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THE RIGIDITY OF GENERALIZED RECTANGULAR FRAMEWORKS

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ABSTRACT. Generally, the rigidity problem of rectangular frameworks consisting of rectangular array of girder beams and riveted joints is determined by the connectivity of the bipartite graph induced by given rectangular frameworks. In this paper, we study on the stability problem of *generalized* rectangular frameworks and also investigate a slightly advanced algorithm for determining the rigidity of rectangular frameworks.

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KEYWORDS AND PHRASES. Rectangular Framework, bipartite graph, rank

1. INTRODUCTION

The contents quoted in this section are mainly excerpted from Wilson and Watkins' book ([7], pp. 59-61). The skyscrapers are supported by a steel framework consisting of rectangular girder beams and welded or riveted joints. However, this structure is treated as a flat (planar rather than space) structure with pin joints rather than rigid welds when joining beams together for various reasons. The simplest structure is a rectangle consisting of four beams and four pin-joints as Figure 1. This structure is unstable because it can be easily deformed under sufficiently high loads. In order for the structure to be stable, it must be braced by extra beam (support).

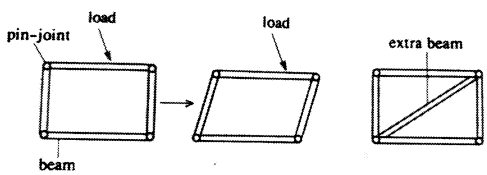

FIGURE 1. Simplest form of rectangular framework([7])

In the case of a larger structure (see, Figure 2) containing many rectangular cells, it is possible to ensure the rigidity by attaching support rods

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(extra beams) to all the rectangular cells, but it is costly in terms of economy. So we have natural mathematical problems; the rigidity problem and the optimization problem

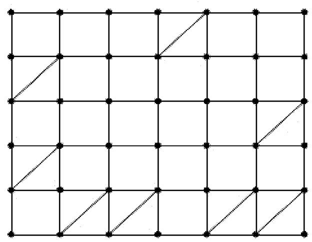

FIGURE 2. 5×6 rectangular framework

It is well known that the above problem can be solved with the connectivity of the bipartite graph induced by a braced rectangular framework (see, [1], [3], [4], [5], [6], [7]). Figure 3 shows a rigid rectangular framework and its connected bipartite graph, and Figure 4 shows a non-rigid rectangular framework and its disconnected bipartite graph.

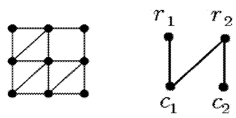

FIGURE 3. Rigid rectangular framework and connected bipartite graph

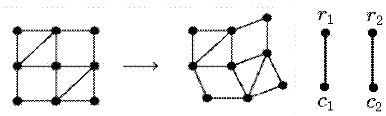

FIGURE 4. Non-rigid rectangular framework and disconnected bipartite graph

In particular, in [4], the rigidity of a rectangular framework is verified in more detail by using the relationship between the connectivity of the bipartite graph induced by given rectangular framework and the variation of angles of parallelograms constituting the rectangular framework.

In this paper, we study how to solve the stability problem using the rank of the matrix induced by the braced rectangular framework. This study is a sequential study of [4] and [2], focusing on the generalized rectangular frameworks (see, Figure 7) and we provide slightly advanced algorithm than that of [2] for determining the rigidity of rectangular frameworks.

2. On the Bracing Rectangular Frameworks

Let R_{mn} be an $m \times n$ rectangular framework (with bracings). For each i, j $(1 \le i \le m, 1 \le j \le n)$, we let θ_{ij} be the angle of the upper left corner of the parallelogram that lies in the i-th row and j-th column of R_{mn}.

If $\theta_{ij} = 90°$ for each i, j $(1 \le i \le m, 1 \le j \le n)$, then the rectangular framework R_{mn} is rigid. Notice that the angle of the upper left corner of a parallelogram with bracing is 90°. To determine the rigidity of the framework, we have to examine whether the corner angles of parallelograms without support are right angles or not.

