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A NOTE ON DEGENERATE MULTI-POLY-BERNOULLI POLYNOMIALS

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Dedicate to retirement of Professor Seog-Hoon Rim

Abstract. Recently, Kim-Kim-Kim-Kwon considered the degenerate multi-poly-Genocchi polynomials which are defined by the degenerate multiple polyexponential functions. Motivated from their work, we consider the degenerate multi-poly-Bernoulli polynomials and investigate some properties of those polynomials. Also, Kim introduced the degenerate multiple polylogarithm functions which are multiple version of the degenerate modified polylogarithm function. We also consider the degenerate type 2 multi-poly-Bernoulli polynomials which are explained by those functions. These are extensions of the degenerate type 2 poly-Bernoulli polynomials which are investigated by Kim-Kim-Kwon-Lee. We obtain explicit expressions and some properties for the degenerate type 2 multi-poly-Bernoulli polynomials

1. Introduction

We recall that, for all $k \in \mathbb{Z}$, the polylogarithm functions are defined by

$$Li_k(x) = \sum_{k=1}^{\infty} \frac{x^n}{n^k}, \quad (|x| < 1), \quad (\text{see [1, 3, 9, 15, 17–20]}).$$

The polyexponential functions were studied by Hardy (cf. [13, 15]). Recently, a slightly different version of those functions, which are called the modified polyexponential functions, are defined as an inverse to polylogarithm functions by Kim-Kim

$$Ei_k(x) = \sum_{n=1}^{\infty} \frac{x^n}{(n-1)!n^k}, \quad (k \in \mathbb{Z}), \quad (\text{see [3, 7, 8, 11, 15–17]}). \tag{1}$$

When $k = 1$, by (1), we get

$$Ei_1(x) = \sum_{n=1}^{\infty} \frac{x^n}{n!} = e^x - 1.$$

In [16] (see also [11]), the degenerate modified polyexponential functions are defined by Kim-Kim

$$Ei_{k,\lambda}(x) = \sum_{n=1}^{\infty} \frac{(1)_{n,\lambda}}{(n-1)!n^k} x^n, \quad (\lambda \in \mathbb{R}), \tag{2}$$

where $(x)_{0,\lambda} = 1$, $(x)_{n,\lambda} = x(x-\lambda)\cdots(x-(n-1)\lambda)$, $(n \geq 1)$.

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From (2), we note that

$$Ei_{1,\lambda}(x) = \sum_{n=1}^{\infty} \frac{(1)_{n,\lambda}}{n!} x^n = e_\lambda(x) - 1.$$

The degenerate exponential functions are given by $e_\lambda^x(t) = (1+\lambda t)^{\frac{x}{\lambda}}$, $e_\lambda(t) = e_\lambda^1(t)$, and the degenerate logarithm functions are defined by $\log_\lambda(t) = \frac{1}{\lambda}(t^\lambda - 1)$, which is the compositional inverse of $e_\lambda(t)$.

The degenerate poly-Bernoulli polynomials are defined by Kim-Kim-Jang in [12] by

$$\frac{Ei_{k,\lambda}(\log_\lambda(1+t))}{e_\lambda(t)-1} e_\lambda^x(t) = \sum_{n=0}^{\infty} \beta_{n,\lambda}^{(k)}(x) \frac{t^n}{n!}.$$

For $x = 0$, $\beta_{n,\lambda}^{(k)} = \beta_{n,\lambda}^{(k)}(0)$ are called the degenerate poly-Bernoulli numbers. Here $\beta_{n,\lambda}^{(1)}(x) = \beta_{n,\lambda}(x)$ are the Carlitz's Bernoulli polynomials given by

$$\frac{t}{e_\lambda(t)-1} e_\lambda^x(t) = \sum_{n=0}^{\infty} \beta_{n,\lambda}(x) \frac{t^n}{n!}, \quad \text{(see [2])}.$$

More generally, for $r \in \mathbb{N}$, the degenerate Bernoulli polynomials of order r are defined by

$$\left(\frac{t}{e_\lambda(t)-1}\right)^r e_\lambda^x(t) = \sum_{n=0}^{\infty} \beta_{n,\lambda}^{(r)}(x) \frac{t^n}{n!}, \quad \text{(see [2, 9])}. \tag{3}$$

From (3), we have the following identity.

$$\beta_{n,\lambda}^{(r)}(x) = \sum_{l=0}^{\infty} \binom{n}{l} \beta_{l,\lambda}^{(k)}(x)_{n-l,\lambda}.$$

We will need the degenerate Genocchi polynomials $G_{n,\lambda}^{(r)}(x)$ of order r, which are defined by

$$\left(\frac{2t}{e_\lambda(t)+1}\right)^r e_\lambda^x(t) = \sum_{n=0}^{\infty} G_{n,\lambda}^{(r)}(x) \frac{t^n}{n!}, \quad \text{(see [11])}. \tag{4}$$

Note here that $G_{0,\lambda}^{(r)}(x) = G_{1,\lambda}^{(r)}(x) = \cdots = G_{r-1,\lambda}^{(r)}(x) = 0$.

