Advanced Studies in Contemporary Mathematics
30 (2020), No. 4, pp. 559 - 570

www.jangjeon.or.kr
http://dx.doi.org/10.17777/ascm2020.30.4.559

A NOTE ON TYPE 2 DEGENERATE POLY-GENOCCHI POLYNOMIALS AND NUMBERS OF COMPLEX VARIABLE ARISING FROM THE DEGENERATE MODIFIED POLYEXPONENTIAL FUNCTION

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ABSTRACT. In recent years, the study of degenerate versions of some special polynomials and numbers regained interests of many mathematicians. Kim et al.(Adv. Differ. Equ., 2019:490, 2019) studied the type 2 degenerate Bernoulli and Euler polynomials of complex variable. In this paper, we introduce the type 2 degenerate poly-Genocchi polynomials and numbers of complex variable, based on Kim and Kim's modified polyexponential function(Russ. J. Math. Phys. 26(1), 2019). We derive several identities related to the type 2 degenerate poly-cosine(sine)-Genocchi polynomials and numbers, including the degenerate and other special polynomials and numbers such as the (degenerate) Stirling numbers of the first kind, the (degenerate) Stirling numbers of the second kind, type 2 cosine-Genocchi polynomials, type 2 sine-Genocchi polynomials, etc.

1. INTRODUCTION

Kim et al. [11] studied the type 2 degenerate Bernoulli and Euler polynomials of complex variable. Moreover, Kim and Kim[10] introduced the modified polyexponential function and constructed type 2 poly-Bernoulli polynomials by using the modified polyexponential function. In particular, the Genocchi numbers have been studied extensively in the field of mathematics and engineering for instance, elementary number theory, complex analytic number theory, differential topology (differential structures on spheres), p-adic analytic number theory (p-adic L-functions), quantum physics (quantum Groups) [1, 12, 25-28]. In the paper, we focus on a new degenerate poly-type 2 Genocchi polynomials and numbers of complex variable, based on modified polyexponential function [10]. We derive some explicit expressions and identities for those numbers and polynomials.

As is well known, the type 2 Genocchi polynomials $G_n(x)$ are defined by

$$\frac{2t}{e^t+e^{-t}}e^{xt}=\sum_{n=0}^{\infty}G_n^*(x)\frac{t^n}{n!},\quad \text{see [1, 12, 25, 28].} \tag{1}$$

The degenerate exponential functions are defined as

$$e_\lambda^x(t)=(1+\lambda t)^{\frac{x}{\lambda}},\quad e_\lambda(t)=e_\lambda^1(t)=(1+\lambda t)^{\frac{1}{\lambda}},\quad (\text{see } [2,\ 7,\ 9-16]).$$

Here we note that

$$e_\lambda^x(t)=\sum_{n=0}^{\infty}(x)_{n,\lambda}\frac{t^n}{n!},\quad (\text{see [7, 9, 10]}), \tag{2}$$

where $(x)_{0,\lambda}=1$, $(x)_{n,\lambda}=x(x-\lambda)(x-2\lambda)\cdots(x-(n-1)\lambda),(n\geq 1)$.

2010 *Mathematics Subject Classification.* 11B73; 11B83; 05A19.

Key words and phrases. type 2 Genocchi polynomials of complex variable; degenerate poly-Genocchi polynomials of complex variable; modified degenerate polyexponential functions; type 2 degenerate poly-cosine-Genocchi polynomials; type 2 degenerate poly-sine-Genocchi polynomials.

In [11], the degenerate type2 Bernoulli polynomials and the degenerate type2 Euler polynomials are given by

$$\frac{2}{e_\lambda(t)+e_\lambda^{-1}(t)}e_\lambda^x(t)=\sum_{n=0}^{\infty}E_{n,\lambda}^*(x)\frac{t^n}{n!},\ \ and\ \ \frac{t}{e_\lambda(t)-e_\lambda^{-1}(t)}e_\lambda^x(t)=\sum_{n=0}^{\infty}B_{n,\lambda}^*(x)\frac{t^n}{n!},$$

respectively.

