18}^4}{f_6 f_9^2}. \tag{12}$$

For a proof, see [5].

Lemma 2.6. [10, p. 212] *We have the following* 5*-dissection:*

$$f_1 = f_{25}\left(a(q^5) - q - q^2/a(q^5)\right), \tag{13}$$

where

$$a := a(q) := \frac{(q^2, q^3; q^5)_\infty}{(q, q^4; q^5)_\infty}. \tag{14}$$

From the equation (13), we have

$$mf_1^3 \equiv mf_{25}^3\left(a^3(q^5) - 3qa^2(q^5) + 5q^3 - 3q^5/a^2(q^5) - q^6/a^3(q^5)\right) \pmod{2m}, \tag{15}$$

where m is a natural number.

Lemma 2.7. [2, p.303, Entry 17(v)] *We have*

$$f_1 = f_{49}\left(\frac{B(q^7)}{C(q^7)} - q\frac{A(q^7)}{B(q^7)} - q^2 + q^5\frac{C(q^7)}{A(q^7)}\right), \tag{16}$$

where $A(q) = f(-q^3, -q^4), B(q) = f(-q^2, q^5)$ *and* $C(q) = f(-q, -q^6)$.

Lemma 2.8. *For positive integers* k *and* m, *we have*

$$f_{2k}^m \equiv f_k^{2m} \pmod 2, \tag{17}$$

$$f_{2k}^{2m} \equiv f_k^{4m} \pmod 4 \tag{18}$$

and

$$f_{2k}^{4m} \equiv f_k^{8m} \pmod 8. \tag{19}$$

We prove the following Theorems:

Theorem 2.9. *Let* $b_1 \in \{7, 31, 79, 103\}$. *Then for any* $n \geq 0$ *and* $\beta \geq 0$, *we have*

$$\sum_{n=0}^{\infty} BPD_{3,4}\left(24\cdot 5^{2\beta}n + 11\cdot 5^{2\beta}\right)q^n \equiv 2f_2f_3^3 \pmod 4, \tag{20}$$

$$\sum_{n=0}^{\infty} BPD_{3,4}\left(24\cdot 5^{2\beta+1}n + 7\cdot 5^{2\beta+1}\right)q^n \equiv 2q^2f_{10}f_{15}^3 \pmod 4, \tag{21}$$

$$BPD_{3,4}\left(24\cdot 5^{2\beta+2}n + b_1\cdot 5^{2\beta+1}\right) \equiv 0 \pmod 4. \tag{22}$$

Theorem 2.10. *Let* $b_2 \in \{11, 59, 83, 107\}$. *Then for any* $n \geq 0$ *and* $\beta \geq 0$, *we have*

$$BPD_{3,4}(24n + 19) \equiv 0 \pmod{16}, \tag{23}$$

$$\sum_{n=0}^{\infty} BPD_{3,4}\left(24\cdot 5^{2\beta}n + 7\cdot 5^{2\beta}\right)q^n \equiv 8f_1^7 \pmod{16}, \tag{24}$$

$$\sum_{n=0}^{\infty} BPD_{3,4}\left(24\cdot 5^{2\beta+1}n + 11\cdot 5^{2\beta+1}\right)q^n \equiv 8qf_5^7 \pmod{16}, \tag{25}$$

$$BPD_{3,4}\left(24\cdot 5^{2\beta+2}n + b_2\cdot 5^{2\beta+1}\right) \equiv 0 \pmod{16}. \tag{26}$$

Theorem 2.11. *Let $b_3 \in \{11,19\}$, $b_4 \in \{51,99\}$ and $b_5 \in \{39,63,87,111\}$. Then for any $n \geq 0$ and $\alpha,\beta,\gamma \geq 0$, we have*

$$\sum_{n=0}^{\infty} BPD_{3,4}\left(24\cdot 3^{2\alpha}\cdot 5^{2\beta}\cdot 7^{2\gamma}n + 3^{2\alpha+1}\cdot 5^{2\beta}\cdot 7^{2\gamma}\right)q^n \equiv 2f_1^3 \pmod 4, \tag{27}$$