For $k_1, k_2, \cdots, k_r \in \mathbb{Z}$, we remind the reader of the degenerate multiple polyexponential functions, which are defined by Kim and Kim in [6],

$$Ei_{k_1,k_2,\cdots,k_r,\lambda}(x) = \sum_{0<n_1<\cdots<n_r} \frac{(1)_{n_1,\lambda}(1)_{n_2,\lambda}\cdots(1)_{n_r,\lambda} x^{n_r}}{(n_1-1)!(n_2-1)!\cdots(n_r-1)! n_1^{k_1} n_2^{k_2} \cdots n_r^{k_r}},$$

where the sum is over all integers $n_1, n_2, \cdots, n_r$, satisfying $0 < n_1 < n_2 < \cdots < n_r$.

2. Degenerate multi-poly-Bernoulli polynomials

For $\lambda \in \mathbb{R}$, the degenerate Stirling numbers of the first kind are defined by

$$(x)_n = \sum_{l=0}^{n} S_{1,\lambda}(n,l)(x)_{l,\lambda}, \quad (n \geq 0), \quad \text{(see [7])},$$

where $(x)_0 = 1$, $(x)_n = x(x-1)\cdots(x-n+1)$, $(n \geq 1)$.

The following identity is well-known in [7]

$$\frac{1}{k!}(\log_\lambda(1+t))^k = \sum_{n=k}^{\infty} S_{1,\lambda}(n,k)\frac{t^n}{n!}, \quad (k \ge 0),$$

and that $\lim_{\lambda\to 0} S_{1,\lambda}(n,l) = S_1(n,l)$, where $S_1(n,l)$ are the Stirling numbers of the first kind.

For $k_1, k_2, \cdots, k_r \in \mathbb{Z}$, the degenerate multi-poly-Bernoulli polynomials are defined by Khan-Khan-Duran in[5]

$$\frac{r!Ei_{k_1,k_2\cdots,k_r,\lambda}(\log_\lambda(1+t))}{(e_\lambda(t)-1)^r}e_\lambda^x(t) = \sum_{n=0}^{\infty}\beta_{n,\lambda}^{(k_1,k_2,\cdots,k_r)}(x)\frac{t^n}{n!}. \tag{5}$$

For $x=0$, $\beta_{n,\lambda}^{(k_1,k_2,\cdots,k_r)} = \beta_{n,\lambda}^{(k_1,k_2,\cdots,k_r)}(0)$ are called the degenerate multi-poly-Bernoulli numbers.

From (5), we have

$$\beta_{n,\lambda}^{(k_1,k_2,\cdots,k_r)}(x) = \sum_{l=0}^{n}\binom{n}{l}\beta_{l,\lambda}^{k_1,k_2,\cdots,k_r}(x)_{n-l,\lambda}, (n \ge 0).$$

For $k_1, k_2, \cdots, k_r \in \mathbb{Z}$, and any integer n, r with $n \ge 0, r \ge 1$, we have

$$\beta_{n,\lambda}^{(k_1,k_2,\cdots,k_r)}(x+y) = \sum_{l=0}^{n}\binom{n}{l}\beta_{l,\lambda}^{(k_1,k_2,\cdots,k_r)}(x)(y)_{n-l,\lambda}.$$

From (5), we can derive the following result.

$$\begin{aligned}
&\sum_{n=0}^{\infty}\beta_{n,\lambda}^{(k_1,k_2,\cdots,k_r)}(x)\frac{t^n}{n!}\\
&= r!\left(\frac{1}{e_\lambda(t)-1}\right)^r e_\lambda^x(t)\sum_{0<n_1<\cdots<n_r}\frac{(1)_{n_1,\lambda}(1)_{n_2,\lambda}\cdots(1)_{n_r,\lambda}\left(\log_\lambda(1+t)\right)^{n_r}}{(n_1-1)!(n_2-1)!\cdots(n_r-1)!n_1^{k_1}n_2^{k_2}\cdots n_r^{k_r}}\\
&= r!\left(\frac{1}{e_\lambda(t)-1}\right)^r e_\lambda^x(t)\sum_{0<n_1<\cdots<n_r\le l}\frac{(1)_{n_1,\lambda}(1)_{n_2,\lambda}\cdots(1)_{n_r,\lambda}S_1(l,n_r)}{(n_1-1)!(n_2-1)!\cdots(n_{r-1}-1)!n_1^{k_1}\cdots n_{r-1}^{k_{r-1}}n_r^{k_r-1}}\frac{t^{l-r}}{l!}\\
&= r!\sum_{m=0}^{\infty}\beta_{m,\lambda}^{(r)}(x)\frac{t^m}{m!}\sum_{0<n_1<\cdots<n_r\le l}\frac{(1)_{n_1,\lambda}(1)_{n_2,\lambda}\cdots(1)_{n_r,\lambda}S_1(l,n_r)}{(n_1-1)!(n_2-1)!\cdots(n_{r-1}-1)!n_1^{k_1}\cdots n_{r-1}^{k_{r-1}}n_r^{k_r-1}}\frac{t^{l-r}}{l!}\\
&= r!\sum_{n=r}^{\infty}\sum_{m=0}^{n-r}\binom{n}{m}\beta_{m,\lambda}^{(r)}(x)\sum_{0<n_1<\cdots<n_r\le n-m}\frac{(1)_{n_1,\lambda}(1)_{n_2,\lambda}\cdots(1)_{n_r,\lambda}S_1(n-m,n_r)}{(n_1-1)!(n_2-1)!\cdots(n_{r-1}-1)!n_1^{k_1}\cdots n_{r-1}^{k_{r-1}}n_r^{k_r-1}}\frac{t^{n-r}}{n!}\\
&=\sum_{n=0}^{\infty}\frac{1}{\binom{n+r}{r}}\sum_{m=0}^{n}\binom{n+r}{m}\beta_{m,\lambda}^{(r)}(x)\sum_{0<n_1<\cdots<n_r\le n+r-m}\frac{(1)_{n_1,\lambda}(1)_{n_2,\lambda}\cdots(1)_{n_r,\lambda}S_{1,\lambda}(n+r-m,n_r)}{(n_1-1)!(n_2-1)!\cdots(n_{r-1}-1)!n_1^{k_1}\cdots n_{r-1}^{k_{r-1}}n_r^{k_r-1}}\frac{t^n}{n!}.
\end{aligned}$$