When $x=0$, $E_{n,\lambda}^*=E_{n,\lambda}^*(0)$ are called the degenerate type 2 Euler numbers, and $B_{n,\lambda}^*=B_{n,\lambda}^*(0)$ are called the degenerate type 2 Bernoulli numbers.
Note that $\lim_{\lambda\to 0}E_{n,\lambda}^*(x)=E_n^*(x),(n\geq 0)$ and $\lim_{\lambda\to 0}B_{n,\lambda}^*(x)=B_n^*(x),(n\geq 0)$, where

$$\frac{2}{e^t+e^{-t}}e^x(t)=\sum_{n=0}^{\infty}E_n^*(x)\frac{t^n}{n!},\ \ and\ \ \frac{t}{e^t-e^{-t}}e^x=\sum_{n=0}^{\infty}B_n^*(x)\frac{t^n}{n!}.$$

Let $\log_\lambda(t)$ be the compositional inverse function of $e_\lambda(t)$ such that

$$e_\lambda(\log_\lambda(t))=\log_\lambda(e_\lambda(t))=t.$$

The degenerate Stirling numbers of the first kind are defined by

$$\frac{1}{k!}\big(\log_\lambda(1+t)\big)^k=\sum_{n=k}^{\infty}S_{1,\lambda}(n,k)\frac{t^n}{n!},\quad (k\geq 0)\quad (\text{see [13, 16]}). \tag{3}$$

Note here that $\lim_{\lambda\to 0}S_{1,\lambda}(n,l)=S_1(n,l)$, where $S_1(n,l)$ are the Stirling numbers of the first kind given by

$$\frac{1}{k!}\big(\log(1+t)\big)^k=\sum_{n=k}^{\infty}S_1(n,k)\frac{t^n}{n!},\quad (k\geq 0),\quad (\text{see [11, 13, 14]}). \tag{4}$$

The degenerate Stirling numbers of the second kind are given by

$$\frac{1}{k!}\big(e_\lambda(t)-1\big)^k=\sum_{n=k}^{\infty}S_{2,\lambda}(n,k)\frac{t^n}{n!},\quad (k\geq 0),\quad (\text{see [13]}). \tag{5}$$

Observe here that $\lim_{\lambda\to 0}S_{2,\lambda}(n,l)=S_2(n,l)$, where $S_2(n,l)$ are the Stirling numbers of the second kind given by

$$\frac{1}{k!}\big(e^t-1\big)^k=\sum_{n=k}^{\infty}S_2(n,k)\frac{t^n}{n!},\quad (k\geq 0),\quad (\text{see [11-13]}). \tag{6}$$

Masjed-Jamei, Beyki and Koepf introduced the new type Euler polynomials which are given by

$$\frac{2e^{pt}}{e^t+1}\cos qt=\sum_{n=0}^{\infty}E_n^{(c)}(p,q)\frac{t^n}{n!},\ \ and\ \ \frac{2e^{pt}}{e^t+1}\sin qt=\sum_{n=0}^{\infty}E_n^{(s)}(p,q)\frac{t^n}{n!}\qquad (\text{see [23]}).$$

The Euler formula is defined by

$$e^{ix}=\cos x+i\sin x,\ \ \text{where } x\in\mathbb{R},\ i=\sqrt{-1}. \tag{7}$$

Thus, by (7), we have

$$\cos qx=\frac{e^{iqx}+e^{-iqx}}{2},\ \ and\ \ \sin qx=\frac{e^{iqx}-e^{-iqx}}{2i}. \tag{8}$$

where $i=\sqrt{-1}$.

We note that

$$e_\lambda^{iy}(t)=e^{\frac{iy}{\lambda}\log(1+\lambda t)}=\sum_{n=0}^{\infty}\left(\sum_{k=0}^{n}\lambda^{n-k}(iy)^kS_1(n,k)\right)\frac{t^n}{n!}, \tag{9}$$

and

$$e_\lambda^{-iy}(t)=e^{\frac{-iy}{\lambda}\log(1+\lambda t)}=\sum_{n=0}^{\infty}\left(\sum_{k=0}^{n}\lambda^{n-k}(-iy)^kS_1(n,k)\right)\frac{t^n}{n!}. \tag{10}$$

