Advanced Studies in Contemporary Mathematics
30 (2020), No. 4, pp. 621 - 624

www.jangjeon.or.kr
http://dx.doi.org/10.17777/ascm2020.30.4.621

PSEUDOREPRESENTATIONS WITH SMALL DEFECT THAT ARE TRIVIAL ON A NORMAL SUBGROUP

A. I. Shtern

Abstract. A sufficient condition for the triviality of the restriction of a pseudorepresentation of a group to a normal subgroup is given.

§ 1. Introduction

Recall that a mapping π of a given group G into the family of invertible operators on a Hilbert space E is said to be a *quasirepresentation* of G on E if $\pi(e_G) = 1_E$, where e_G stands for the identity element of G and 1_E for the identity operator on the space E, and

$$\|\pi(g_1g_2) - \pi(g_1)\pi(g_2)\| \leq \varepsilon, \qquad g_1, g_2 \in G,$$

for some ε, which is usually assumed to be sufficiently small and (its greatest lower bound for π) is referred to as a *defect* of π, and a quasirepresentation π of G is said to be a *pseudorepresentation* of G if $\pi(g^n)$ is conjugate to $\pi(g)^n$, $n \in \mathbb{Z}$, with the help of an operator sufficiently close to the identity operator. For the generalities concerning pseudorepresentations and quasirepresentations of groups, see [1–6].

§ 2. Preliminaries

If a pseudorepresentation π of G is one-dimensional and has a sufficiently small defect (less than 0.24) and if π is trivial on a normal subgroup N of G (i.e., $\pi(n) = 1_E$ for every $n \in N$, where 1_E stands for the identity operator

2010 *Mathematics Subject Classification.* Primary 22A99, Secondary 22A25, 22A10.
Key words and phrases. Pseudorepresentation, normal subgroup.

Typeset by $\mathcal{A}\mathcal{M}\mathcal{S}$-TEX on E), then, as was shown in [7], the one-dimensional pseudorepresentation π is completely determined by some pseudorepresentation of the quotient group G/N. It is unclear whether or not a similar assertion holds for general pseudorepresentations. However, the conditions under which the restriction of a given pseudorepresentation to a given normal subgroup is trivial in the above sense are of interest. In the note, we give a sufficient condition for the triviality of such a restriction of a pseudorepresentation of a group to a normal subgroup.

§ 3. Main theorem

The following theorem is a triviality theorem, i.e., establishes conditions under which a given pseudorepresentation is the identity representation on the corresponding Hilbert space. The theorem uses qualitative results of [2].

Theorem. *Let G be a group, let π be a pseudorepresentation of G on a Hilbert space E with a defect ε, let $\|\pi(g)\| \le C$ and $\|(\pi(g))^{-1}\| \le C$ for all $g \in G$, and let $\varepsilon < (C+\varepsilon)^{-2}/2$. Let $\|\pi(g)-1_E\| \le \delta$ and $\|(\pi(g))^{-1}-1_E\| \le \delta$ for all $g \in G$ and some δ such that $\varepsilon < (1+\delta+\varepsilon)^{-2}/2$, $\delta < 1$, and*

$$(1) \qquad (1+2\varepsilon((1-\delta)^{-1}+\varepsilon)^2)\varepsilon(1-2\varepsilon((1-\delta)^{-1}+\varepsilon)^2)^{-1} < \sqrt{3}-\varepsilon.$$

Then $\pi(g) = 1_E$ for all $g \in G$.

Proof. Recall that, by Theorem 5.3 of [2], if $\|\pi(g)\| \le C$ and $\|(\pi(g))^{-1}\| \le C$ for all $g \in G$, then the operator Q implementing the conjugacy of $\pi(g^n)$ and $(\pi(g))^n$ can be chosen to satisfy the inequality

$$\|Q - 1_E\| \le 2\varepsilon(C+\varepsilon)^2,$$

where ε stands for the defect of π. Since, in our case, the inequality

$$C \le \max((1+\delta),(1-\delta)^{-1}) = (1-\delta)^{-1}$$

holds, it follows from what was said above that the norm of $Q\pi(g^n)Q^{-1}-1_E$ does not exceed the left-hand side of (1). Therefore, $\|\pi(g)^n - 1_E\|$ does not exceed the left-hand side of (1) increased by ε, and hence is less than $\sqrt{3}$. The representation $\{\pi(g)^n, n \in \mathbb{Z}\}$ of the cyclic subgroup generated by $\pi(g)$, $g \in G$ (the group $G(g)$ is commutative and hence amenable), turns out to be bounded. Therefore, this representation is conjugate to a unitary representation. It follows from the inequality