Comparing the coefficients on both sides of (6), we have the following theorem.

Theorem 2.1 (cf. [5], Theorem 2)**.** *For $k_1, k_2, \cdots, k_r \in \mathbb{Z}$, and $n, r \in \mathbb{N}$, we have*

$$\begin{aligned}
\beta_{n,\lambda}^{(k_1,k_2,\cdots,k_r)}(x) &= \frac{1}{\binom{n+r}{r}}\sum_{m=0}^{n}\binom{n+r}{m}\beta_{m,\lambda}^{(r)}(x)\\
&\times\sum_{0<n_1<\cdots<n_r\le n+r-m}\frac{(1)_{n_1,\lambda}(1)_{n_2,\lambda}\cdots(1)_{n_r,\lambda}S_{1,\lambda}(n+r-m,n_r)}{(n_1-1)!(n_2-1)!\cdots(n_{r-1}-1)!n_1^{k_1}\cdots n_{r-1}^{k_{r-1}}n_r^{k_r-1}}.
\end{aligned}$$

Now we relate the degenerate Genocchi polynomials of order r and the degenerate Bernoulli polynomials of order r. From (3) and (4), we consider the following

$$\begin{aligned}\sum_{n=0}^{\infty} G_{n,\lambda}^{(r)}(x)\frac{t^n}{n!} &= \left(\frac{2t}{e_\lambda(t)+1}\right)^r e_\lambda^x(t) \\ &= \left(\frac{2t}{e_\lambda^2(t)-1}\right)^r e_\lambda^x(t)(e_\lambda(t)-1)^r \\ &= \sum_{l=0}^{r}\binom{r}{l}(-1)^{r-l}\left(\frac{2t}{e_\lambda^2(t)-1}\right)^r e_\lambda^{x+l}(t) \\ &= \sum_{l=0}^{r}\binom{r}{l}(-1)^{r-l}\left(\frac{2t}{e_{\frac{\lambda}{2}}(2t)-1}\right)^r e_{\frac{\lambda}{2}}^{\frac{x+l}{2}}(2t) \\ &= \sum_{n=0}^{\infty}\sum_{l=0}^{r}\binom{r}{l}(-1)^{r-l}\beta_{n,\frac{\lambda}{2}}^{(r)}\left(\frac{x+l}{2}\right)\frac{2^n t^n}{n!}.\end{aligned} \tag{6}$$

By comparing the coefficients on both sides of (6), we have the following proposition.

Proposition 2.2. *For and any integer n, r with $n \geq 0, r \geq 1$, we have*

$$G_{n,\lambda}^{(r)}(x) = 2^n \sum_{l=0}^{r}\binom{r}{l}(-1)^{r-l}\beta_{n,\frac{\lambda}{2}}^{(r)}\left(\frac{x+l}{2}\right)$$

Specially, for $r = 1$ case, the degenerate Genocchi polynomials are given by

$$G_{n,\lambda}(x) = 2^n\left(\beta_{n,\frac{\lambda}{2}}\left(\frac{x+1}{2}\right) - \beta_{n,\frac{\lambda}{2}}\left(\frac{x}{2}\right)\right).$$

For $k_1, k_2, \cdots, k_r \in \mathbb{Z}$, Kim-Kim-Kim-Kwon considered the degenerate multi-poly-Genocchi polynomials which are given by

$$\frac{2^r Ei_{k_1,k_2\cdots,k_r,\lambda}(\log_\lambda(1+t))}{(e_\lambda(t)+1)^r} e_\lambda^x(t) = \sum_{n=0}^{\infty} g_{n,\lambda}^{(k_1,k_2,\cdots,k_r)}(x)\frac{t^n}{n!}, \quad \text{(see [15])}. \tag{7}$$

Note that $g_{0,\lambda}^{(k_1,k_2,\cdots,k_r)}(x) = \cdots = g_{r-1,\lambda}^{(k_1,k_2,\cdots,k_r)}(x) = 0$. For $x = 0$, $g_{n,\lambda}^{(k_1,k_2,\cdots,k_r)} = g_{n,\lambda}^{(k_1,k_2,\cdots,k_r)}(0)$ are called the degenerate multi-poly-Genocchi numbers.

