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A FAMILY OF ASSOCIATED SEQUENCES AND THEIR REPRESENTATIONS BY APPELL POLYNOMIALS

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ABSTRACT. This paper studies the problem of representing any polynomial in certain family of associated sequences in terms of Appell polynomials by using umbral calculus. Further, simple examples are provided to illustrate the results.

1. INTRODUCTION AND BRIEF REVIEW OF UMBRAL CALCULUS

The classical linearization problem concerns with determining the coefficients in the expansion of the product $q_n(x)r_m(x)$ of two polynomials $q_n(x)$ and $r_m(x)$ in terms of an arbitrary polynomial sequence $\{p_k(x)\}_{k\geq 0}$:

$$q_n(x)r_m(x) = \sum_{k=0}^{n+m} c_k(nm)p_k(x).$$

A special case of this is when $p_n(x) = q_n(x) = r_n(x)$, which is called either the standard linearization or Clebsch-Gordan-type problem. Another particular case is when $r_m(x) = 1$, which is so called the connection problem. If further $q_n(x) = x^n$, it is called the inversion problem for the sequence $\{p_k(x)\}_{k\geq 0}$.

The aim of this paper is to represent any polynomial in certain family of associated sequences in terms of Appell polynomials by making use of umbral calculus. So this is the classical connection problem just mentioned. In addition, we will illustrate our results by simple examples.

In recent years, representations by Bernoulli polynomials have been done for certain sums of finite products of quite a few Appell and non-Appell polynomials. Actually, these were done by deriving Fourier series expansions for functions closely related to those sums of finite products. Indeed, those were done for certain sums of finite products of Bernoulli and Euler polynomials in [1,6] and of quite a few non-Appell polynomials such as Chebyshev polynomials of the first, second, third and fourth kinds, and Legendre, Laguerre, Fibonacci and Lucas polynomials (see in [3-5,7]).

Lastly, we would like the mention some of the previous results that are related to the present work. Certain sums of finite products of Chebyshev polynomials of the first, second, third and fourth kinds, and of Legendre, Laguerre, Fibonacci and Lucas polynomials are expressed in terms of all four kinds of Chebyshev polynomials and also in terms of Hermite, extended Laguerre, Legendre, Gegenbauer and Jacobi polynomials (see [2 and the references therein]).

Before we start our investigation in the next section, we are going to briefly review some basic facts about umbral calculus. We let the readers refer to [9] for a complete treatment of umbral calculus.

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Let $\mathbb{C}$ be the field of complex numbers. Then $\mathscr{F}$ denotes the algebra of all formal power series in the variable t with the coefficients in $\mathbb{C}$:

$$\mathscr{F} = \left\{ f(t) = \sum_{k=0}^{\infty} a_k \frac{t^k}{k!} \;\middle|\; a_k \in \mathbb{C} \right\}.$$

Let $\mathbb{P} = \mathbb{C}[x]$ be the ring of polynomials in x with the coefficients in $\mathbb{C}$, and let $\mathbb{P}^*$ denote the vector space of all linear functionals on $\mathbb{P}$. For $L \in \mathbb{P}^*$ and $p(x) \in \mathbb{P}$, by the notation $< L|p(x) >$ we will denote the action of the linear functional L on $p(x)$.

For $f(t) = \sum_{k=0}^{\infty} a_k \frac{t^k}{k!} \in \mathscr{F}$, the linear functional $< f(t)|\cdot >$ on $\mathbb{P}$ is defined by

$$< f(t) \mid x^n >= a_n, \quad (n \geq 0). \tag{1}$$

From (1), in particular we see that

$$< t^k \mid x^n >= n!\delta_{n,k}, \quad (n,k \geq 0),$$

where $\delta_{n,k}$ is the Kronecker symbol.

For $L \in \mathbb{P}^*$, let $f_L(t) = \sum_{k=0}^{\infty} < L \mid x^k > \frac{t^k}{k!} \in \mathscr{F}$. Then we see that $< f_L(t)|x^n >=< L|x^n >$, and the map $L \mapsto f_L(t)$ is a vector space isomorphism from $\mathbb{P}^*$ to $\mathscr{F}$. Henceforth, $\mathscr{F}$ denotes both the algebra of all formal power series in t and the vector space of all linear functionals on $\mathbb{P}$. Hence an element $f(t)$ of $\mathscr{F}$ will be thought of as both a formal power series and a linear functional on $\mathbb{P}$. $\mathscr{F}$ is called the umbral algebra, the study of which is the umbral calculus.

The order $O(f(t))$ of $0 \neq f(t) \in \mathscr{F}$ is the smallest integer k such that the coefficient of t^k does not vanish. In particular, $f(t) \in \mathscr{F}$ is called an invertible series if $O(f(t)) = 0$ and a delta series if $O(f(t)) = 1$.

Let $f(t), g(t) \in \mathscr{F}$ with $O(g(t)) = 0,\ O(f(t)) = 1$. Then there exists a unique sequence of polynomials $s_n(x)$ $(deg\ s_n(x) = n)$ such that

$$< g(t)f(t)^k \mid s_n(x) >= n!\delta_{n,k}, \quad \text{for } n,k \geq 0.$$

Such a sequence is called the Sheffer sequence for the Sheffer pair $(g(t),\ f(t))$, which is indicated by

$$s_n(x) \sim (g(t),\ f(t)). \tag{2}$$

Then $s_n(x) \sim (g(t),\ f(t))$ if and only if

$$\frac{1}{g(\bar{f}(t))} e^{x\bar{f}(t)} = \sum_{n=0}^{\infty} s_n(x) \frac{t^n}{n!}, \tag{3}$$

where $\bar{f}(t)$ is the compositional inverse of $f(t)$ satisfying $f(\bar{f}(t)) = \bar{f}(f(t)) = t$.
In particular, if $f(t) = t$, then $s_n(x)$ is the Appell sequence for $g(t)$; if $g(t) = 1$, then $s_n(x)$ is called the associated sequence to $f(t)$. Assume that $s_n(x) \sim (g(t),\ f(t)),\ r_n(x) \sim (h(t),\ l(t))$. Then

$$s_n(x) = \sum_{k=0}^{n} C_{n,k} r_k(x),$$

where

$$C_{n,k} = \frac{1}{k!} \left\langle \frac{h(\bar{f}(t))}{g(\bar{f}(t))} l(\bar{f}(t))^k \mid x^n \right\rangle. \tag{4}$$

The following theorem is stated as Theorem 3.5.6 in [8] and important for our purpose.