Lemma 2.1. ([4]) *Let R_{mn} be an $m \times n$ rectangular framework. For each $i, j (1 \le i < m, 1 \le j < n)$, we have*

$$\theta_{ij} + \theta_{i+1j+1} = \theta_{ij+1} + \theta_{i+1j}$$

By Lemma 2.1, we have a matrix equation of the form

$$FY = O, \tag{2.1}$$

where

$$F = \begin{bmatrix} T & -T & O & O & \cdots & O & O & O \\ O & T & -T & O & \cdots & O & O & O \\ & & & & \ddots & & & \\ O & O & O & O & \cdots & O & T & -T \end{bmatrix}_{\{(m-1)\times(n-1)\}\times(m\times n)}$$

with

$$T = \begin{bmatrix} 1 & -1 & 0 & 0 & \cdots & 0 & 0 & 0 \\ 0 & 1 & -1 & 0 & \cdots & 0 & 0 & 0 \\ & & & & \ddots & & & \\ 0 & 0 & 0 & 0 & \cdots & 0 & 1 & -1 \end{bmatrix}_{(n-1)\times n},$$

$O = [0]_{(n-1)\times n}$, and

$$Y = [\theta_{11}\theta_{12}\cdots\theta_{1n}\theta_{21}\theta_{22}\cdots\theta_{2n}\cdots\theta_{m1}\theta_{m2}\cdots\theta_{mn}]^T.$$

Now, if we consider the bracings of given rectangular framework R_{mn}, then we have the following system of linear equations from (2.1),

$$F_R X = B, \tag{2.2}$$

where F_R is the matrix obtained by deleting θ_{ij} columns in F corresponding to braced parallelograms in R_{mn}, X is the matrix obtained by deleting θ_{ij} columns in Y corresponding to braced parallelograms in R_{mn}. That is to say, X is determined by non-braced parallelograms in R_{mn}, and B is the $\{(m-1)(n-1)\} \times 1$ matrix obtained by deleted θ_{ij} columns in F corresponding to braced parallelograms in R_{mn}.

For example, in Figure 5, we have the following equations

$$FY = O, \tag{2.3}$$

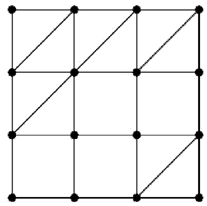

FIGURE 5. R_{33}

where O is zero matrix,

$$F = \begin{bmatrix} T & -T & O \\ O & T & -T \end{bmatrix}$$

with

$$T = \begin{bmatrix} 1 & -1 & 0 \\ 0 & 1 & -1 \end{bmatrix},$$

and

$$Y = [\theta_{11}\,\theta_{12}\,\theta_{13}\,\theta_{21}\,\theta_{22}\,\theta_{23}\,\theta_{31}\,\theta_{32}\,\theta_{33}]^T.$$

Equivalently, if we consider bracings in Figure 5, then we have

$$F_R X = B \tag{2.4}$$

where

$$F_R = \begin{bmatrix} 1 & 0 & 0 & 0 \\ -1 & 1 & 0 & 0 \\ -1 & 0 & -1 & 1 \\ 1 & -1 & 0 & -1 \end{bmatrix},$$

$X = [\theta_{22}\,\theta_{23}\,\theta_{31}\,\theta_{32}]^T$, and $B = [90^\circ\, 0^\circ\, - 90^\circ\, - 90^\circ]^T$

Lemma 2.2. *Let L be a $m \times n$ matrix. Then the linear system $LX = B$ is consistent if and only if rank(L) = rank$([L \mid B])$, where $[L \mid B]$ is the augmented matrix of the system. Also, if rank$(L) = n$, then $LX = B$ has a unique solution.*

By Lemma 2.2, the rectangular framework in Figure 5 is rigid if and only if the rank of the matrix F_R is 4.

Let R_{mn} be a $m \times n$ $(m, n \geq 2)$ rectangular framework and let

$$E_{F_R} = \begin{bmatrix} C_1 & & & & \\ & \ddots & & \mathbf{O} & \\ & & C_i & & \\ & \mathbf{O} & & \ddots & \\ & & & & C_m \end{bmatrix} \tag{2.5}$$