$$\begin{aligned} &BPD_{3,4}\left(24\cdot 3^{2\alpha+1}\cdot 5^{2\beta}\cdot 7^{2\gamma}n + 3^{2\alpha+1}\cdot 5^{2\beta}\cdot 7^{2\gamma}\right)\\ &\equiv \begin{cases} 2 & \text{if } n = k(3k+1)/2 \text{ for some } k\in\mathbb{Z},\\ 0 & \text{otherwise,}\end{cases} \pmod 4, \end{aligned} \tag{28}$$

$$\sum_{n=0}^{\infty} BPD_{3,4}\left(24\cdot 3^{2\alpha+1}\cdot 5^{2\beta}\cdot 7^{2\gamma}n + 3^{2\alpha+3}\cdot 5^{2\beta}\cdot 7^{2\gamma}\right)q^n \equiv 2f_3^3 \pmod 4, \tag{29}$$

$$BPD_{3,4}\left(24\cdot 3^{2\alpha+1}\cdot 5^{2\beta}\cdot 7^{2\gamma}n + 17\cdot 3^{2\alpha+1}\cdot 5^{2\beta}\cdot 7^{2\gamma}\right) \equiv 0 \pmod 4, \tag{30}$$

$$BPD_{3,4}\left(24\cdot 3^{2\alpha+2}\cdot 5^{2\beta}\cdot 7^{2\gamma}n + b_3\cdot 3^{2\alpha+2}\cdot 5^{2\alpha}\cdot 7^{2\gamma}\right) \equiv 0 \pmod 4, \tag{31}$$

$$\sum_{n=0}^{\infty} BPD_{3,4}\left(24\cdot 3^{2\alpha}\cdot 5^{2\beta+1}\cdot 7^{2\gamma}n + 3^{2\alpha+1}\cdot 5^{2\beta+2}\cdot 7^{2\gamma}\right)q^n \equiv 2f_5^3 \pmod 4, \tag{32}$$

$$BPD_{3,4}\left(24\cdot 3^{2\alpha}\cdot 5^{2\beta+1}\cdot 7^{2\gamma}n + b_4\cdot 3^{2\alpha}\cdot 5^{2\beta+1}\cdot 7^{2\gamma}\right) \equiv 0 \pmod 4, \tag{33}$$

$$BPD_{3,4}\left(24\cdot 3^{2\alpha}\cdot 5^{2\beta+2}\cdot 7^{2\gamma}n + b_5\cdot 3^{2\alpha}\cdot 5^{2\beta+1}\cdot 7^{2\gamma}\right) \equiv 0 \pmod 4, \tag{34}$$

$$\sum_{n=0}^{\infty} BPD_{3,4}\left(24\cdot 3^{2\alpha}\cdot 5^{2\beta}\cdot 7^{2\gamma+1}n + 3^{2\alpha+1}\cdot 5^{2\beta}\cdot 7^{2\gamma+2}\right)q^n \equiv 2f_7^3 \pmod 4. \tag{35}$$

Theorem 2.12. *Let $b_7 \in \{19,31,43,55,67,79\}$. Then for any $n \geq 0$ and $\beta,\gamma \geq 0$, we have*

$$BPD_{3,4}\left(12\cdot 5^{2\beta+2}n + 5^{2\beta+2}\right) \equiv 3^{\beta+1}\cdot BPD_{3,4}(12n+1) \pmod 8, \tag{36}$$

$$\sum_{n=0}^{\infty} BPD_{3,4}\left(12\cdot 7^{2\gamma}n + 7^{2\gamma}\right)q^n \equiv 2f_1^2 \pmod 8, \tag{37}$$

$$\sum_{n=0}^{\infty} BPD_{3,4}\left(12\cdot 7^{2\gamma+1}n + 7^{2\gamma+2}\right)q^n \equiv 2f_7^2 \pmod 8, \tag{38}$$

$$BPD_{3,4}\left(12\cdot 7^{2\gamma+2}n + b_7\cdot 7^{2\gamma+1}\right) \equiv 0 \pmod 8. \tag{39}$$

Theorem 2.13. *For any $n \geq 0$ and $\alpha \geq 0$, we have*

$$BPD_{3,4}\left(4\cdot 3^{2\alpha+3}n + 3^{2\alpha+4}\right) \equiv 3^{\alpha+1}\cdot BPD_{3,4}(12n+9) \pmod{16}. \tag{40}$$