Theorem 1.1. *If $s_n(x) \sim (g(t),\ f(t))$, then, for any power series $h(t)$, we have*

$$< h(t) \mid s_n(x) >=< g(\bar{f}(t))^{-1} h(\bar{f}(t)) \mid x^n > .$$

The falling factorial $(x)_n$ and the rising factorial $< x >_n$ are respectively givnen by

$$(5)\qquad \begin{aligned} &(x)_0 = 1,\ (x)_n = x(x-1)\cdots(x-n+1),\ (n \geq 1),\\ &< x >_0 = 1,\ < x >_n = x(x+1)\cdots(x+n-1),\ (n \geq 1). \end{aligned}$$

The two factorials are related by

$$(6)\qquad (-1)^n (x)_n = < -x >_n,\ (-1)^n < x >_n = (-x)_n.$$

The signed Stirling numbers of the first kind $S_1(n,k)$ and the unsigned Stirling numbers of the first kind $|S_1(n,k)|$ are respectively given by

$$(7)\qquad \begin{aligned} &(x)_n = \sum_{k=0}^{n} S_1(n,k) x^k,\ < x >_n = \sum_{k=0}^{n} |S_1(n,k)| x^k,\\ &|S_1(n,k)| = (-1)^{n-k} S_1(n,k). \end{aligned}$$

2. Main results

For any positive integer r and complex numbers $a_1, a_2, \ldots, a_r$, with $a_1, a_r \neq 0$, assume that the generating function of the polynomials $s_n(x; a_1, \ldots, a_r)$ is given by

$$(8)\qquad \begin{aligned} G(t,x) &= \left(\frac{1}{1 + a_1 t + \cdots + a_r t^r} \right)^x \\ &= e^{-x \log(1 + a_1 t + \cdots + a_r t^r)} \\ &= \sum_{n=0}^{\infty} s_n(x; a_1, \ldots, a_r) \frac{t^n}{n!}. \end{aligned}$$

Thus, in the notation (2), $s_n(x; a_1, \ldots, a_r) \sim (1, f(t))$, with $\bar{f}(t) = -\log(1 + a_1 t + \cdots + a_r t^r)$.
Let $r_n(x)$ be an Appell sequence with $r_n(x) \sim (h(t),\ t)$, and let $s_n(x; a_1, \ldots, a_r) = \sum_{k=0}^{n} C_{n,k} r_k(x)$.
Then, from (4) and Theorem 1.1, we note that

$$(9)\qquad \begin{aligned} C_{n,k} &= \frac{1}{k!} \left\langle \left(\frac{1}{h(\bar{f}(t))} \right)^{-1} \bar{f}(t)^k \;\middle|\; x^n \right\rangle \\ &= \frac{1}{k!} < t^k \mid q_n(x) >, \end{aligned}$$

where $q_n(x) \sim (\frac{1}{h(t)},\ f(t))$.
This means that $C_{n,k}$ is the coefficient of x^k in $q_n(x)$. We state this important fact as the next theorem.

Theorem 2.1. *Let r be a positive integer and let $a_1, a_2, \ldots, a_r$ be complex numbers with $a_1, a_r \neq 0$. Assume that the generating function of the polynomials $s_n(x; a_1, \ldots, a_r)$ is given by*

$$\left(\frac{1}{1 + a_1 t + \cdots + a_r t^r} \right)^x = e^{x \bar{f}(t)} = \sum_{n=0}^{\infty} s_n(x; a_1, \ldots, a_r) \frac{t^n}{n!}.$$

Let $r_n(x)$ be an Appell sequence for $h(t)$, and let $s_n(x; a_1, \ldots, a_r) = \sum_{k=0}^{n} C_{n,k} r_k(x)$. Then $C_{n,k}$ is the coefficient of x^k in $q_n(x)$, where $q_n(x) \sim (\frac{1}{h(t)},\ f(t))$.

To proceed further, assume that

$$(10)\qquad 1 + a_1 t + \cdots + a_r t^r = (1 - \alpha_1 t) \cdots (1 - \alpha_r t).$$

Then we observe from (5) and (6) that

$$
\begin{aligned}
G(t,x) &= \sum_{n=0}^{\infty} s_n(x;a_1,\ldots,a_r)\frac{t^n}{n!} \\
&= e^{-x\log(1-\alpha_1 t)}\cdots e^{-x\log(1-\alpha_r t)} \\
&= (1-\alpha_1 t)^{-x}\cdots(1-\alpha_r t)^{-x} \\
&= \sum_{k_1=0}^{\infty} \alpha_1^{k_1} <x>_{k_1} \frac{t^{k_1}}{k_1!}\cdots\sum_{k_r=0}^{\infty} \alpha_r^{k_r} <x>_{k_r} \frac{t^{k_r}}{k_r!} \\
&= \sum_{n=0}^{\infty} \sum_{k_1+\cdots+k_r=n} \binom{n}{k_1,\ldots,k_r} \alpha_1^{k_1}\cdots\alpha_r^{k_r} <x>_{k_1}\cdots <x>_{k_r} \frac{t^n}{n!}.
\end{aligned} \tag{11}
$$

Hence, from (11) and (7), we have

$$
\begin{aligned}
s_n(x;a_1,\ldots,a_r) &= \sum_{k_1+\cdots+k_r=n} \binom{n}{k_1,\ldots,k_r} \alpha_1^{k_1}\cdots\alpha_r^{k_r} <x>_{k_1}\cdots <x>_{k_r} \\
&= \sum_{k_1+\cdots+k_r=n} \sum_{0\le i_1\le k_1,\ldots,0\le i_r\le k_r} \binom{n}{k_1,\ldots,k_r} \alpha_1^{k_1}\cdots\alpha_r^{k_r} |S_1(k_1,i_1)\cdots S_1(k_r,i_r)|\, x^{i_1+\cdots+i_r}.
\end{aligned} \tag{12}
$$

Now, assume that $h(\bar{f}(t)) = \sum_{n=0}^{\infty} \gamma_n \frac{t^n}{n!}$. Then, from Theorem 2.1, (3) and (11), we note that

$$
\sum_{n=0}^{\infty} q_n(x)\frac{t^n}{n!} = h(\bar{f}(t))e^{x\bar{f}(t)} = h(\bar{f}(t))G(t,x). \tag{13}
$$