The size of the matrix C_i is $n \times p$, where p is the number of non-braced parallelograms in i-th row in R_{mn}. Here, in the case of all parallelograms in

i-th row in R_{mn} are bacings, for convenience, we consider the size of C_i to $n \times 1$. The entries of C_i $(i = 1, \ldots, m)$ are constructed by the informations of R_{mn}; first, if we give a cell with a support in R_{mn} a value of zero, a cell without one, each row in R_{mn} is expressed as the sum of one-hot encoding except zero rows. And then the transpose matrix of the matrix consisting of one-hot encodings in i-row of R_{mn} is C_i. If the components of i-th row are all 0 (all parallelograms in i-th row in R_{mn} are bacings), the entries of C_i are all zeros (see, the 1st row of the rectangular framework in Figure 5). From the Figure 5, we have the following matrix of the form

$$\begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 1 \\ 1 & 1 & 0 \end{bmatrix}$$

FIGURE 6. Matrix with information of supports

Thus we obtain C_i $(i = 1, 2, 3)$ from the matrix in Figure 6;

- 1st row: [0 0 0]
$$C_1 = \begin{bmatrix} 0 & 0 & 0 \end{bmatrix}^T$$
- 2nd row: $[0\ 1\ 1] = [0\ 1\ 0] + [0\ 0\ 1]$
$$C_2 = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}^T$$
- 3rd row: $[1\ 1\ 0] = [1\ 0\ 0] + [0\ 1\ 0]$
$$C_3 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}^T$$

And also we have
$$E_{F_R} = \begin{bmatrix} C_1 & O & O \\ O & C_2 & O \\ O & O & C_3 \end{bmatrix}$$

Let $\overline{F}_R = FE$. Then we have
$$\overline{F}_R = \begin{bmatrix} 0 & 1 & 0 & 0 & 0 \\ 0 & -1 & 1 & 0 & 0 \\ 0 & -1 & 0 & -1 & 1 \\ 0 & 1 & -1 & 0 & -1 \end{bmatrix}$$

Lemma 2.3. *Let $\overline{F}_R = FE_{F_R}$. Then the rank of $\overline{F}_R$ is equal to the rank of F_R.*

Proof. Notice that $\overline{F}_R$ is equal to F_R or just a matrix with a few zero columns added to matrix F_R. Thus the rank of $\overline{F}_R$ is equal to the rank of F_R. □

Corollary 2.4. ([2]) *Let R_{mn} be an $m \times n$ $(m, n \geq 2)$ rectangular framework. Then R_{mn} is rigid if and only if rank(F_R) = rank(E_{F_R}).*

Proof. We note that

$$\begin{aligned}\operatorname{rank}(F_R) &= \operatorname{rank}(\overline{F}_R) = \operatorname{rank}(FE_{F_R})\\ &\le \operatorname{rank}(E_{F_R}) = \operatorname{rank}(C_1) + \cdots + \operatorname{rank}(C_m)\\ &= \text{The number of columns of } F_R\end{aligned}$$

Thus we have that the rectangular framework R_{mn} is rigid if and only if $\operatorname{rank}(F_R) = \operatorname{rank}(E_{F_R})$. □

Corollary 2.5. ([2]) *Let R be a rectangular framework obtained by permute the rows (or columns) of any rigid rectangular framework. Then R is also rigid.*

Proof. Notice that two matrices that are row(or column) equivalent have the same rank. □

Let R_{mn} be an $m \times n$ rectangular framework (with bracings). Then

$$F = \left[\begin{array}{ccccc|c} & & \tilde{F} & & & O \\ & & & & & O \\ & & & & & \vdots \\ & & & & & O \\ \hline O & O & \cdots & O & T & -T \end{array}\right]_{\{(m-1)\times(n-1)\}\times(m\times n)}$$

where

$$\tilde{F} = \begin{bmatrix} T & -T & O & O & \cdots & O & O & O \\ O & T & -T & O & \cdots & O & O & O \\ & & & & \ddots & & & \\ O & O & O & O & \cdots & O & T & -T \end{bmatrix}_{\{(m-2)\times(n-1)\}\times\{(m-1)\times n\}}$$

Notice that $\tilde{F}$ is the matrix induced by $(m-1)\times n$ rectangular framework $R_{(m-1)n}$. That is to say, $\tilde{F}$ is the matrix corresponding to a rectangular framework with one row removed from given rectangular framework. For convenience, we consider an augmented matrix given by

$$F^* = [\tilde{F} \mid \mathbf{O}], \ \mathbf{O} = [O\ O\ \cdots O]^T$$

Lemma 2.6. *Let R_{mn} be an $m \times n$ rectangular framework (with bracings). Then we have*

(1) *rank*$(T) = n - 1$.
(2) *rank*$(F) = (m-1)$ *rank*$(T) = (m-1)(n-1)$.
(3) *rank*$(F^*) =$ *rank*$(\tilde{F}) = (m-2)(n-1)$.