Theorem 2.14. *Let $b_8 \in \{99, 171\}$, $b_9 \in \{153, 297\}$ and $b_{10} \in \{117, 189, 261, 333\}$. Then for any $n \geq 0$ and $\alpha, \beta, \gamma \geq 0$, we have*

$$\sum_{n=0}^{\infty} BPD_{3,4}\left(72 \cdot 3^{2\alpha} \cdot 5^{2\beta} \cdot 7^{2\gamma} n + 3^{2\alpha+2} \cdot 5^{2\beta} \cdot 7^{2\gamma}\right) q^n \equiv 4f_1^3 \pmod 8, \tag{41}$$

$$\begin{aligned} &BPD_{3,4}\left(72 \cdot 3^{2\alpha+1} \cdot 5^{2\beta} \cdot 7^{2\gamma} n + 3^{2\alpha+2} \cdot 5^{2\beta} \cdot 7^{2\gamma}\right) \\ &\equiv \begin{cases} 4 & \text{if } n = k(3k+1)/2 \text{ for some } k \in \mathbb{Z}, \\ 0 & \text{otherwise,} \end{cases} \pmod 8, \end{aligned} \tag{42}$$

$$\sum_{n=0}^{\infty} BPD_{3,4}\left(72 \cdot 3^{2\alpha+1} \cdot 5^{2\beta} \cdot 7^{2\gamma} n + 3^{2\alpha+4} \cdot 5^{2\beta} \cdot 7^{2\gamma}\right) q^n \equiv 4f_3^3 \pmod 8, \tag{43}$$

$$BPD_{3,4}\left(72 \cdot 3^{2\alpha+1} \cdot 5^{2\beta} \cdot 7^{2\gamma} n + 51 \cdot 3^{2\alpha+1} \cdot 5^{2\beta} \cdot 7^{2\gamma}\right) \equiv 0 \pmod 8, \tag{44}$$

$$BPD_{3,4}\left(72 \cdot 3^{2\alpha+2} \cdot 5^{2\beta} \cdot 7^{2\gamma} n + b_8 \cdot 3^{2\alpha+2} \cdot 5^{2\beta} \cdot 7^{2\gamma}\right) \equiv 0 \pmod 8, \tag{45}$$

$$\sum_{n=0}^{\infty} BPD_{3,4}\left(72 \cdot 3^{2\alpha} \cdot 5^{2\beta+1} \cdot 7^{2\gamma} n + 3^{2\alpha+2} \cdot 5^{2\beta+2} \cdot 7^{2\gamma}\right) q^n \equiv 4f_5^3 \pmod 8, \tag{46}$$

$$BPD_{3,4}\left(72 \cdot 3^{2\alpha} \cdot 5^{2\beta+1} \cdot 7^{2\gamma} n + b_9 \cdot 3^{2\alpha} \cdot 5^{2\beta} \cdot 7^{2\gamma}\right) \equiv 0 \pmod 8, \tag{47}$$

$$BPD_{3,4}\left(72 \cdot 3^{2\alpha} \cdot 5^{2\beta+2} \cdot 7^{2\gamma} n + b_{10} \cdot 3^{2\alpha} \cdot 5^{2\beta+2} \cdot 7^{2\gamma}\right) \equiv 0 \pmod 8, \tag{48}$$

$$\sum_{n=0}^{\infty} BPD_{3,4}\left(72 \cdot 3^{2\alpha} \cdot 5^{2\beta} \cdot 7^{2\gamma+1} n + 3^{2\alpha+2} \cdot 5^{2\beta} \cdot 7^{2\gamma+2}\right) q^n \equiv 4f_7^3 \pmod 8. \tag{49}$$

Theorem 2.15. *Let $b_{12} \in \{21, 93, 237, 309\}$. Then for any $n \geq 0$ and $\beta \geq 0$, we have*

$$BPD_{3,4}(72n + 69) \equiv 0 \pmod{16}, \tag{50}$$

$$\sum_{n=0}^{\infty} BPD_{3,4}\left(72 \cdot 5^{2\beta} n + 33 \cdot 5^{2\beta}\right) q^n \equiv 8f_2 f_3^3 \pmod{16}, \tag{51}$$

$$\sum_{n=0}^{\infty} BPD_{3,4}\left(72 \cdot 5^{2\beta+1} n + 21 \cdot 5^{2\beta+1}\right) q^n \equiv 8q^2 f_{10} f_{15}^3 \pmod{16}, \tag{52}$$

$$BPD_{3,4}\left(72 \cdot 5^{2\beta+2} n + b_{12} \cdot 5^{2\beta+1}\right) \equiv 0 \pmod{16}. \tag{53}$$