Then, from (11) – (13), we see that

$$
\begin{aligned}
q_n(x) &= \sum_{m=0}^{n} \binom{n}{m} \gamma_{n-m} s_m(x;a_1,\ldots,a_r) \\
&= \sum_{m=0}^{n} \sum_{k_1+\cdots+k_r=m} \binom{n}{m}\binom{m}{k_1,\ldots,k_r} \gamma_{n-m}\alpha_1^{k_1}\cdots\alpha_r^{k_r} <x>_{k_1}\cdots <x>_{k_r} \\
&= \sum_{m=0}^{n} \sum_{k_1+\cdots+k_r=m} \sum_{0\le i_1\le k_1,\ldots,0\le i_r\le k_r} \binom{n}{m}\binom{m}{k_1,\ldots,k_r} \gamma_{n-m} \\
&\quad \times \alpha_1^{k_1}\cdots\alpha_r^{k_r} |S_1(k_1,i_1)\cdots S_1(k_r,i_r)|\, x^{i_1+\cdots+i_r}.
\end{aligned} \tag{14}
$$

Finally, from Theorem 2.1 and (14), the next theorem follows.

Theorem 2.2. *Let r be a positive integer and let $a_1,a_2,\ldots,a_r$ be complex numbers with $a_1,a_r \neq 0$. Assume that the generating function of the polynomials $s_n(x;a_1,\ldots,a_r)$ is given by*

$$
\left(\frac{1}{1+a_1 t+\cdots+a_r t^r}\right)^x = e^{x\bar{f}(t)} = \sum_{n=0}^{\infty} s_n(x;a_1,\ldots,a_r)\frac{t^n}{n!}.
$$

Let $r_n(x)$ be the Appell sequence for $h(t)$, $h(\bar{f}(t)) = \sum_{n=0}^{\infty} \gamma_n \frac{t^n}{n!}$, $s_n(x;a_1,\ldots,a_r) = \sum_{k=0}^{n} C_{n,k} r_k(x)$, and let

$$
1+a_1 t+\cdots+a_r t^r = (1-\alpha_1 t)\cdots(1-\alpha_r t).
$$

Then $C_{n,k}$ is given by

$$
\begin{aligned}
C_{n,k} &= \sum_{m=k}^{n} \sum_{k_1+\cdots+k_r=m} \sum \binom{n}{m}\binom{m}{k_1,\ldots,k_r} \gamma_{n-m} \\
&\quad \times \alpha_1^{k_1}\cdots\alpha_r^{k_r} |S_1(k_1,i_1)\cdots S_1(k_r,i_r)|,
\end{aligned}
$$

where the innermost sum is over all integers $i_1,\dots,i_r$, *satisfying* $0\le i_1\le k_1,\dots,0\le i_r\le k_r$, *with* $i_1+\cdots+i_r=k$.

Next, we will consider the following more general case.
Let r,s be integers with $0\le s\le r,\ r\ge 1$, and let $a_1,a_2,\dots,a_r,\ b_1,b_2,\dots,b_s$ be complex numbers, with $a_1,a_r,b_1,b_s\ne 0$. Further, $a_1\ne b_1$, if $s\ge 1$. Assume that

$$\begin{aligned}F(t,x)&=\left(\frac{1+b_1t+\cdots+b_st^s}{1+a_1t+\cdots+a_rt^r}\right)^x\\&=e^{x(\log(1+b_1t+\cdots+b_st^s)-\log(1+a_1t+\cdots+a_rt^r))}\\&=\sum_{n=0}^{\infty}s_n(x;a_1,\cdots,a_r;b_1,\cdots,b_s)\frac{t^n}{n!}.\end{aligned}\tag{15}$$

Thus $s_n(x;a_1,\cdots,a_r;b_1,\cdots,b_s)\sim(1,f(t))$, where

$$\bar{f}(t)=\log\left(\frac{1+b_1t+\cdots+b_st^s}{1+a_1t+\cdots+a_rt^r}\right).$$

Let $r_n(x)$ be an Appell sequence, with $r_n(x)\sim(h(t),\ t)$, and let $s_n(x;a_1,\cdots,a_r;b_1,\cdots,b_s)=\sum_{k=0}^{n}C_{n,k}r_k(x)$. Then, as in (9), we have

$$C_{n,k}=\frac{1}{k!}<t^k\mid q_n(x)>,$$

where $q_n(x)\sim(\frac{1}{h(t)},\ f(t))$. So $C_{n,k}$ is the coefficient of x^k in $q_n(x)$.
To proceed further, we assume that

$$1+a_1t+\cdots+a_rt^r=(1-\alpha_1t)\cdots(1-\alpha_rt),\ 1+b_1t+\cdots+b_st^s=(1-\beta_1t)\cdots(1-\beta_st).\tag{16}$$

Then we note from (5) and (6) that

$$\begin{aligned}F(t,x)&=\sum_{n=0}^{\infty}s_n(x;a_1,\dots,a_r;b_1,\dots,b_s)\frac{t^n}{n!}\\&=e^{x\log(1-\beta_1t)}\cdots e^{x\log(1-\beta_st)}e^{-x\log(1-\alpha_1t)}\cdots e^{-x\log(1-\alpha_rt)}\\&=(1-\beta_1t)^x\cdots(1-\beta_st)^x(1-\alpha_1t)^{-x}\cdots(1-\alpha_rt)^{-x}\\&=\sum_{l_1=0}^{\infty}\beta_1^{l_1}<-x>_{l_1}\frac{t^{l_1}}{l_1!}\cdots\sum_{l_s=0}^{\infty}\beta_s^{l_s}<-x>_{l_s}\frac{t^{l_s}}{l_s!}\\&\quad\times\sum_{k_1=0}^{\infty}\alpha_1^{k_1}<x>_{k_1}\frac{t^{k_1}}{k_1!}\cdots\sum_{k_r=0}^{\infty}\alpha_r^{k_r}<x>_{k_r}\frac{t^{k_r}}{k_r!}\\&=\sum_{n=0}^{\infty}\sum_{k_1+\cdots+k_r+l_1+\cdots+l_s=n}\binom{n}{k_1,\dots,k_r,l_1,\dots,l_s}\alpha_1^{k_1}\cdots\alpha_r^{k_r}\beta_1^{l_1}\cdots\beta_s^{l_s}\\&\quad\times<x>_{k_1}\cdots<x>_{k_r}<-x>_{l_1}\cdots<-x>_{l_s}\frac{t^n}{n!}.\end{aligned}\tag{17}$$