Lemma 2.7. *Let R_{mn} be an $m\times n$ $(m, n \ge 2)$ rectangular framework. Then R_{mn} is rigid if and only if the rank of F_R is the number of columns of F_R.*

Corollary 2.8. *Let R_{mn} be an $m \times n$ $(m, n \geq 2)$ rectangular framework. Then we have*

(1) *The nullity of F, the dimension of the solution space of $FX = O$, is $m - n + 1$.*
(2) *The quantity $m + n - 1$ is the minimum number of supports for R_{mn} to be rigid.*

Proof. Notice that the size of the matrix F is

$$(m-1)(n-1) \times mn$$

and

$$\begin{aligned} mn - (m+n-1) &= (m-1)(n-1) \\ &= \operatorname{rank}(F) \geq \operatorname{rank}(F_R). \end{aligned}$$

This completes the proof. □

Theorem 2.9. *Let R_{mn} be an $m \times n$ $(m, n \geq 2)$ rectangular framework with bracings and let $\tilde{R}_{(m-1)\,n}$ be an $(m-1) \times n$ rectangular framework that removes one row with only one support in R_{mn}. Then the rigidity of R_{mn} depends on the rigidity of $\tilde{R}_{(m-1)\,n}$.*

Proof. By Lemma 2.7, R_{mn} is rigid if and only if the rank of F_R is the number of columns of F_R. Notice that

$$\begin{aligned} \operatorname{rank}(F_R) &= \operatorname{rank}(E_{F_R}) \\ &= \operatorname{rank}(E_{\tilde{F}_R}) + (n-1) \\ &\geq \operatorname{rank}(\tilde{F}_R) + (n-1) \end{aligned} \tag{2.6}$$

Now, we also have

$$\operatorname{rank}(\tilde{F}_R) \leq \operatorname{rank}(F_R^*) = \operatorname{rank}(F_R) - (n-1) \tag{2.7}$$

By (2.6) and (2.7), we have

$$\operatorname{rank}(F_R) = \operatorname{rank}(\tilde{F}_R) + (n-1)$$

□

3. Generalized rectangular frameworks

In general, a rectangular framework has connected rows of rectangular cells and also the size of a rectangular framework has $n \times m$. In this secton, we investigate the rigidity of generalized rectangular frameworks. A *generalized rectangular framework* means a structure connected by rectangular cells that do not make holes (see, Figure 7 and Figure 8). It can have a row of cells that are not connected consecutively (see, 1st row of Figure 7) and the size of the structure is not $m \times n$ in general (see, Figure 7).

In Figure 7, the first row is disconnected with respect to cells. It has two connected cell components and the 2nd row has only one connected cell component.

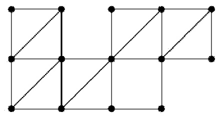

FIGURE 7. Generalized rectangular framework

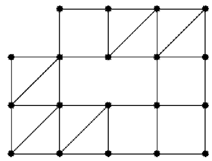

FIGURE 8. A structure connected by rectangular cells with hole

Lemma 3.1. *The continuous connection of rectangular cells without supports to cell components does not affect the rigidity of a given rectangular frameworks if it is not to connect two row or column cell components.*

Proof. The continuous connection of rectangular cells without supports to cell components does not effect of the connectivity of the bipartite graph induced by the given rectangular framework (see, Figure 9). □

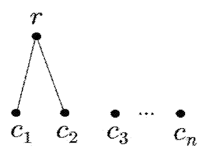

FIGURE 9. Cell component extention and bipartite graph

For example, Figure 7 is equivalent to Figure 10 with respect to the rigidity of generalized rectangular frameworks.