Theorem 2.16. *Let $b_{13} \in \{33, 177, 249, 321\}$. Then for any $n \geq 0$ and $\beta \geq 0$, we have*

$$\sum_{n=0}^{\infty} BPD_{3,4}\left(72 \cdot 5^{2\beta} n + 21 \cdot 5^{2\beta}\right) q^n \equiv 8 f_1^7 \pmod{16}, \tag{54}$$

$$\sum_{n=0}^{\infty} BPD_{3,4}\left(72 \cdot 5^{2\beta+1} n + 33 \cdot 5^{2\beta+1}\right) q^n \equiv 8 q f_5^7 \pmod{16}, \tag{55}$$

$$BPD_{3,4}\left(72 \cdot 5^{2\beta+2} n + b_{13} \cdot 5^{2\beta+1}\right) \equiv 0 \pmod{16}. \tag{56}$$

Theorem 2.17. *Let $b_{14} \in \{11, 59, 83, 107, 131, 155\}$. Then for any $n \geq 0$ and $\gamma \geq 0$, we have*

$$\sum_{n=0}^{\infty} BPD_{3,4}\left(24 \cdot 7^{2\gamma} n + 5 \cdot 7^{2\gamma}\right) q^n \equiv 2 f_1 f_4 \pmod{4}, \tag{57}$$

$$\sum_{n=0}^{\infty} BPD_{3,4}\left(24 \cdot 7^{2\gamma+1} n + 11 \cdot 7^{2\gamma+1}\right) q^n \equiv 2 q f_7 f_{28} \pmod{4}, \tag{58}$$

$$BPD_{3,4}\left(12 \cdot 7^{2\gamma+2} n + b_{14} \cdot 7^{2\gamma+1}\right) \equiv 0 \pmod{4}. \tag{59}$$

3. Proof of theorem 2.9

From the equation (6), we have

$$\sum_{n=0}^{\infty} BPD_{3,4}(n) q^n = \frac{f_4^2 f_6^4 f_8^2 f_9^2 f_{72}^2}{f_1^2 f_2^2 f_{18}^2 f_{24}^4 f_{36}^2}. \tag{60}$$

Using (8) in (60) and then comparing the terms involving q^{2n+1} on both sides, we arrive at

$$\sum_{n=0}^{\infty} BPD_{3,4}(2n+1) q^n = 2 \frac{f_2^4 f_3^4 f_4^2 f_6^2 f_{36}^2}{f_1^7 f_9 f_{12}^4 f_{18}^2}. \tag{61}$$

By using the Lemma 2.8, (61) reduces to

$$\sum_{n=0}^{\infty} BPD_{3,4}(2n+1) q^n \equiv 2 \frac{f_1 f_3^4 f_4^2 f_6^2 f_{36}^2}{f_9 f_{12}^4 f_{18}^2} \pmod{16}. \tag{62}$$

Using (7) and (9) in (62) and then comparing the coefficients of q^{2n} and q^{2n+1} on both sides of the resultant equation, we get

$$\sum_{n=0}^{\infty} BPD_{3,4}(4n+1) q^n \equiv 2 \frac{f_1 f_2 f_6^9 f_{18}^2}{f_3 f_9^4 f_{12}^4} + 8 q^2 \frac{f_2^3 f_3^5 f_{12}^4 f_{18}^4}{f_6^7 f_9^5} \pmod{16} \tag{63}$$

and

$$\sum_{n=0}^{\infty} BPD_{3,4}(4n+3) q^n \equiv 14 \frac{f_2^3 f_3 f_6^5 f_{18}^4}{f_9^5 f_{12}^4} + 8 q \frac{f_1 f_2 f_3^3 f_{12}^4 f_{18}^2}{f_6^3 f_9^4} \pmod{16}. \tag{64}$$