From (7), we have

$$\begin{aligned}
&\sum_{n=0}^{\infty} g_{n,\lambda}^{(k_1,k_2,\cdots,k_r)}(x)\frac{t^n}{n!} \\
&= \left(\frac{2}{e_\lambda(t)+1}\right)^r e_\lambda^x(t) Ei_{k_1,k_2\cdots,k_r,\lambda}(\log_\lambda(1+t)) \\
&= \left(\frac{2}{e_\lambda^2(t)-1}\right)^r e_\lambda^x(t)(e_\lambda(t)-1)^r Ei_{k_1,k_2\cdots,k_r,\lambda}(\log_\lambda(1+t)) \\
&= 2^r \sum_{l=0}^{r}\binom{r}{l}(-1)^{r-l}\left(\frac{1}{e_{\frac{\lambda}{2}}(2t)-1}\right)^r Ei_{k_1,k_2\cdots,k_r,\lambda}(\log_\lambda(1+t)) e_{\frac{\lambda}{2}}^{\frac{x+l}{2}}(2t) \\
&= 2^r \sum_{m=0}^{\infty}\sum_{l=0}^{r}(-1)^{r-l}\binom{r}{l}\beta_{m,\frac{\lambda}{2}}^{(r)}\left(\frac{x+l}{2}\right)\frac{2^m t^m}{m!} \\
&\quad\times \sum_{0<n_1<\cdots<n_r\le j}\frac{(1)_{n_1,\lambda}(1)_{n_2,\lambda}\cdots(1)_{n_r,\lambda}S_{1,\lambda}(j,n_r)}{(n_1-1)!(n_2-1)!\cdots(n_{r-1}-1)!n_1^{k_1}\cdots n_{r-1}^{k_{r-1}}n_r^{k_r-1}}\frac{t^j}{j!} \\
&= 2^r \sum_{n=r}^{\infty}\sum_{m=0}^{n-r}\binom{n}{m}\sum_{l=0}^{r}(-1)^{r-l}\binom{r}{l}\beta_{m,\frac{\lambda}{2}}\left(\frac{x+l}{2}\right)2^m \\
&\quad\times \sum_{0<n_1<\cdots<n_r\le n-m}\frac{(1)_{n_1,\lambda}(1)_{n_2,\lambda}\cdots(1)_{n_r,\lambda}S_{1,\lambda}(n-m,n_r)}{(n_1-1)!(n_2-1)!\cdots(n_{r-1}-1)!n_1^{k_1}\cdots n_{r-1}^{k_{r-1}}n_r^{k_r-1}}\frac{t^n}{n!}.
\end{aligned} \tag{8}$$

Thus comparing the coefficients on both sides of (8), we have the following theorem.

Theorem 2.3. *For $k_1, k_2, \cdots, k_r \in \mathbb{Z}$, and $n, r \in \mathbb{N}$ with $n \geq r$, we have*

$$\begin{aligned}
g_{n,\lambda}^{(k_1,k_2,\cdots,k_r)}(x) &= 2^r \sum_{m=0}^{n-r}\binom{n}{m}\sum_{l=0}^{r}(-1)^{r-l}\binom{r}{l}\beta_{m,\frac{\lambda}{2}}\left(\frac{x+l}{2}\right)2^m \\
&\quad\times \sum_{0<n_1<\cdots<n_r\le n-m}\frac{(1)_{n_1,\lambda}(1)_{n_2,\lambda}\cdots(1)_{n_r,\lambda}S_{1,\lambda}(n-m,n_r)}{(n_1-1)!(n_2-1)!\cdots(n_{r-1}-1)!n_1^{k_1}\cdots n_{r-1}^{k_{r-1}}n_r^{k_r-1}}.
\end{aligned}$$

3. Degenerate type 2 multi-poly-Bernoulli polynomials

Kim and Kim defined in [7] the degenerate polylogarithm function which is given by

$$l_{k,\lambda}(x) = \sum_{n=1}^{\infty}\frac{(-\lambda)^{n-1}(1)_{n,\frac{1}{\lambda}}}{(n-1)!n^k}x^n, \quad (|x|<1).$$

Then we note that

$$\frac{d}{dx}l_{k,\lambda}(x) = \frac{1}{x}l_{k-1,\lambda}(x),$$

and

$$l_{1,\lambda}(x) = -\log_\lambda(1-x).$$

In [10], Kim considered the degenerate Stirling numbers of the second kind which are defined as

$$(x)_{n,\lambda} = \sum_{k=0}^{n} S_{2,\lambda}(x)_k, \quad (n \geq 0).$$

Note that

$$\lim_{\lambda \to 0} S_{2,\lambda}(n,k) = S_2(n,k).$$

The generating function of the degenerate Stirling numbers of the second kind is given by

$$\frac{1}{k!}(e_\lambda(t) - 1)^k = \sum_{n=k}^{\infty} S_{2,\lambda}(n,k)\frac{t^n}{n!}, \quad (k \geq 0) \quad (\text{see [10]}).$$