By using (9) and (10), we have (see [7])

$$\cos_\lambda^{(y)}(t)=\frac{e_\lambda^{iy}(t)+e_\lambda^{-iy}(t)}{2}=\sum_{n=0}^{\infty}\left(\sum_{k=0}^{[\frac{n}{2}]}\lambda^{n-2k}(-1)^ky^{2k}S_1(n,2k)\right)\frac{t^n}{n!}, \tag{11}$$

and

$$\sin_\lambda^{(y)}(t)=\frac{e_\lambda^{iy}(t)-e_\lambda^{-iy}(t)}{2i}=\sum_{n=1}^{\infty}\left(\sum_{k=0}^{[\frac{n-1}{2}]}\lambda^{n-2k-1}(-1)^ky^{2k+1}S_1(n,2k+1)\right)\frac{t^n}{n!}. \tag{12}$$

Note that

$$\lim_{\lambda\to0}\cos_\lambda^{(y)}(t)=\sum_{k=0}^{\infty}(-1)^ky^{2k}\frac{t^{2k}}{2k!}=\cos yt,\quad \lim_{\lambda\to0}\sin_\lambda^{(y)}(t)=\sum_{k=0}^{\infty}(-1)^ky^{2k+1}\frac{t^{2k+1}}{(2k+1)!}=\sin yt.$$

Kim et al. [11] recently studied the type 2 degenerate Bernoulli polynomials and Euler polynomials of complex variable respectively as follows:

$$\frac{t}{e_\lambda^{\frac{1}{2}}(t)-e_\lambda^{-\frac{1}{2}}(t)}e_\lambda^{x+iy}(t)=\sum_{n=0}^{\infty}B_{n,\lambda}^{*}(x+iy)\frac{t^n}{n!},\quad \frac{t}{e_\lambda^{\frac{1}{2}}(t)-e_\lambda^{-\frac{1}{2}}(t)}e_\lambda^{x-iy}(t)=\sum_{n=0}^{\infty}B_{n,\lambda}^{*}(x-iy)\frac{t^n}{n!},$$

and

$$\frac{2}{e_\lambda^{\frac{1}{2}}(t)+e_\lambda^{-\frac{1}{2}}(t)}e_\lambda^{x+iy}(t)=\sum_{n=0}^{\infty}E_{n,\lambda}^{*}(x+iy)\frac{t^n}{n!},\quad \frac{2}{e_\lambda^{\frac{1}{2}}(t)+e_\lambda^{-\frac{1}{2}}(t)}e_\lambda^{x-iy}(t)=\sum_{n=0}^{\infty}E_{n,\lambda}^{*}(x-iy)\frac{t^n}{n!}.$$

Naturally, we consider the type 2 degenerate Genocchi polynomials of complex variable as follows:

$$\frac{2t}{e_\lambda(t)+e_\lambda^{-1}(t)}e_\lambda^{x+iy}(t)=\sum_{n=0}^{\infty}G_{n,\lambda}^{*}(x+iy)\frac{t^n}{n!}, \tag{13}$$

and

$$\frac{2t}{e_\lambda(t)+e_\lambda^{-1}(t)}e_\lambda^{x-iy}(t)=\sum_{n=0}^{\infty}G_{n,\lambda}^{*}(x-iy)\frac{t^n}{n!}, \tag{14}$$

where $i=\sqrt{-1}$.

Recently, Kim-Kim introduced the modified polyexponential function as

$$\mathrm{Ei}_k(x)=\sum_{n=1}^{\infty}\frac{x^n}{(n-1)!n^k},\ (k\in\mathbb{Z}),\quad (\text{see [8]}). \tag{15}$$

Kim et al. [16] also considered the degenerate modified polyexponential function given by

$$\mathrm{Ei}_{k,\lambda}(x)=\sum_{n=1}^{\infty}\frac{(1)_{n,\lambda}}{(n-1)!n^k}x^n,\quad (\lambda\in\mathbb{R}). \tag{16}$$

Note that $\mathrm{Ei}_{1,\lambda}(x)=\sum_{n=1}^{\infty}(1)_{n,\lambda}\frac{x^n}{n!}=e_\lambda(x)-1.$

Kim-Jang considered the degenerate poly-Genocchi polynomials given by

$$\frac{\mathrm{Ei}_{k,\lambda}(\log_\lambda(1+2t))}{e_\lambda(t)+1}e_\lambda^x(t)=\sum_{n=0}^{\infty}G_{n,\lambda}^{(k)}(x)\frac{t^n}{n!},\qquad \text{(see [12])}. \tag{17}$$

When $x=0$, $G_{n,\lambda}^{(k)}=G_{n,\lambda}^{(k)}(0)$ are called the degenerate poly-Genocchi numbers.