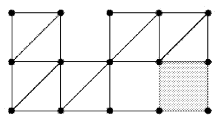

FIGURE 10. Cell component extention

Lemma 3.2. *The connection using cells without support of the two cell components can affect the rigidity of a given generalized rectangular framework.*

Proof. If each cell component has different connected components in the bipartite graph, connecting the two cell components with cells without supports affects the rigidity of a given generalized rectangular framework (see, Figure 11 and Figure 12). □

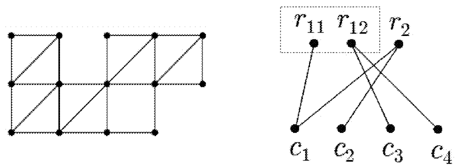

FIGURE 11. Non-rigid generalized rectangular framework; the 1st row has two connected cell componets: r_{11} and r_{12} and its bipartite graph is disconnected

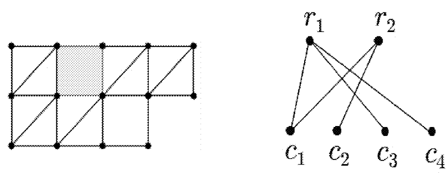

FIGURE 12. Connection of two connected cell componets: the structure becomes stable and its bipartite graph is connected

Theorem 3.3. *Any generalized rectangular framework can be made into a rectangular framework without affecting its rigidity.*

Proof. By Lemma 3.1, if two row (or column) cell components are not connected by rectangular cells without supports, continuous connections do not affect the rigidity of a given generalized rectangular framework. Thus in a given generalized rectangular framework, we can make the size of column equal for each row. By Lemma 3.2, the connection of a connected cell components with rectangular cells without supports affects the rigidity of a generalized rectangular framework, but the connecting cell components are independent of each other unless the connecting cell components are connected. Thus consideration of each cell connected component as a single row does not affect the rigidity of a given generalized rectangular frame structure (see, Figure 13). This completes the proof.

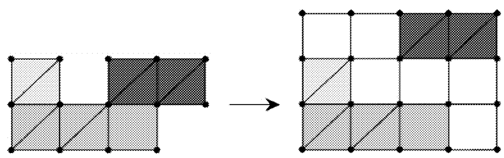

FIGURE 13. Create Rectangular Framework

□

For example, let's create a rectangular framework with the structure of Figure 14.

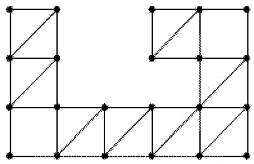

FIGURE 14. Generalized rectangular framework

In Figure 14, we cannot make a rectangular cell in column 4th of row 2nd since two cell components of colunm 4th are connected. Thus the two connecting components of the fourth column shall be independently made (see, Figure 15).

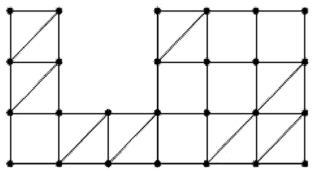

FIGURE 15

Now, make the two cell components of 2nd row independent, and then the two cell components of the first row independent (see, Figure 16).

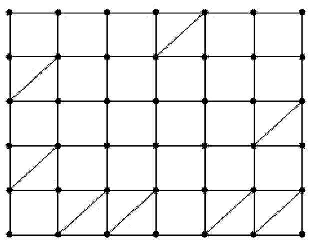

FIGURE 16. Rectangular framework made from generalized rectangular framework given in Figure 14

4. Algorithm

Step 1: Make a rectangular framework R_{mn} from a given generalized rectangular framework.

Step 2:

- If $m = 0$, then R_{mn} is NOT rigid.
- For $m = 1$ or $n = 1$, if all cells are braced then R_{mn} is rigid, otherwise R_{mn} is NOT rigid.

Otherwise, go to Step 3.

Step 3: If R_{mn} has a row that all its cells are not braced, then R_{mn} is NOT rigid. Otherwise, go to Step 4.

Step 4: If R_{mn} has a row that its only one entry has braced, then delete the row. Let Q be a rectangular framework made by the process of Step 4.