For $k_1, k_2, \cdots, k_r \in \mathbb{Z}$, the degenerate multi-polylogarithm function is given by Jang in [4]

$$l_{k_1,k_2,\cdots,k_r,\lambda}(x) = \sum_{0<n_1<\cdots<n_r} \frac{(-\lambda)^{n_1+n_2+\cdots+n_r-r}(1)_{n_1,\frac{1}{\lambda}}(1)_{n_2,\frac{1}{\lambda}}\cdots(1)_{n_r,\frac{1}{\lambda}}x^{n_r}}{(n_1-1)!(n_2-1)!\cdots(n_r-1)!n_1^{k_1}n_2^{k_2}\cdots n_r^{k_r}},$$

where the sum is over all integers $n_1, n_2, \cdots, n_r$, satisfying $0 < n_1 < n_2 < \cdots < n_r$.

Now we consider the degenerate type 2 multi-poly-Bernoulli polynomials which are given by

$$r!\left(\frac{1}{e_\lambda(t)-1}\right)^r l_{k_1,k_2,\cdots,k_r,\lambda}(1-e_\lambda(-t))e_\lambda^x(t) = \sum_{n=0}^{\infty} \mathfrak{B}_{n,\lambda}^{(k_1,k_2,\cdots,k_r)}(x)\frac{t^n}{n!}. \tag{9}$$

For $x = 0$, $\mathfrak{B}_{n,\lambda}^{(k_1,k_2,\cdots,k_r)} = \mathfrak{B}_{n,\lambda}^{(k_1,k_2,\cdots,k_r)}(0)$ are called the degenerate type 2 multi-poly-Bernoulli numbers.

From (9), we note that

$$\mathfrak{B}_{n,\lambda}^{(k_1,k_2,\cdots,k_r)}(x) = \sum_{l=0}^{n} \binom{n}{l} \mathfrak{B}_{l,\lambda}^{(k_1,k_2,\cdots,k_r)}(x)_{n-l,\lambda}, \quad (n \geq 0).$$

For the case $r = 1$, the degenerate type 2 multi-poly-Bernoulli polynomials are the degenerate poly-Bernoulli polynomials which are defined by Kim-Kim-Kim-Lee-Jang in [14] as

$$\frac{l_k(1-e_\lambda(-t))}{e_\lambda(t)-1}e_\lambda^x(t) = \sum_{n=0}^{\infty} \mathfrak{B}_{n,\lambda}^{(k)}(x)\frac{t^n}{n!}.$$

Now we consider the following

$$l_{k_1,k_2,\cdots,k_r,\lambda}(1-e_\lambda(-t)) = \sum_{0<n_1<\cdots<n_r} \frac{(-\lambda)^{n_1+n_2+\cdots+n_r-r}(1)_{n_1,\frac{1}{\lambda}}(1)_{n_2,\frac{1}{\lambda}}\cdots(1)_{n_r,\frac{1}{\lambda}}(1-e_\lambda(-t))^{n_r}}{(n_1-1)!(n_2-1)!\cdots(n_r-1)!n_1^{k_1}n_2^{k_2}\cdots n_r^{k_r}}. \tag{10}$$

From (9) and (10), we note that

$$
\begin{aligned}
&\sum_{n=0}^{\infty}\mathfrak{B}_{n,\lambda}^{(k_1,k_2,\cdots,k_r)}(x)\frac{t^n}{n!}\\
&=r!\left(\frac{1}{e_\lambda(t)-1}\right)^r e_\lambda^x(t)\sum_{0<n_1<\cdots<n_r}\frac{(-\lambda)^{n_1+n_2+\cdots+n_r-r}(1)_{n_1,\frac{1}{\lambda}}\cdots(1)_{n_r,\frac{1}{\lambda}}(1-e_\lambda(-t))^{n_r}}{(n_1-1)!\cdots(n_r-1)!n_1^{k_1}n_2^{k_2}\cdots n_r^{k_r}}\\
&=r!\left(\frac{t}{e_\lambda(t)-1}\right)^r e_\lambda^x(t)\sum_{0<n_1<\cdots<n_r\le l}\frac{(-\lambda)^{n_1+n_2+\cdots+n_r-r}(1)_{n_1,\frac{1}{\lambda}}\cdots(1)_{n_r,\frac{1}{\lambda}}}{(n_1-1)!\cdots(n_{r-1}-1)!n_1^{k_1}\cdots n_{r-1}^{k_{r-1}}n_r^{k_r-1}}S_{2,\lambda}(l,n_r)\frac{(-1)^{l+n_r}t^{l-r}}{l!}\\
&=r!\left(\sum_{m=0}^{\infty}\beta_{m,\lambda}^{(r)}(x)\frac{t^m}{m!}\right)\left(\sum_{0<n_1<\cdots<n_r\le l}\frac{(-\lambda)^{n_1+n_2+\cdots+n_r-r}(1)_{n_1,\frac{1}{\lambda}}\cdots(1)_{n_r,\frac{1}{\lambda}}}{(n_1-1)!\cdots(n_{r-1}-1)!n_1^{k_1}\cdots n_{r-1}^{k_{r-1}}n_r^{k_r-1}}(-1)^{l+n_r}S_{2,\lambda}(l,n_r)\frac{t^{l-r}}{l!}\right)\\
&=\sum_{n=0}^{\infty}\left(\frac{1}{\binom{n+r}{r}}\sum_{m=0}^{n}\binom{n+r}{m}\beta_{m,\lambda}^{(r)}(x)\right.\\
&\times\left.\sum_{0<n_1<\cdots<n_r\le n+r-m}\frac{(-1)^{n+r-m+n_r}(-\lambda)^{n_1+n_2+\cdots+n_r-r}(1)_{n_1,\frac{1}{\lambda}}(1)_{n_2,\frac{1}{\lambda}}\cdots(1)_{n_r,\frac{1}{\lambda}}S_{2,\lambda}(n+r-m,n_r)}{(n_1-1)!\cdots(n_{r-1}-1)!n_1^{k_1}\cdots n_{r-1}^{k_{r-1}}n_r^{k_r-1}}\right)\frac{t^n}{n!}.
\end{aligned}
\tag{11}
$$