Using (11) and (12) in (64) and then collecting the coefficients of q^{3n} , q^{3n+1} and q^{3n+2}, we obtain

$$\sum_{n=0}^{\infty} BPD_{3,4}\left(12n+3\right)q^n \equiv 14\frac{f_1 f_3^3 f_4 f_6^6}{f_2^4 f_{12}^3} + 8q^2\frac{f_1 f_3^3 f_{12}^6}{f_2 f_4^2 f_6^3} \pmod{16}, \tag{65}$$

$$\sum_{n=0}^{\infty} BPD_{3,4}\left(12n+7\right)q^n \equiv 8\frac{f_1^2 f_4^4}{f_2^2} \pmod{16} \tag{66}$$

and

$$\sum_{n=0}^{\infty} BPD_{3,4}\left(12n+11\right)q^n \equiv 6\frac{f_1 f_3^3 f_6^3}{f_2^3} + 8f_1 f_3 f_4^3 f_6 \pmod{16}. \tag{67}$$

The equation (67) reduces to

$$\sum_{n=0}^{\infty} BPD_{3,4}\left(12n+11\right)q^n \equiv 2\frac{f_3^3 f_6^3}{f_1 f_2^2} \pmod{4}. \tag{68}$$

Using (10) in (68) and then collecting the terms involving q^{2n} from both sides of the resultant equation, we get

$$\sum_{n=0}^{\infty} BPD_{3,4}\left(24n+11\right)q^n \equiv 2\frac{f_2^3 f_3^5}{f_1^4 f_6} \pmod{4}, \tag{69}$$

which becomes

$$\sum_{n=0}^{\infty} BPD_{3,4}\left(24n+11\right)q^n \equiv 2f_2 f_3^3 \pmod{4}, \tag{70}$$

which is $\beta = 0$ case of (20). Suppose that the congruence (20) is true for some integer $\beta \geq 0$. Employing (13) and (15) with $m = 2$ in (20) and then comparing the coefficients of q^{5n+1}, we obtain

$$\sum_{n=0}^{\infty} BPD_{3,4}\left(24\cdot 5^{2\beta+1}n + 7\cdot 5^{2\beta+1}\right)q^n \equiv 2q^2 f_{10} f_{15}^3 \pmod{4}, \tag{71}$$

which implies

$$\sum_{n=0}^{\infty} BPD_{3,4}\left(24\cdot 5^{2\beta+2}n + 11\cdot 5^{2\beta+2}\right)q^n \equiv 2f_2 f_3^3 \pmod{4}, \tag{72}$$

which implies that the congruence (20) is true for $\beta+1$. By mathematical induction, the congruence (20) is true for all integer $\beta \geq 0$.

Using (13) and (15) with $m = 2$ in (20) and then collecting the coefficients of q^{5n+1} from both sides of the resultant equation, we get (21).

Extracting the terms involving q^{5n+i} for $i = 0, 1, 3, 4$ from the equation (21) , we get (22).

4. Proof of theorem 2.10

The equation (66) becomes

$$\sum_{n=0}^{\infty} BPD_{3,4}(12n+7)\, q^n \equiv 8f_2^7 \pmod{16}. \tag{73}$$

Extracting the terms involving q^{2n+1} from both sides of the above equation, we get (23).

The equation (73) implies

$$\sum_{n=0}^{\infty} BPD_{3,4}(24n+7)\, q^n \equiv 8f_1^7 \pmod{16}, \tag{74}$$

which is $\beta = 0$ case of (24). Suppose that the congruence (24) is true for some integer $\beta \geq 0$, we have

$$\begin{aligned}
&\sum_{n=0}^{\infty} BPD_{3,4}\left(24\cdot 5^{2\beta}n + 7\cdot 5^{2\beta}\right) q^n \\
&\equiv 8f_1^7 \\
&\equiv 8f_{25}^7\left(a(q^5) - q - q^2/a(q^5)\right)^7 \\
&\equiv 8f_{25}^7(a^6(q^5)q + a^4(q^5)q^3 + a^3(q^5)q^4 + q^8/a(q^5) + q^{10}/a^3(q^5) + q^{11}/a^4(q^5) \\
&\quad + a(q^5)q^6 + q^{13}/a^6(q^5) + q^{14}/a^7(q^5) + q^7 + a^7(q^5) \pmod{16}.
\end{aligned} \tag{75}$$

Comparing the coefficients of q^{5n+2} on both sides of the above equation, we obtain

$$\sum_{n=0}^{\infty} BPD_{3,4}\left(24\cdot 5^{2\beta+1}n + 11\cdot 5^{2\beta+1}\right) q^n \equiv 8qf_5^7 \pmod{16}, \tag{76}$$

which implies

$$\sum_{n=0}^{\infty} BPD_{3,4}\left(24\cdot 5^{2\beta+2}n + 7\cdot 5^{2\beta+2}\right) q^n \equiv 8f_1^7 \pmod{16}, \tag{77}$$

which implies that the congruence (24) is true for $\beta+1$. By mathematical induction, the congruence (24) is true for all integer $\beta \geq 0$.