Step 5: If $R_{mn} = Q$, then construct a Toeplitz type matrix T :

$$T = \begin{bmatrix} \underline{1 \quad -1} & 0 & 0 & \cdots & 0 & 0 & 0 \\ 0 & \underline{1 \quad -1} & 0 & \cdots & 0 & 0 & 0 \\ & & & \ddots & & & \\ 0 & 0 & 0 & 0 & \cdots & 0 & \underline{1 \quad -1} \end{bmatrix}_{(n-1)\times n}$$

Step 6: Construct a block Toeplitz type matrix F from T :

$$F = \begin{bmatrix} \underline{T \quad -T} & O & O & \cdots & O & O & O \\ O & \underline{T \quad -T} & O & \cdots & O & O & O \\ & & & \ddots & & & \\ O & O & O & O & \cdots & O & \underline{T \quad -T} \end{bmatrix}_{(m-1)\times m}$$

Here, the size $(m-1) \times m$ of F means that when we consider the entries of F as $(n-1) \times n$ matrices.

Step 7: Construct the matrix $\overline{F}_R$ from the block diagonal matrix E: $\overline{F}_R = F\,E$

$$E = \begin{bmatrix} C_1 & & & & \\ & \ddots & & \mathbf{O} & \\ & & C_i & & \\ & \mathbf{O} & & \ddots & \\ & & & & C_m \end{bmatrix} \qquad \text{(see, (2.5))}$$

Step 8: Find rank($\overline{F}_R$) and rank(E) : if rank($\overline{F}_R$) = rank(E), then rectangular framework R_{mn} is rigid. Otherwise, R_{mn} is non-rigid.

Step 9: If Q is $p \times n$ matrix, then replaces m with p and apply from Step 5 to Step 8.

A python code related to this algorithm is shown in the Figure 17. In Figure 17,

- R_{mn} : $m \times n$ rectangular framework,
- args: the position in which without brace in each row vector, for example, $[2, 3, 6]$ indicates that there are no supports in the second, third, and sixth parallelograms in the given row, input [0] if there is a row that its every entries have braces.

For example, for the rectangular framework in Figure 13,

```
>>> Rec_Frame(3, 4, [1, 2], [2, 3, 4], [3, 4])
The rectangular framework is NOT rigid
```

For the rectangular framework in Figure 5,

```
>>> Rec_Frame(3, 3, [0], [2, 3], [1, 2])
The rectangular framework is rigid with minimum bracings
```

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Appendix

```
import scipy.linalg
import numpy as np
from scipy.linalg import block_diag

def Rec_Frame(m,n,*args):
    if m == 0:
        print('The rectangular framework is NOT rigid')
    elif m == 1:
        for i in args:
            if i == [0]:
                print('The rectangular framework is rigid with minimum bracings')
            else:
                print('The rectangular framework is NOT rigid')
    elif n == 1:
        L=[]
        for i in args:
            if i == [0]:
                L.append(i)
        if m == len(L):
            print('The rectangular framework is rigid with minimum bracings')
        else:
            print('The rectangular framework is NOT rigid')
    else:
        for i in args:
            if len(i) == n:
                print('The rectangular framework is NOT rigid')
                return
        Q = []
        for i in range(0,m):
            if len(args[i]) != n-1 or args[i] == [0]:   # args[i] == [0]: consider 2 x 2 structure
                Q.append(args[i])
        if len(Q) == m:
            T = scipy.linalg.toeplitz([np.block([1,np.zeros(n-2)])],
                                      [np.block([1,-1,np.zeros(n-2)])])
            E = []
            for a in args:
                C = []
                for j in range(len(a)):
                    if a[j] == 0:
                        C.append(np.zeros(n))
                    else:
                        C.append(np.block([np.zeros(a[j]-1),1, np.zeros(n-a[j])]))
                E.append(np.transpose(C))
            D = block_diag(*B)
            F = np.zeros((m-1, n-1, m, n), dtype=T.dtype)
            for i in range(m-1):
                F[i,:,i,:] = T
                F[i,:,i+1,:] = -T
            F.shape = ((m-1)*(n-1), m*n)
            F_R = np.dot(F,D)
            if  np.linalg.matrix_rank(F_R) == np.linalg.matrix_rank(D):
                if m * n - np.linalg.matrix_rank(D) == m + n -1:
                    print('The rectangular framework is rigid with minimum bracings')
                else:
                    print('The rectangular framework is rigid')
            else:
                print('The rectangular framework is NOT rigid')
        else:
            return Rec_Frame(len(Q),n,*Q)
```

Figure 17. Python code

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