Comparing the coefficients on both sides of (11), we have the following theorem.

Theorem 3.1. *For $k_1,k_2,\cdots,k_r\in\mathbb{Z}$, and $n,r\in\mathbb{N}$, we have*

$$
\begin{aligned}
\mathfrak{B}_{n,\lambda}^{(k_1,k_2,\cdots,k_r)}(x)&=\frac{1}{\binom{n+r}{r}}\sum_{m=0}^{n}\binom{n+r}{m}\beta_{m,\lambda}^{(r)}(x)\\
&\times\sum_{0<n_1<\cdots<n_r\le n+r-m}\frac{(-1)^{n+r-m+n_r}(-\lambda)^{n_1+n_2+\cdots+n_r-r}(1)_{n_1,\frac{1}{\lambda}}(1)_{n_2,\frac{1}{\lambda}}\cdots(1)_{n_r,\frac{1}{\lambda}}S_{2,\lambda}(n+r-m,n_r)}{(n_1-1)!\cdots(n_{r-1}-1)!n_1^{k_1}\cdots n_{r-1}^{k_{r-1}}n_r^{k_r-1}}.
\end{aligned}
$$

For $k_1,k_2,\cdots,k_r\in\mathbb{Z}$, we consider the type 2 degenerate multi-poly-Genocchi polynomials, which are defined and investigated by Jang in [4]

$$
\sum_{n=0}^{\infty}\mathcal{G}_{n,\lambda}^{(k_1,k_2,\cdots,k_r)}(x)\frac{t^n}{n!}=\frac{2^r l_{k_1,k_2,\cdots,k_r,\lambda}(1-e_\lambda(-t))}{(e_\lambda(t)+1)^r}e_\lambda^x(t). \tag{12}
$$

By (9) and (12), we consider the following identity.

$$
\begin{aligned}
&\sum_{n=0}^{\infty}\mathcal{G}_{n,\lambda}^{(k_1,k_2,\cdots,k_r)}(x)\frac{t^n}{n!}\\
&=2^r\left(\frac{1}{e_\lambda^2(t)-1}\right)^r e_\lambda^x(t)(e_\lambda(t)-1)^r l_{k_1,k_2,\cdots,k_r,\lambda}(1-e_\lambda(-t))\\
&=\sum_{j=0}^{r}\binom{r}{j}(-1)^{r-j}2^r\left(\frac{1}{e_\lambda^2(t)-1}\right)^r e_\lambda^{x+j}(t)l_{k_1,k_2,\cdots,k_r,\lambda}(1-e_\lambda(-t))\\
&=\sum_{j=0}^{r}\binom{r}{j}(-1)^{r-j}2^r\left(\frac{1}{e_\lambda^2(2t)-1}\right)^r e_{\frac{\lambda}{2}}^{\frac{x+j}{2}}(2t)
\end{aligned}
$$

$$\times \sum_{0<n_1<\cdots<n_r} \frac{(-\lambda)^{n_1+n_2+\cdots+n_r-r}(1)_{n_1,\frac{1}{\lambda}}(1)_{n_2,\frac{1}{\lambda}}\cdots(1)_{n_r,\frac{1}{\lambda}}(1-e_\lambda(-t))^{n_r}}{(n_1-1)!(n_2-1)!\cdots(n_r-1)!n_1^{k_1}n_2^{k_2}\cdots n_r^{k_r}} \tag{13}$$

$$= \sum_{j=0}^{r}\binom{r}{j}(-1)^{r-j}2^r\left(\frac{t}{e_{\frac{\lambda}{2}}(2t)-1}\right)^r e_{\frac{\lambda}{2}}^{\frac{x+j}{2}}(2t)$$