$$\|\pi(g)^n - 1_E\| < \sqrt{3}, \qquad g \in G, \quad n \in \mathbb{Z},$$

that there are no spectral points of $\pi(g^n)$ outside the arc on the unit circle formed by the complex numbers z, $|z| = 1$, with $|z-1| < \sqrt{3}$. This arc contains no subgroups of the unit circle except for the identity subgroup $\{1\}$. Hence, the unitary operators of the representation

$$g^n \mapsto \pi(g)^n, \qquad g \in G, \quad n \in \mathbb{Z},$$

are identity operators, and the corresponding representation of the cyclic group $\{g^n, n \in \mathbb{Z}\}$ takes all elements to the identity operator 1_E. Since $g \in G$ is arbitrary, it follows that $\pi(g) = 1_E$ for all $g \in G$, as was to be proved.

§ 4. Concluding remarks

The above theorem implies immediately the following corollary which gives sufficient conditions under which a given pseudorepresentation of a group is trivial on a given normal subgroup of the group.

Corollary. *Let G be a group, let π be a pseudorepresentation of G on a Hilbert space E with a defect ε, let N be a normal subgroup of G, let $\|\pi(n)\| \le C$ and $\|(\pi(n))^{-1}\| \le C$ for all $n \in N$, and let $\varepsilon < (C+\varepsilon)^{-2}/2$. Let*

$$\|\pi(n) - 1_E\| \le \delta \quad \textit{and} \quad \|(\pi(n))^{-1} - 1_E\| \le \delta$$

for all $n \in G$ and some δ such that

$$\varepsilon < (1+\delta+\varepsilon)^{-2}/2, \qquad \delta < 1,$$

and, as in (1),

$$(1 + 2\varepsilon((1-\delta)^{-1} + \varepsilon)^2)\varepsilon(1 - 2\varepsilon((1-\delta)^{-1} + \varepsilon)^2)^{-1} < \sqrt{3} - \varepsilon.$$

Then $\pi(n) = 1_E$ for all $n \in N$.

In contrast to the case of one-dimensional pseudorepresentations (see [7]), it is unclear whether or not a pseudorepresentation trivial on a normal subgroup is determined by a pseudorepresentation of the corresponding quotient group.

Acknowledgments

I thank Professor Taekyun Kim for the invitation to publish this paper in the Advanced Studies of Contemporary Mathematics.

Funding

The research was partially supported by the Scientific Research Institute of System Analysis, Russian Academy of Sciences (FGU FNTs NIISI RAN), theme NIR 0065-2019-0007.

References

1. A. I. Shtern, Quasireprezentations and pseudorepresentations, Funktsional. Anal. i Prilozhen. **25** (2), 70–73 (1991) [Funct. Anal. Appl. **25** (2), 140–143 (1991)].
2. A. I. Shtern, Quasisymmetry. I. Russ. J. Math. Phys. **2** (3), 353–382 (1994).
3. A. I. Shtern, A version of van der Waerden's theorem and a proof of Mishchenko's conjecture on homomorphisms of locally compact groups. Izv. Ross. Akad. Nauk Ser. Mat. **72** (1), 183–224 (2008) [Izv. Math. **72** (1), 169–205 (2008)].
4. A. I. Shtern, Finite-dimensional quasirepresentations of connected Lie groups and Mishchenko's conjecture. Fundam. Prikl. Mat. **13** (7), 85–225 (2007) [J. Math. Sci. (N. Y.) **159** (5), 653–751 (2009)].
5. A. I. Shtern, "Locally Bounded Finally Precontinuous Finite-Dimensional Quasirepresentations of Locally Compact Groups," Mat. Sb. **208** (10), 149–170 (2017) [Sb. Math. **208** (10), 1557–1576 (2017)].
6. A. I. Shtern, "Specific properties of one-dimensional pseudorepresentations of groups," Fundam. Prikl. Mat. **21** (1), 247–255 (2016) [J. Math. Sci. (N.Y.) **233** (5), 770–776 (2018)].
7. A. I. Shtern, "One-Dimensional Pseudorepresentations with Small Defect that Are Trivial on a Normal Subgroup," Proc. Jangjeon Math. Soc. **23** (4), 553–556 (2020).

Moscow Center for Fundamental and Applied Mathematics, Moscow, 119991 Russia
Faculty of Mechanics and Mathematics, Lomonosov Moscow State University, Moscow, 119991 Russia
Scientific Research Institute of System Analysis, Russian Academy of Sciences (FGU FNTs NIISI RAN),
Moscow, 117312 Russia
E-mail: aishtern@mtu-net.ru, rroww@mail.ru