Employing (13) in (24) and then collecting the coefficients of q^{5n+2} from both sides of the resultant equation, we obtain (25).

From the equation (25), we obtain (26).

5. Proof of theorem 2.11

From the equation (65), we have

$$\sum_{n=0}^{\infty} BPD_{3,4}(12n+3)\, q^n \equiv 2\frac{f_3^3}{f_1 f_2} \pmod{4}. \tag{78}$$

Using (10) in (78) and then comparing the coefficients of q^{2n} on both sides, we get

$$\sum_{n=0}^{\infty} BPD_{3,4}\,(24n+3)\,q^n \equiv 2\frac{f_2^3 f_3^2}{f_1^3 f_6} \pmod 4, \tag{79}$$

which becomes

$$\sum_{n=0}^{\infty} BPD_{3,4}\,(24n+3)\,q^n \equiv 2f_1^3 \pmod 4, \tag{80}$$

which is $\alpha=\beta=\gamma=0$ case of (27). Suppose that the congruence (27) is true for $\alpha \geq 0$ with $\beta=\gamma=0$, we have

$$\sum_{n=0}^{\infty} BPD_{3,4}\left(24\cdot 3^{2\alpha}n+3^{2\alpha+1}\right) q^n \equiv 2f_1^3 \pmod 4. \tag{81}$$

Using (11) in (81), we arrive at

$$\sum_{n=0}^{\infty} BPD_{3,4}\left(24\cdot 3^{2\alpha+1}n+3^{3\alpha+3}\right) q^n \equiv 2f_3^3 \pmod 4, \tag{82}$$

which implies

$$\sum_{n=0}^{\infty} BPD_{3,4}\left(24\cdot 3^{2\alpha+2}n+3^{2\alpha+3}\right) q^n \equiv 2f_1^3 \pmod 4, \tag{83}$$

which implies that the congruence (27) is true for $\alpha+1$ with $\beta=\gamma=0$. By mathematical induction, the congruence (27) is true for all $\alpha \geq 0$. Suppose that the congruence (27) holds for $\alpha,\beta \geq 0$ with $\gamma=0$. Using (15) with $m=2$ in (27) with $\gamma=0$, we get

$$\sum_{n=0}^{\infty} BPD_{3,4}\left(24\cdot 3^{2\alpha}\cdot 5^{2\beta+1}n+3^{2\alpha+1}\cdot 5^{2\beta+2}\right) q^n \equiv 2f_5^3 \pmod 4, \tag{84}$$

which implies

$$\sum_{n=0}^{\infty} BPD_{3,4}\left(24\cdot 3^{2\alpha}\cdot 5^{2\beta+2}n+3^{2\alpha+1}\cdot 5^{2\beta+2}\right) q^n \equiv 2f_1^3 \pmod 4, \tag{85}$$

which implies that the congruence (27) is true for $\beta+1$ with $\gamma=0$. By mathematical induction, the congruence (27) is true for all non-negative integers $\alpha,\beta \geq 0$ with $\gamma=0$. Suppose that the congruence (27) holds for all $\alpha,\beta,\gamma \geq 0$. Employing (16) in (27), we get

$$\sum_{n=0}^{\infty} BPD_{3,4}\left(24\cdot 3^{2\alpha}\cdot 5^{2\beta}\cdot 7^{2\gamma+1}n+3^{2\alpha+1}\cdot 5^{2\beta}\cdot 7^{2\gamma+2}\right) q^n \equiv 2f_7^3 \pmod 4, \tag{86}$$

which implies

$$\sum_{n=0}^{\infty} BPD_{3,4}\left(24\cdot 3^{2\alpha}\cdot 5^{2\beta}\cdot 7^{2\gamma+2}n+3^{2\alpha+1}\cdot 5^{2\beta}\cdot 7^{2\gamma+2}\right) q^n \equiv 2f_1^3 \pmod 4, \tag{87}$$

which implies that the congruence (27) is true for $\gamma+1$. By mathematical induction, the congruence (27) is true for all integers $\alpha,\beta,\gamma \geq 0$.