Hence, from (17) and (7), we have

$$\begin{aligned} s_n(x;a_1,\dots,a_r;b_1,\dots,b_s) &= \sum_{k_1+\cdots+k_r+l_1+\cdots+l_s=n} \binom{n}{k_1,\dots,k_r,l_1,\dots,l_s} \alpha_1^{k_1}\cdots\alpha_r^{k_r}\beta_1^{l_1}\cdots\beta_s^{l_s} \\ &\quad \times < x >_{k_1} \cdots < x >_{k_r} < -x >_{l_1} \cdots < -x >_{l_s} \\ &= \sum_{k_1+\cdots+k_r+l_1+\cdots+l_s=n} \sum \binom{n}{k_1,\dots,k_r,l_1,\dots,l_s} \alpha_1^{k_1}\cdots\alpha_r^{k_r}\beta_1^{l_1}\cdots\beta_s^{l_s} \\ &\quad \times (-1)^{j_1+\cdots+j_s} |S_1(k_1,i_1)\cdots S_1(k_r,i_r) S_1(l_1,j_1)\cdots S_1(l_s,j_s)| \\ &\quad \times x^{i_1+\cdots+i_r+j_1+\cdots+j_s}, \end{aligned} \tag{18}$$

where the inner sum is over all integers $i_1,\dots,i_r,j_1,\dots,j_s$ satisfying $0 \le i_1 \le k_1,\dots,0 \le i_r \le k_r$, $0 \le j_1 \le l_1,\dots,0 \le j_s \le l_s$.

Now, assume that $h(\bar{f}(t)) = \sum_{n=0}^{\infty} \gamma_n \frac{t^n}{n!}$. Then, from (3), we note that

$$\sum_{n=0}^{\infty} q_n(x)\frac{t^n}{n!} = h(\bar{f}(t))e^{x\bar{f}(t)} = h(\bar{f}(t))F(t,x). \tag{19}$$

Then, from from (17) – (19), we see that

$$\begin{aligned} q_n(x) &= \sum_{m=0}^{n} \binom{n}{m} \gamma_{n-m} s_m(x;a_1,\dots,a_r;b_1,\dots,b_s) \\ &= \sum_{m=0}^{n} \sum_{k_1+\cdots+k_r+l_1+\cdots+l_s=m} \binom{n}{m} \binom{m}{k_1,\dots,k_r,l_1,\dots,l_s} \gamma_{n-m}\alpha_1^{k_1}\cdots\alpha_r^{k_r}\beta_1^{l_1}\cdots\beta_s^{l_s} \\ &\quad \times < x >_{k_1} \cdots < x >_{k_r} < -x >_{l_1} \cdots < -x >_{l_s} \\ &= \sum_{m=0}^{n} \sum_{k_1+\cdots+k_r+l_1+\cdots+l_s=m} \sum \binom{n}{m} \binom{m}{k_1,\dots,k_r,l_1,\dots,l_s} \gamma_{n-m}\alpha_1^{k_1}\cdots\alpha_r^{k_r}\beta_1^{l_1}\cdots\beta_s^{l_s} \\ &\quad \times (-1)^{j_1+\cdots+j_s} |S_1(k_1,i_1)\cdots S_1(k_r,i_r) S_1(l_1,j_1)\cdots S_1(l_s,j_s)| \\ &\quad \times x^{i_1+\cdots+i_r+j_1+\cdots+j_s}, \end{aligned} \tag{20}$$

where the innermost sum is over all integers $i_1,\dots,i_r,j_1,\dots,j_s$ satisfying $0 \le i_1 \le k_1,\dots,0 \le i_r \le k_r$, $0 \le j_1 \le l_1,\dots,0 \le j_s \le l_s$.

Finally, from (20), the next theorem follows.

Theorem 2.3. *Let r,s be integers with $0 \le s \le r$, $r \ge 1$, and let $a_1,a_2,\dots,a_r$, $b_1,b_2,\dots,b_s$ be complex numbers, with $a_1,a_r,b_1,b_s \ne 0$. Further, $a_1 \ne b_1$, if $s \ge 1$. Assume that the generating function of the polynomials $s_n(x;a_1,\dots,a_r;b_1,\dots,b_s)$ is given by*

$$\left(\frac{1+b_1t+\cdots+b_st^s}{1+a_1t+\cdots+a_rt^r}\right)^x = e^{x\bar{f}(t)} = \sum_{n=0}^{\infty} s_n(x;a_1,\dots,a_r;b_1,\dots,b_s)\frac{t^n}{n!}.$$

Let $r_n(x)$ be the Appell sequence for $h(t)$, $h(\bar{f}(t)) = \sum_{n=0}^{\infty} \gamma_n \frac{t^n}{n!}$, $s_n(x;a_1,\dots,a_r;b_1,\dots,b_s) = \sum_{k=0}^{n} C_{n,k} r_k(x)$, and let

$$1+a_1t+\cdots+a_rt^r = (1-\alpha_1 t)\cdots(1-\alpha_r t),\ 1+b_1t+\cdots+b_st^s = (1-\beta_1 t)\cdots(1-\beta_s t).$$

Then $C_{n,k}$ is given by

$$\begin{aligned} C_{n,k} &= \sum_{m=k}^{n} \sum_{k_1+\cdots+k_r+l_1+\cdots+l_s=m} \sum \binom{n}{m} \binom{m}{k_1,\dots,k_r,l_1,\dots,l_s} \gamma_{n-m}\alpha_1^{k_1}\cdots\alpha_r^{k_r}\beta_1^{l_1}\cdots\beta_s^{l_s} \\ &\quad \times (-1)^{j_1+\cdots+j_s} |S_1(k_1,i_1)\cdots S_1(k_r,i_r) S_1(l_1,j_1)\cdots S_1(l_s,j_s)|, \end{aligned}$$

where the inner sum is over all integers $i_1,\dots,i_r, j_1,\dots,j_s$ *satisfying* $0 \le i_1 \le k_1,\dots,0 \le i_r \le k_r$, $0 \le j_1 \le l_1,\dots,0 \le j_s \le l_s$, *with* $i_1 \cdots + i_r + j_1 + \cdots + j_s = k$.

3. Examples

In this section, we are going to illustrate our results in the previous section by some examples.