$$\times \sum_{0<n_1<\cdots<n_r} \frac{(-\lambda)^{n_1+n_2+\cdots+n_r-r}(1)_{n_1,\frac{1}{\lambda}}(1)_{n_2,\frac{1}{\lambda}}\cdots(1)_{n_r,\frac{1}{\lambda}}}{(n_1-1)!\cdots(n_{r-1}-1)!n_1^{k_1}\cdots n_{r-1}^{k_{r-1}}n_r^{k_r-1}} \sum_{l=n_r}^{\infty}(-1)^{l+n_r}S_{2,\lambda}(l,n_r)\frac{t^{l-r}}{l!}$$

$$= \sum_{j=0}^{r}\binom{r}{j}(-1)^{r-j}2^r\left(\sum_{m=0}^{\infty}\beta_{m,\frac{\lambda}{2}}^{(r)}\left(\frac{x+j}{2}\right)\frac{2^m t^m}{m!}\right)$$

$$\times \sum_{0<n_1<\cdots<n_r\le l}(-1)^{l+n_r}\frac{(-\lambda)^{n_1+n_2+\cdots+n_r-r}(1)_{n_1,\frac{1}{\lambda}}(1)_{n_2,\frac{1}{\lambda}}\cdots(1)_{n_r,\frac{1}{\lambda}}}{(n_1-1)!\cdots(n_{r-1}-1)!n_1^{k_1}\cdots n_{r-1}^{k_{r-1}}n_r^{k_r-1}}\frac{t^{l-r}}{l!}$$

$$= \sum_{n=r}^{\infty}\left(\sum_{m=0}^{n-r}\binom{n}{m}\sum_{j=0}^{r}\binom{r}{j}(-1)^{r-j}2^{r+m}\beta_{m,\frac{\lambda}{2}}^{(r)}\left(\frac{x+j}{2}\right)\right.$$

$$\left.\times \sum_{0<n_1<\cdots<n_r\le n-m}(-1)^{n-m+n_r}\frac{(-\lambda)^{n_1+n_2+\cdots+n_r-r}(1)_{n_1,\frac{1}{\lambda}}(1)_{n_2,\frac{1}{\lambda}}\cdots(1)_{n_r,\frac{1}{\lambda}}S_{2,\lambda}(n-m,n_r)}{(n_1-1)!\cdots(n_{r-1}-1)!n_1^{k_1}\cdots n_{r-1}^{k_{r-1}}n_r^{k_r-1}}\right)\frac{t^{n-r}}{n!}$$

$$= \frac{1}{r!}\sum_{n=0}^{\infty}\left(\sum_{m=0}^{n}\frac{\binom{n+r}{m}}{\binom{n+r}{r}}\sum_{j=0}^{r}\binom{r}{j}(-1)^{r-j}2^{r+m}\beta_{m,\frac{\lambda}{2}}^{(r)}\left(\frac{x+j}{2}\right)\right.$$

$$\left.\times \sum_{0<n_1<\cdots<n_r\le n+r-m}\frac{(-1)^{n+r-m+n_r}(-\lambda)^{n_1+n_2+\cdots+n_r-r}(1)_{n_1,\frac{1}{\lambda}}(1)_{n_2,\frac{1}{\lambda}}\cdots(1)_{n_r,\frac{1}{\lambda}}S_{2,\lambda}(n-m+r,n_r)}{(n_1-1)!\cdots(n_{r-1}-1)!n_1^{k_1}\cdots n_{r-1}^{k_{r-1}}n_r^{k_r-1}}\right)\frac{t^n}{n!}.$$

Thus by comparing the coefficients on both sides of (13), we have the following theorem.

Theorem 3.2. *For $k_1, k_2, \cdots, k_r \in \mathbb{Z}$, and $n, r \in \mathbb{N}$, we have*

$$\mathcal{G}_{n,\lambda}^{(k_1,k_2,\cdots,k_r)}(x) = \frac{1}{r!}\sum_{m=0}^{n}\frac{\binom{n+r}{m}}{\binom{n+r}{r}}\sum_{j=0}^{r}\binom{r}{j}(-1)^{r-j}2^{r+m}\beta_{m,\frac{\lambda}{2}}^{(r)}\left(\frac{x+j}{2}\right)$$

$$\times \sum_{0<n_1<\cdots<n_r\le n+r-m}\frac{(-1)^{n+r-m+n_r}(-\lambda)^{n_1+n_2+\cdots+n_r-r}(1)_{n_1,\frac{1}{\lambda}}(1)_{n_2,\frac{1}{\lambda}}\cdots(1)_{n_r,\frac{1}{\lambda}}S_{2,\lambda}(n-m+r,n_r)}{(n_1-1)!\cdots(n_{r-1}-1)!n_1^{k_1}\cdots n_{r-1}^{k_{r-1}}n_r^{k_r-1}}.$$

By (9), we get

$$\begin{aligned}
&\sum_{n=0}^{\infty}\mathfrak{B}_{n,\lambda}^{(k_1,k_2,\cdots,k_r)}(x+y)\frac{t^n}{n!} \\
&= r!\left(\frac{1}{e_\lambda(t)-1}\right)^r l_{k_1,k_2,\cdots,k_r,\lambda}(1-e_\lambda(-t))e_\lambda^x(t)e_\lambda^y(t) \\
&= \sum_{l=0}^{\infty}\mathfrak{B}_{l,\lambda}^{(k_1,k_2,\cdots,k_r)}(x)\frac{t^l}{l!}\sum_{m=0}^{\infty}(y)_{m,\lambda}\frac{t^m}{m!} \\
&= \sum_{n=0}^{\infty}\left(\sum_{l=0}^{n}\binom{n}{l}\mathfrak{B}_{l,\lambda}^{(k_1,k_2,\cdots,k_r)}(x)(y)_{n-l,\lambda}\right)\frac{t^n}{n!}.
\end{aligned} \tag{14}$$

Thus by comparing the coefficients on both sides of (14), we have the following proposition.