Using (11) in (27) and then comparing the coefficients of q^{3n}, q^{3n+1} and q^{3n+2}, we obtain (28), (29) and (30) respectively.

Extracting the terms involving q^{3n+1} and q^{3n+2} from (29), we get (31).

Using (15) with $m = 2$ in (27) and then extracting the terms involving q^{5n+3}, we get (32).

Comparing the coefficients of q^{5n+2} and q^{5n+4} on both sides of the equation (27) along with (15) with $m = 2$, we obtain (33).

Extracting the terms involving q^{5n+i} for $i = 1, 2, 3, 4$ from (32), we arrive at (34).

Utilizing (16) in (27) and then comparing the coefficients of q^{7n+6}, we get (35).

6. Proof of theorem 2.12

Using (11) and (12) in (63) and then extracting the terms involving q^{3n}, q^{3n+1} and q^{3n+2} from both sides of the resultant equation, we have

$$\sum_{n=0}^{\infty} BPD_{3,4}\left(12n+1\right)q^n \equiv 2\frac{f_2^2}{f_1^2} \pmod{16}, \tag{88}$$

$$\sum_{n=0}^{\infty} BPD_{3,4}\left(12n+5\right)q^n \equiv 14\frac{f_2 f_6^3}{f_1 f_3^3} + 8qf_1^7 f_3^9 \pmod{16} \tag{89}$$

and

$$\sum_{n=0}^{\infty} BPD_{3,4}\left(12n+9\right)q^n \equiv 12f_3^2 f_6^2 + 8f_1^5 f_3^3 f_4 \pmod{16}. \tag{90}$$

The equation (88) becomes

$$\sum_{n=0}^{\infty} BPD_{3,4}\left(12n+1\right)q^n \equiv 2f_1^2 \pmod{8}. \tag{91}$$

Using (13) in (91) and then extracting the terms involving q^{5n+2}, we obtain

$$\sum_{n=0}^{\infty} BPD_{3,4}\left(60n+25\right)q^n \equiv 6f_5^2 \pmod{8}, \tag{92}$$

which implies

$$\sum_{n=0}^{\infty} BPD_{3,4}\left(300n+25\right)q^n \equiv 6f_1^2 \pmod{8}. \tag{93}$$

In view of the congruences (91) and (93), we arrive at

$$BPD_{3,4}\left(300n+25\right) \equiv 3\cdot BPD_{3,4}\left(12n+1\right) \pmod{8}. \tag{94}$$

By using the above relation and by induction on β, we get (36).

The congruence (91) is $\gamma = 0$ case of (37). Suppose that the congruence (37) is true for $\gamma \geq 0$. Using (16) in (37), we get

$$\sum_{n=0}^{\infty} BPD_{3,4}\left(12\cdot 7^{2\gamma+1}n + 7^{2\gamma+2}\right)q^n \equiv 2f_7^2 \pmod{8}, \tag{95}$$

which implies

$$\sum_{n=0}^{\infty} BPD_{3,4}\left(12\cdot 7^{2\gamma+2}n+7^{2\gamma+2}\right)q^n \equiv 2f_1^2 \pmod 8, \tag{96}$$

which implies that the congruence (37) is true for $\gamma+1$. By induction, the congruence (37) holds for all integer $\gamma \geq 0$.

Using (16) in (37) and then extracting the terms involving q^{7n+4} from both sides of the resultant equation, we get (38).

From the equation (38), we get (39).

7. Proof of theorem 2.13

The equation (90) implies

$$\sum_{n=0}^{\infty} BPD_{3,4}(12n+9)\,q^n \equiv 12f_3^2f_6^2+8f_1^3f_2^3f_3^3 \pmod{16}. \tag{97}$$

Using (12) in (97) and then extracting the terms involving q^{3n}, q^{3n+1} and q^{3n+2} respectively, we have

$$\sum_{n=0}^{\infty} BPD_{3,4}(36n+9)\,q^n \equiv 8f_2^3+8qf_1^3f_3^3f_6^3+12f_1^2f_2^2 \pmod{16}, \tag{98}$$

$$\sum_{n=0}^{\infty} BPD_{3,4}(36n+21)\,q^n \equiv 8f_1f_2^2f_3^3 \pmod{16} \tag{99}$$

and

$$\sum_{n=0}^{\infty} BPD_{3,4}(36n+33)\,q^n \equiv 8f_1^2f_2f_3^6 \pmod{16}. \tag{100}$$