3.1. **Representation of the rising factorial by Euler polynomials.** Here we represent the rising factorial $< x >_n$ in terms of the Euler polybomials $E_k(x)$. First, we observe that

$$\left(\frac{1}{1-t}\right)^x = e^{-x\log(1-t)} = \sum_{n=0}^{\infty} s_n(x;-1)\frac{t^n}{n!} = \sum_{n=0}^{\infty} < x >_n \frac{t^n}{n!}.$$

Thus $< x >_n \sim (1, f(t))$, with $\bar{f}(t) = -\log(1-t)$. Let $< x >_n = \sum_{k=0}^{n} C_{n,k}E_k(x)$. As $E_n(x) \sim \left(\frac{e^t+1}{2},\, t\right) = (h(t),\, t)$, it is immediate to see that

$$h(\bar{f}(t)) = \frac{1}{2}(e^{\bar{f}(t)}+1) = 1 + \sum_{l=1}^{\infty} \frac{l!}{2}\frac{t^l}{l!},$$

which says that

$$\gamma_0 = 1,\ \gamma_l = \frac{1}{2}l!,\ (l \ge 1). \tag{21}$$

Thus, from Theorem 2.2 and (21), we get

$$\begin{aligned} C_{n,k} &= \sum_{m=k}^{n} \binom{n}{m} \gamma_{n-m} |S_1(m,k)| \\ &= \frac{1}{2}n! \sum_{m=k}^{n} \frac{\varepsilon_{n,m}}{m!} |S_1(m,k)|, \end{aligned} \tag{22}$$

where

$$\varepsilon_{n,m} = \begin{cases} 1, & \text{if } n \ne m, \\ 2, & \text{if } n = m. \end{cases}$$

Hence we obtain the following representation from (22).

$$< x >_n = \frac{1}{2}n! \sum_{k=0}^{n} \sum_{m=k}^{n} \frac{\varepsilon_{n,m}}{m!} |S_1(m,k)| E_k(x). \tag{23}$$

In order to obtain another expression, we compute $C_{n,k}$ in a different manner. From Theorem 2.1, we note that

$$\begin{aligned} \sum_{n=0}^{\infty} q_n(x)\frac{t^n}{n!} &= \frac{e^{\bar{f}(t)}+1}{2} e^{x\bar{f}(t)} \\ &= (1-t)^{-x-1} - \frac{1}{2}t(1-t)^{-x-1} \\ &= \sum_{n=0}^{\infty} < x+1 >_n \frac{t^n}{n!} - \frac{1}{2}t \sum_{m=0}^{\infty} < x+1 >_m \frac{t^m}{m!} \\ &= 1 + \sum_{n=1}^{\infty} (< x+1 >_n - \frac{1}{2}n < x+1 >_{n-1})\frac{t^n}{n!} \\ &= 1 + \sum_{n=1}^{\infty} < x+1 >_{n-1} \left(x + \frac{1}{2}n\right)\frac{t^n}{n!}. \end{aligned} \tag{24}$$

Hence we see from (24) that

(25)
$$q_0(x)=1,\ q_n(x)=<x+1>_{n-1}\left(x+\frac{1}{2}n\right) = \sum_{k=0}^{n}(|S_1(n,k)|+\frac{1}{2}n|S_1(n,k+1)|)x^k,\ (n\geq 1).$$

Equivalently, (25) says that, for all $n \geq 0$, we have

$$C_{n,k}=|S_1(n,k)|+\frac{1}{2}n|S_1(n,k+1)|,$$

and from this we have another representation

(26)
$$<x>_n=\sum_{k=0}^{n}(|S_1(n,k)|+\frac{1}{2}n|S_1(n,k+1)|)E_k(x).$$

From (23) and (26), we get the identity

(27)
$$\frac{1}{2}n!\sum_{m=k}^{n}\frac{\varepsilon_{n,m}}{m!}|S_1(m,k)|=|S_1(n,k)|+\frac{1}{2}n|S_1(n,k+1)|.$$

Let us check the validity of (27), for $n=4, k=1$. Here we have to verify that

(28)
$$12\sum_{m=1}^{4}\frac{\varepsilon_{4,m}}{m!}|S_1(m,1)|=|S_1(4,1)|+2|S_1(4,2)|.$$

Indeed, the left hand side of (28) is

$$12\left\{\frac{1}{1!}|S_1(1,1)|+\frac{1}{2!}|S_1(2,1)|+\frac{1}{3!}|S_1(3,1)|+\frac{2}{4!}|S_1(4,1)|\right\} =12\left(\frac{1}{1!}\times 1+\frac{1}{2!}\times 1+\frac{1}{3!}\times 2+\frac{2}{4!}\times 6\right)=28.$$

Also, the right hand side of (28) is

$$|S_1(4,1)|+2|S_1(4,2)|=6+2\times 11=28.$$

3.2. **Representation of the rising factorial by Bernoulli polynomials.** Here we express the rising factorial $<x>_n$ in terms of Bernoulli polynomials $B_n(x)$. As before, $<x>_n\sim(1,f(t))$, with $\bar{f}(t)=-\log(1-t)$. Recalling that $B_n(x)\sim\left(\frac{e^t-1}{t},\ t\right)=(h(t),\ t)$, let $<x>_n=\sum_{k=0}^{n}C_{n,k}B_k(x)$. Then we first observe that

$$h(\bar{f}(t))=\frac{e^{\bar{f}(t)}-1}{\bar{f}(t)}=\frac{-t}{\log(1-t)}\sum_{l=0}^{\infty}t^l=n!\sum_{n=0}^{\infty}\sum_{l=0}^{n}(-1)^l\frac{b_l}{l!}\frac{t^n}{n!},$$

where the Bernoulli polynomials of the second kind are given by

(29)
$$\frac{t}{\log(1+t)}(1+t)^x=\sum_{n=0}^{\infty}b_n(x)\frac{t^n}{n!},$$

and $b_n=b_n(0)$ are the Bernoulli numbers of the second kind. Thus we have shown that

(30)
$$\gamma_n=n!\sum_{l=0}^{n}(-1)^l\frac{b_l}{l!}$$

From (30) and Theorem 2.2, we get

$$\begin{aligned} C_{n,k} &= \sum_{m=k}^{n} \binom{n}{m} \gamma_{n-m} |S_1(m,k)| \\ &= n! \sum_{m=k}^{n} \sum_{l=0}^{n-m} \frac{(-1)^l b_l}{m! l!} |S_1(m,k)|. \end{aligned}$$

Therefore we have the following expression

(31)
$$< x >_n = n! \sum_{k=0}^{n} \sum_{m=k}^{n} \sum_{l=0}^{n-m} \frac{(-1)^l b_l}{m! l!} |S_1(m,k)| B_k(x).$$

In order to derive another expression, we compute $C_{n,k}$ in a different way. From Theorem 2.1, we note that