2. Type 2 degenerate poly-Genocchi numbers and polynomials of complex variable

In this section, we introduce a new type 2 degenerate poly-cosine-Genocchi polynomials and type 2 degenerate poly-sine-Genocchi polynomials, respectively. Furthermore, we derive several combinatorial identities related to the type 2 degenerate poly-cosine-Genocchi polynomials and the type 2 degenerate poly-sine-Genocchi polynomials.

The type2 cosine-Genocchi and the type 2 sine-Genocchi polynomials respectively are given by

$$\frac{2t}{e^t+e^{-t}}e^{xt}\cos(yt)=\sum_{n=0}^{\infty}G_n^{*(c)}(x,y)\frac{t^n}{n!},\quad and\quad \frac{2t}{e^t+e^{-t}}e^{xt}\sin(yt)=\sum_{n=0}^{\infty}G_n^{*(s)}(x,y)\frac{t^n}{n!}. \tag{18}$$

Naturally, we consider the degenerate type2 cosine-Genocchi and the degenerate type 2 sine-Genocchi polynomials respectively given by

$$\frac{2t}{e_\lambda(t)+e_\lambda^{-1}(t)}e_\lambda^x(t)\cos_\lambda^{(y)}(t)=\sum_{n=0}^{\infty}G_{n,\lambda}^{*(c)}(x,y)\frac{t^n}{n!} \tag{19}$$

and

$$\frac{2t}{e_\lambda(t)+e_\lambda^{-1}(t)}e_\lambda^x(t)\sin_\lambda^{(y)}(t)=\sum_{n=0}^{\infty}G_{n,\lambda}^{*(s)}(x,y)\frac{t^n}{n!}. \tag{20}$$

In addition,

$$\frac{2t}{e_\lambda(t)+e_\lambda^{-1}(t)}e_\lambda^x(t)=\sum_{n=0}^{\infty}G_{n,\lambda}^{*(c)}(x)\frac{t^n}{n!}\quad and\quad \frac{2t}{e_\lambda(t)+e_\lambda^{-1}(t)}e_\lambda^x(t)=\sum_{n=0}^{\infty}G_{n,\lambda}^{*(s)}(x)\frac{t^n}{n!}.$$

Furthermore, we consider the type 2 degenerate poly-Genocchi polynomials (c.f (17)) as follows:

$$\frac{\mathrm{Ei}_{k,\lambda}(\log_\lambda(1+2t))}{e_\lambda(t)+e_\lambda^{-1}(t)}e_\lambda^x(t)=\sum_{n=0}^{\infty}G_{n,\lambda}^{*(k)}(x)\frac{t^n}{n!}. \tag{21}$$

We introduce a new type of type 2 degenerate poly-cosine-Genocchi and type 2 degenerate poly-sine-Genocchi polynomials respectively as follows.

$$\frac{\mathrm{Ei}_{k,\lambda}(\log_\lambda(1+2t))}{e_\lambda(t)+e_\lambda^{-1}(t)}e_\lambda^x(t)\cos_\lambda^{(y)}(t)=\sum_{n=0}^{\infty}G_{n,\lambda}^{*(c,k)}(x,y)\frac{t^n}{n!}, \tag{22}$$

and

$$\frac{\mathrm{Ei}_{k,\lambda}(\log_\lambda(1+2t))}{e_\lambda(t)+e_\lambda^{-1}(t)}e_\lambda^x(t)\sin_\lambda^{(y)}(t)=\sum_{n=0}^{\infty}G_{n,\lambda}^{*(s,k)}(x,y)\frac{t^n}{n!}. \tag{23}$$

When $k=1$,

$$\lim_{\lambda\to 0}G_{n,\lambda}^{*(c,1)}(x,y)=G_n^{*(c)}(x,y)\quad and\quad \lim_{\lambda\to 0}G_{n,\lambda}^{*(s,1)}(x,y)=G_n^{*(s)}(x,y).$$

When $x=0$ and $y=0$,

$$G_{n,\lambda}^{*(c,k)} = G_{n,\lambda}^{*(c,k)}(0,0) \quad and \quad G_{0,\lambda}^{*(c,k)} = 0.$$