Again, using (11) and (12) in (98) and then collecting the coefficients of q^{3n+2}, we arrive at

$$\sum_{n=0}^{\infty} BPD_{3,4}(108n+81)\,q^n \equiv 4f_3^2f_6^2+8f_1^3f_2^3f_3^3 \pmod{16}. \tag{101}$$

In view of the congruences (97) and (101), we get

$$BPD_{3,4}(108n+81) \equiv 3\cdot BPD_{3,4}(12n+9) \pmod{16}. \tag{102}$$

By using the above relation and by induction on α, we obtain (40).

7.1. **Proof of Theorem 2.14.**

The equation (97) becomes

$$\sum_{n=0}^{\infty} BPD_{3,4}(12n+9)\,q^n \equiv 4f_6^3 \pmod 8, \tag{103}$$

which implies

$$\sum_{n=0}^{\infty} BPD_{3,4}(72n+9)\,q^n \equiv 4f_1^3 \pmod 8, \tag{104}$$

which is $\alpha = \beta = \gamma = 0$ case of (41). The rest of the proofs of the identities (41)-(49) are similar to the proofs of the identities (27)- (35). So, we omit the details.

7.2. **Proof of Theorem 2.15.**
The congruence (100) becomes

$$\sum_{n=0}^{\infty} BPD_{3,4}\,(36n+33)\,q^n \equiv 8f_4f_6^3 \pmod{16}. \tag{105}$$

Extracting the terms involving q^{2n+1} from both sides of the above equation, we obtain (50).
The congruence (105) implies

$$\sum_{n=0}^{\infty} BPD_{3,4}\,(72n+33)\,q^n \equiv 8f_2f_3^3 \pmod{16}, \tag{106}$$

which is $\beta = 0$ case of (51). The rest of the proofs of the identities (51)-(53) are similar to the proofs of the identities (20)-(22). So, we omit the details.

7.3. **Proof of Theorem 2.16.**
From the equation (99), we have

$$\sum_{n=0}^{\infty} BPD_{3,4}\,(36n+21)\,q^n \equiv 8\frac{f_2^3f_3^3}{f_1} \pmod{16}. \tag{107}$$

Employing (10) in (107) and then collecting the coefficients of q^{2n} from both sides of the resultant equation, we arrive at

$$\sum_{n=0}^{\infty} BPD_{3,4}\,(72n+21)\,q^n \equiv 8f_1^7 \pmod{16}, \tag{108}$$

which is $\beta = 0$ case of (54). The rest of the proofs of the identities (54)-(56) are similar to the proofs of the identities (24)-(26). So, we omit the details.

8. Proof of theorem 2.17

From the equation (89), we arrive at

$$\sum_{n=0}^{\infty} BPD_{3,4}\,(12n+5)\,q^n \equiv 2\frac{f_2f_3^3}{f_1} \pmod{4}. \tag{109}$$

Utilizing (10) in (109) and then collecting the coefficients of q^{2n} from both sides of the resultant equation, we get

$$\sum_{n=0}^{\infty} BPD_{3,4}\,(24n+5)\,q^n \equiv 2f_1f_4 \pmod{4}, \tag{110}$$

which is $\gamma = 0$ case of (57). Suppose that the congruence (57) is true for $\gamma \geq 0$. Using (16) in (57), we obtain

$$\sum_{n=0}^{\infty} BPD_{3,4}\left(24\cdot 7^{2\gamma+1}n + 11\cdot 7^{2\gamma+1}\right) q^n \equiv 2qf_7f_{28} \pmod{4}, \tag{111}$$

which implies

$$\sum_{n=0}^{\infty} BPD_{3,4}\left(24\cdot 7^{2\gamma+2}n + 5\cdot 7^{2\gamma+2}\right) q^n \equiv 2f_1f_4 \pmod 4, \tag{112}$$

which implies that the congruence (57) is true for $\gamma+1$. Hence, by induction, the congruence (57) holds for all $\gamma \geq 0$.

Employing (16) in (57) and then comparing the coefficients of q^{7n+3} on both sides of the resultant equation, we obtain (58).

From the congruence (58), we arrive at (59).

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