(32)
$$\begin{aligned} \sum_{n=0}^{\infty} q_n(x) \frac{t^n}{n!} &= \frac{e^{\bar{f}(t)} - 1}{\bar{f}(t)} e^{x\bar{f}(t)} \\ &= \frac{-t}{\log(1-t)} (1-t)^{-x-1}. \end{aligned}$$

Now, from (29) and (32), we see that

(33)
$$q_n(x) = (-1)^n b_n(-x-1).$$

From Theorem 2.1 and (33), we have

(34)
$$C_{n,k} = \frac{1}{k!} (-1)^{n-k} b_n^{(k)}(-1),$$

where $b_n^{(k)}(x)$ is the kth derivative of $b_n(x)$. We would like to obtain more explicit expression for $C_{n,k}$. For this, we need an explicit expression of $b_n(x)$. From (7) and (29), we see that

$$\begin{aligned} b_n(x) &= \sum_{l=0}^{n} \binom{n}{l} b_l (x)_{n-l} \\ &= \sum_{l=0}^{n} \binom{n}{l} b_l \sum_{m=0}^{n-l} S_1(n-l,m) x^m \\ &= \sum_{m=0}^{n} \sum_{l=0}^{n-m} \binom{n}{l} b_l S_1(n-l,m) x^m, \end{aligned}$$

from which we see that

(35)
$$b_n^{(k)}(-1) = \sum_{m=k}^{n} \sum_{l=0}^{n-m} \binom{n}{l} b_l S_1(n-l,m) (m)_k (-1)^{m-k}.$$

Now, from (34) and (35), we get another expression of $C_{n,k}$ given by

$$C_{n,k} = \sum_{m=k}^{n} \sum_{l=0}^{n-m} (-1)^l \binom{n}{l} \binom{m}{k} b_l |S_1(n-l,m)|,$$

and hence we also have

(36)
$$< x >_n = \sum_{k=0}^{n} \sum_{m=k}^{n} \sum_{l=0}^{n-m} (-1)^l \binom{n}{l} \binom{m}{k} b_l |S_1(n-l,m)| B_k(x).$$

In addition, from (31) and (36) we have the following identity:

$$\sum_{m=k}^{n}\sum_{l=0}^{n-m}(-1)^l\binom{n}{l}\binom{m}{k}b_l|S_1(n-l,m)| = n!\sum_{m=k}^{n}\sum_{l=0}^{n-m}\frac{(-1)^l b_l}{m!l!}|S_1(m,k)|.$$

3.3. **Representation of some orthogonal polynomials by Bernoulli polynomials.** Here we will express some orthogonal polynomials and products of such polynomials in terms of Bernoulli polynomials. Consider the associated sequences given by the following generating function given by

$$\left(\frac{1}{1-2yt+t^2}\right)^x = \sum_{n=0}^{\infty} s_n(x;-2y,1)\frac{t^n}{n!}. \tag{37}$$

We note that various special values of x in (37) give the following identities:

$$\begin{aligned}
U_n(y) &= \frac{1}{n!}s_n(1;-2y,1),\\
P_n(y) &= \frac{1}{n!}s_n(\frac{1}{2};-2y,1),\\
C_n^{(\lambda)}(y) &= \frac{1}{n!}s_n(\lambda;-2y,1),\\
\sum_{i_1+\cdots+i_r=n} U_{i_1}(y)\cdots U_{i_r}(y) &= \frac{1}{n!}s_n(r;-2y,1),\\
\sum_{i_1+\cdots+i_r=n} P_{i_1}(y)\cdots P_{i_r}(y) &= \frac{1}{n!}s_n(\frac{r}{2};-2y,1),
\end{aligned} \tag{38}$$

where $U_n(y), P_n(y)$, and $C_n^{(\lambda)}(y)$ are respectively the Chebyshev polynomials of the second kind, the Legendre polynomials and the Gegenbauer polynomials. Here $s_n(x;-2y,1) \sim (1, f(t))$, with $\bar{f}(t) = -\log(1-2yt+t^2)$. Write $1-2yt+t^2 = (1-\alpha_1 t)(1-\alpha_2 t)$. Then we see that

$$\alpha_1 = y+\sqrt{y^2-1},\ \alpha_2 = y-\sqrt{y^2-1}. \tag{39}$$

Thus we have

Recalling that $B_n(x) \sim \left(\frac{e^t-1}{t},\ t\right) = (h(t),\ t)$, let $s_n(x;-2y,1) = \sum_{k=0}^{n} C_{n,k}B_k(x)$. Here $C_{n,k} = C_{n,k}(y)$ depends on y. Then, with $u = -2yt+t^2$, we have

$$\begin{aligned}
h(\bar{f}(t)) = \frac{e^{\bar{f}(t)}-1}{\bar{f}(t)} &= \frac{u}{\log(1+u)}(1+u)^{-1} = \sum_{m=0}^{\infty} b_m(-1)\frac{u^m}{m!}\\
&= \sum_{m=0}^{\infty} b_m(-1)\frac{1}{m!}\sum_{l=0}^{m}\binom{m}{l}(-2y)^{m-l}t^{l+m}\\
&= \sum_{n=0}^{\infty}\sum_{l=0}^{[\frac{n}{2}]} b_{n-l}(-1)\frac{n!}{l!(n-2l)!}(-2y)^{n-2l}\frac{t^n}{n!}.
\end{aligned}$$

Hence we have shown that

$$\gamma_n = \sum_{l=0}^{[\frac{n}{2}]} b_{n-l}(-1)\frac{n!}{l!(n-2l)!}(-2y)^{n-2l}. \tag{40}$$

From Theorem 2.2, we also have

$$C_{n,k} = \sum_{m=k}^{n}\sum_{k_1+k_2=m}\sum\binom{n}{m}\binom{m}{k_1,k_2}\gamma_{n-m}\alpha_1^{k_1}\alpha_2^{k_2}|S_1(k_1,i_1)S_1(k_2,i_2)|. \tag{41}$$

Finally, we have

$$(42)\qquad s_n(x;-2y,1)=\sum_{k=0}^{n}\sum_{m=k}^{n}\sum_{k_1+k_2=m}\sum\binom{n}{m}\binom{m}{k_1,k_2}\gamma_{n-m}\alpha_1^{k_1}\alpha_2^{k_2}|S_1(k_1,i_1)S_1(k_2,i_2)|B_k(x).$$