Proposition 3.3. *For* $k_1, k_2, \cdots, k_r \in \mathbb{Z}$, *and* $n, r \in \mathbb{N}$, *we have*

$$\mathfrak{B}_{n,\lambda}^{(k_1,k_2,\cdots,k_r)}(x+y) = \sum_{l=0}^{n} \binom{n}{l} \mathfrak{B}_{l,\lambda}^{(k_1,k_2,\cdots,k_r)}(x)(y)_{n-l,\lambda}.$$

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References

[1] A. Bayad, Y. Hamahata, *Multiple polylogarithm and multi-poly-Bernoulli polynomials*, Funct. Approx. Comment. Math. **46** (2016), part 1, 45–61

[2] L. Carlitz, *Degenerate Stirling, Bernoulli and Eulerian numbers*, Utilitas Math. **15** (1979), 51-88.

[3] D. V. Dolgy, D. S. Kim, T. Kim, T. Komatsu, *Barnes' multiple Bernoulli and polyBernoulli mixed-type polynomials*, J. Comput. Anal. Appl. **18** (2015), no. 5, 933–951.

[4] L.-C. Jang, *A note on degenerate type* 2 *multi-poly-Genocchi polynomials*, preprint.

[5] W. A. Khan, A. Khan, U. Duran, *A Note on Type* 2 *Degenerate Multi-Poly-Bernoulli Polynomials and Numbers*. Preprints 2020, 2020080057

[6] D. S. Kim, T. Kim, *A note on polyexponential and unipoly functions*, Russ. J. Math. Phys.**26** (2019), no. 1, 40–49.

[7] D. S. Kim, T. Kim, *A note on a new type of degenerate Bernoulli numbers*, Russ. J. Math. Phys.**27** (2020), no. 2, 227–235.

[8] D. S. Kim, T. Kim, *A note on degenerate poly-Bernoulli numbers and polynomials*, Adv. Difference Equ. (2015), **2015**:258.

[9] M.-S. Kim, T. Kim, *An explicit formula on the generalized Bernoulli number with order n*, Indian J. Pure Appl. Math. **31** (2000), no. 11, 1455–1461.

[10] T. Kim, *A note on degenerate Stirling polynomials of the second kind*, Proc. Jangjeon Math. Soc. **20** (2017), no. 3, 319–331.

[11] T. Kim, D. S. Kim, *Degenerate polyexponential functions and degenerate Bell polynomials*, J. Math. Anal. Appl. **487** (2020), no. 2, 124017.

[12] T. Kim, D. S. Kim, H. Y. Kim, L.-C. Jang, *Degenerate poly-Bernoulli numbers and polynomials*, Informatica **31** (2020), no. 3, 2—8.

[13] T. Kim, D. S. Kim, H. Kim, J. Kwon *A note on degenerate multi-poly-Genocchi polynomials*, Adv. Stud. Contemp. Math. (Kyungshang) **30** (2020), no. 3, 449–454.

[14] T. Kim, D. S. Kim, H.-Y. Kim, H. Lee, L.-C. Jang, *Degenerate poly-Bernoulli polynomials arising from degenerate polylogarithm*, Adv. Difference Equ. (2020), **2020**:444.

[15] T. Kim, D. S. Kim, J. Kwon, H. Y. Kim, *A note on degenerate Genocchi and polyGenocchi numbers and polynomials*, J. Inequal. Appl. (2020), **2020**:110.

[16] T. Kim, D. S. Kim, J. Kwon, H. Lee, *Degenerate polyexponential functions and type* 2 *degenerate poly-Bernoulli numbers and polynomials*, Adv. Difference Equ. (2020), **2020**:168.

[17] D. S. Lee, H. K. Kim, L.-C. Jang, *Type* 2 *degenerate poly-Euler polynomials*, Symmetry (2020), **12**:6, 1011.

[18] L. Lewin, *Polylogarithms and associated functions. With a foreword by A. J. Van der Poorten*, North-Holland Publishing Co., New York-Amsterdam, 1981
[19] F. Qi, D. S. Kim, T. Kim, D. V. Dolgy, *Multiple poly-Bernoulli polynomials of the second kind*, Adv. Stud. Contemp. Math. (Kyungshang) **25** (2015), 1-–7
[20] M. Waldschmidt, *Multiple polylogarithms: an introduction. Number theory and discrete mathematics*, (Chandigarh, 2000), 112, Trends Math., Birkhauser, Basel, 2002.

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