Here the innermost sums in (41) and (42) are over $0\le i_1\le k_1, 0\le i_2\le k_2$, with $i_1+i_2=k$, and α_1,α_2, and γ_n are as in (39) and (40). Substituting $1,\frac{1}{2},\lambda,r,\frac{r}{2}$ for x in (42) give expressions of the polynomials on the left hand sides of (38). For example, we have

$$\sum_{i_1+\cdots+i_r=n}P_{i_1}(y)\cdots P_{i_r}(y)=\sum_{k=0}^{n}\sum_{m=k}^{n}\sum_{k_1+k_2=m}\sum\binom{n}{m}\binom{m}{k_1,k_2}\gamma_{n-m}\alpha_1^{k_1}\alpha_2^{k_2}|S_1(k_1,i_1)S_1(k_2,i_2)|B_k\big(\frac{r}{2}\big).$$

3.4. **Representation of some orthogonal polynomials by Euler polynomials.** Here we represent the orthogonal polynomials and products of such polynomials in the previous subsection in terms of Euler polynomials. As before, $s_n(x;-2y,1)\sim(1,f(t))$, with $\bar{f}(t)=-\log(1-2yt+t^2)$. Also, let α_1,α_2 be as in (39), so that $1-2yt+t^2=(1-\alpha_1 t)(1-\alpha_2 t)$. Recalling that $E_n(x)\sim\big(\frac{e^t+1}{2},\,t\big)=(h(t),\,t)$, we note that, with $u=2yt-t^2$,

$$\begin{aligned}h(\bar{f}(t))&=\frac{1}{2}\big(e^{\bar{f}(t)}+1\big)=\frac{1}{2}\big(\sum_{m=0}^{\infty}u^m+1\big)\\&=\frac{1}{2}\big(\sum_{m=0}^{\infty}\sum_{l=0}^{m}(-1)^l\binom{m}{l}(2y)^{m-l}t^{m+l}\big)\\&=\sum_{n=1}^{\infty}\frac{n!}{2}\sum_{l=0}^{[\frac{n}{2}]}(-1)^l\binom{n-l}{l}(2y)^{n-2l}\frac{t^n}{n!}+1.\end{aligned}$$

Therefore we obtain

$$(43)\qquad \gamma_n=\frac{n!}{2}\varepsilon_{n,0}\sum_{l=0}^{[\frac{n}{2}]}(-1)^l\binom{n-l}{l}(2y)^{n-2l},$$

where

$$\varepsilon_{n,0}=\begin{cases}1, & \text{if } n\ge 1,\\ 2, & \text{if } n=0.\end{cases}$$

Using $1-u=(1-\alpha_1 t)(1-\alpha_2 t)$, it is immediate to show that γ_n is alternatively given by

$$(44)\qquad \gamma_n=\frac{n!}{2}\varepsilon_{n,0}\frac{\alpha_1^{n+1}-\alpha_2^{n+1}}{\alpha_1-\alpha_2}.$$

Thus from (43) and (44) we obtain

$$\sum_{l=0}^{[\frac{n}{2}]}(-1)^l\binom{n-l}{l}(2y)^{n-2l}=\frac{\alpha_1^{n+1}-\alpha_2^{n+1}}{\alpha_1-\alpha_2}.$$

Now, we finally get

$$(45)\qquad C_{n,k}=\sum_{m=k}^{n}\sum_{k_1+k_2=m}\sum\binom{n}{m}\binom{m}{k_1,k_2}\gamma_{n-m}\alpha_1^{k_1}\alpha_2^{k_2}|S_1(k_1,i_1)S_1(k_2,i_2)|,$$

and

$$(46)\qquad s_n(x;-2y,1)=\sum_{k=0}^{n}\sum_{m=k}^{n}\sum_{k_1+k_2=m}\sum\binom{n}{m}\binom{m}{k_1,k_2}\gamma_{n-m}\alpha_1^{k_1}\alpha_2^{k_2}|S_1(k_1,i_1)S_1(k_2,i_2)|E_k(x).$$

Here the innermost sums in (45) and (46) are over all i_1,i_2, with $0\le i_1\le k_1, 0\le i_2\le k_2$, $i_1+i_2=k$ and α_1,α_2, and γ_n are as in (39) and (44).

3.5. **Representation of the Mittag-Leffler polynomial by Euler and Bernoulli polynomials.** First, we represent the Mittag-Leffler polynomial $M_n(x)$ in terms of the Euler polybomials $E_k(x)$. First, we note that

$$\left(\frac{1+t}{1-t}\right)^x = e^{x(\log(1+t)-\log(1-t))} = \sum_{n=0}^{\infty} s_n(x;-1;1)\frac{t^n}{n!} = \sum_{n=0}^{\infty} M_n(x)\frac{t^n}{n!}.$$

Hence $M_n(x) \sim (1,\ f(t))$, with $\bar{f}(t) = \log(1+t) - \log(1-t)$. Let $M_n(x) = \sum_{k=0}^{n} C_{n,k}E_k(x)$. Then, from $E_n(x) \sim \left(\frac{e^t+1}{2},\ t\right) = (h(t),\ t)$, we see that

$$h(\bar{f}(t)) = \sum_{n=0}^{\infty} \gamma_n \frac{t^n}{n!} = \frac{1}{2}(e^{\bar{f}(t)}+1) = \sum_{n=0}^{\infty} t^n.$$

Thus $\gamma_n = n!$, and from Theorem 2.3, we have

$$\begin{aligned} C_{n,k} &= \sum_{m=k}^{n} \sum_{k_1+l_1=m} \sum \binom{n}{m}\binom{m}{k_1,l_1}\gamma_{n-m}(-1)^{l_1+j_1}|S_1(k_1,i_1)S_1(l_1,j_1)| \\ &= n!\sum_{m=k}^{n} \sum_{k_1+l_1=m} \sum \frac{1}{k_1!l_1!}(-1)^{l_1+j_1}|S_1(k_1,i_1)S_1(l_1,j_1)| \end{aligned} \tag{47}$$

and hence we get

$$M_n(x) = n!\sum_{k=0}^{n}\sum_{m=k}^{n} \sum_{k_1+l_1=m} \sum \frac{1}{k_1!l_1!}(-1)^{l_1+j_1}|S_1(k_1,i_1)S_1(l_1,j_1)|E_k(x), \tag{48}$$

where the innermost sums in (47) and (48) are over all i_1, j_1, with $0 \le i_1 \le k_1, 0 \le j_1 \le l_1, i_1+j_1 = k$.

Next, we represent the Mittag-Leffler polynomial $M_n(x)$ in terms of the Bernoulli polybomials $B_k(x)$. Recalling that $B_n(x) \sim \left(\frac{e^t-1}{t},\ t\right)$, let $M_n(x) = \sum_{k=0}^{n} C_{n,k}B_k(x)$. Then, with $u = \frac{2t}{1-t}$, we observe that

$$\begin{aligned} h(\bar{f}(t)) &= \frac{e^{\bar{f}(t)}-1}{\bar{f}(t)} = \frac{u}{\log(1+u)} = \sum_{s=0}^{\infty} b_s \frac{1}{s!}u^s \\ &= 1 + \sum_{s=1}^{\infty} b_s \frac{1}{s!}2^s t^s \sum_{l=0}^{\infty}\binom{s+l-1}{l}t^l \\ &= 1 + \sum_{n=1}^{\infty}\sum_{s=1}^{n} 2^s \frac{n!}{s!}\binom{n-1}{s-1}b_s \frac{t^n}{n!}, \end{aligned} \tag{49}$$

where b_s are the Bernoulli numbers of the second kind in (29).

This shows that $\gamma_0 = 1$, $\gamma_n = \sum_{s=1}^{n} 2^s \frac{n!}{s!}\binom{n-1}{s-1}b_s$, $(n \ge 1)$. Thus, $\gamma_n = \sum_{s=0}^{n} 2^s \frac{n!}{s!}\binom{n-1}{s-1}b_s$, with the convention that $\binom{-1}{-1} = 1$, $\binom{m}{-1} = 0$, $(m \ge 0)$. Now, from this observation and Theorem 2.3, we have

$$\begin{aligned} C_{n,k} &= \sum_{m=k}^{n} \sum_{k_1+l_1=m} \sum \binom{n}{m}\binom{m}{k_1,l_1}\gamma_{n-m}(-1)^{l_1+j_1}|S_1(k_1,i_1)S_1(l_1,j_1)| \\ &= n!\sum_{m=k}^{n} \sum_{k_1+l_1=m} \sum_{s=0}^{n-m} \sum \frac{2^s}{k_1!l_1!s!}\binom{n-m-1}{s-1} b_s(-1)^{l_1+j_1}|S_1(k_1,i_1)S_1(l_1,j_1)| \end{aligned} \tag{50}$$

and hence we get

$$\begin{aligned} M_n(x) &= n!\sum_{k=0}^{n}\sum_{m=k}^{n} \sum_{k_1+l_1=m} \sum_{s=0}^{n-m} \sum \frac{2^s}{k_1!l_1!s!}\binom{n-m-1}{s-1} \\ &\quad \times b_s(-1)^{l_1+j_1}|S_1(k_1,i_1)S_1(l_1,j_1)|B_k(x), \end{aligned} \tag{51}$$

where the innermost sums in (50) and (51) are over all i_1, j_1, with $0 \le i_1 \le k_1, 0 \le j_1 \le l_1, i_1 + j_1 = k$.

4. Further remark

Here we will mention a few examples that can be expressed by some polynomials in the family of associated sequences considered in Section 2.

Consider the associated sequences given by the following generating functions given by

$$\begin{aligned}
\left(\frac{1-yt}{1-2yt+t^2}\right)^x &= \sum_{n=0}^{\infty} s_n(x;-2y,1;-y)\frac{t^n}{n!},\\
\left(\frac{1-t}{1-2yt+t^2}\right)^x &= \sum_{n=0}^{\infty} s_n(x;-2y,1;-1)\frac{t^n}{n!},\\
\left(\frac{1+t}{1-2yt+t^2}\right)^x &= \sum_{n=0}^{\infty} s_n(x;-2y,1;1)\frac{t^n}{n!},\\
\left(\frac{1}{1-yt-t^2}\right)^x &= \sum_{n=0}^{\infty} s_n(x;-y,-1)\frac{t^n}{n!},\\
\left(\frac{1-\frac{y}{2}t}{1-yt-t^2}\right)^x &= \sum_{n=0}^{\infty} s_n(x;-y,-1;-\frac{y}{2})\frac{t^n}{n!}.
\end{aligned} \tag{52}$$

We note here that some special values of x in (52) give the following:

$$\begin{aligned}
&T_n(y) = \frac{1}{n!}s_n(1;-2y,1;-y),\\
&\sum_{i_1+\cdots+i_r=n} T_{i_1}(y)\cdots T_{i_r}(y) = \frac{1}{n!}s_n(r;-2y,1;-y),\\
&V_n(y) = \frac{1}{n!}s_n(1;-2y,1;-1),\\
&\sum_{i_1+\cdots+i_r=n} V_{i_1}(y)\cdots V_{i_r}(y) = \frac{1}{n!}s_n(r;-2y,1;-1),\\
&W_n(y) = \frac{1}{n!}s_n(1;-2y,1;1),\\
&\sum_{i_1+\cdots+i_r=n} W_{i_1}(y)\cdots W_{i_r}(y) = \frac{1}{n!}s_n(r;-2y,1;1),\\
&L_n(y) = \frac{2}{n!}s_n(1;-y,-1;-\frac{y}{2}),\\
&\sum_{i_1+\cdots+i_r=n} L_{i_1}(y)\cdots L_{i_r}(y) = \frac{2^r}{n!}s_n(r;-y,-1;-\frac{y}{2}),\\
&F_{n+1}(y) = \frac{1}{n!}s_n(1;-y,-1),\\
&\sum_{i_1+\cdots+i_r=n} F_{i_1+1}(y)\cdots F_{i_r+1}(y) = \frac{1}{n!}s_n(r;-y,-1),
\end{aligned}$$

where $T_n(y), V_n(y), W_n(y), L_n(y)$, and $F_n(y)$ $(n = 1, 2, \dots)$ are respectively the Chebyshev polunomials of the first, third and fourth kinds, the Lucas polynomials given by $\frac{2-yt}{1-yt-t^2} = \sum_{n=0}^{\infty} L_n(y)t^n$, and the Fibobacci polynomials given by $\frac{1}{1-yt-t^2} = \sum_{n=0}^{\infty} F_{n+1}t^n$.

In [8], the reader can find more examples that can be represented by certain polynomials in the family of associated sequences considered in Section 2. These include Sinha polynomials, Humbert polynomials, first kind Chebyshev polynomials in several variables, second kind Chebyshev polynomials in several variables and tribonacci polynomials. For more detail, we let the reader refer to [8].

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