By (3) and (16), we note that

$$\begin{aligned}\mathrm{Ei}_{k,\lambda}(\log_\lambda(1+2t)) &= \sum_{l=1}^{\infty} \frac{(1)_{l,\lambda}(\log_\lambda(1+2t))^l}{(l-1)!l^k} \\ &= \sum_{l=1}^{\infty} \frac{(1)_{l,\lambda}}{l^{k-1}} \frac{1}{l!}(\log_\lambda(1+2t))^l \\ &= \sum_{l=1}^{\infty} \frac{(1)_{l,\lambda}}{l^{k-1}} \sum_{m=l}^{\infty} S_{1,\lambda}(m,l)\frac{2^m t^m}{m!} \\ &= \sum_{m=1}^{\infty}\left(\sum_{l=1}^{m} \frac{(1)_{l,\lambda}2^m}{l^{k-1}} S_{1,\lambda}(m,l)\right)\frac{t^m}{m!}.\end{aligned} \tag{24}$$

When $k=1$,

$$\mathrm{Ei}_{1,\lambda}(\log_\lambda(1+2t)) = \sum_{n=1}^{\infty} \frac{(1)_{n,\lambda}(\log_\lambda(1+2t))^n}{n!} = e_\lambda(\log_\lambda(1+2t)) - 1 = 2t.$$

Lemma 1. *For $n,k \geq 0$, we have*

$$G_{n,\lambda}^{*(c)}(x,y) = G_{n,\lambda}^{*}(x+iy) + G_{n,\lambda}^{*}(x-iy), \tag{25}$$

and

$$G_{n,\lambda}^{*(s)}(x,y) = -i(G_{n,\lambda}^{*}(x+iy) - G_{n,\lambda}^{*}(x-iy)). \tag{26}$$

Proof. From (13) and (14) , it is easy to see that

$$\begin{aligned}\sum_{n=0}^{\infty}(G_{n,\lambda}^{*}(x+iy)+G_{n,\lambda}^{*}(x-iy))\frac{t^n}{n!} &= \frac{2t}{e_\lambda(t)+e_\lambda^{-1}(t)}e_\lambda^{x+iy}(t) + \frac{2t}{e_\lambda(t)+e_\lambda^{-1}(t)}e_\lambda^{x-iy}(t) \\ &= \frac{2t}{e_\lambda(t)+e_\lambda^{-1}(t)}e_\lambda^{x}(t)\cos_\lambda^{(y)}(t) = \sum_{n=0}^{\infty} G_{n,\lambda}^{*(c)}(x,y)\frac{t^n}{n!},\end{aligned} \tag{27}$$

and

$$\begin{aligned}\sum_{n=0}^{\infty}(G_{n,\lambda}^{*}(x+iy)-G_{n,\lambda}^{*}(x-iy))\frac{t^n}{n!} &= \frac{2t}{e_\lambda(t)+e_\lambda^{-1}(t)}e_\lambda^{x+iy}(t) - \frac{2t}{e_\lambda(t)+e_\lambda^{-1}(t)}e_\lambda^{x-iy}(t) \\ &= \frac{2t}{e_\lambda(t)+e_\lambda^{-1}(t)}e_\lambda^{x}(t)i\sin_\lambda^{(y)}(t) = i\sum_{n=0}^{\infty} G_{n,\lambda}^{*(s)}(x,y)\frac{t^n}{n!}.\end{aligned} \tag{28}$$

Therefore, by comparing the coefficients on both sides of (27) and (28) respectively, we get the desired result. □

Theorem 2. *For n, $k \geq 0$, we have*

$$G_{n,\lambda}^{*(c,k)}(x,y) = \sum_{m=0}^{n}\sum_{l=1}^{m+1}\binom{n}{m}\frac{(1)_{l,\lambda}2^m S_{1,\lambda}(m+1,l)}{(m+1)l^{k-1}}(G_{n-m,\lambda}^{*}(x+iy)+G_{n-m,\lambda}^{*}(x-iy)).$$

In addition,

$$G^{*(s,k)}_{n,\lambda}(x,y)=\sum_{m=0}^{n}\sum_{l=1}^{m+1}\binom{n}{m}\frac{(1)_{l,\lambda}2^mS_{1,\lambda}(m+1,l)}{(m+1)l^{k-1}}(-i)(G^{*}_{n-m,\lambda}(x+iy)-G^{*}_{n-m,\lambda}(x-iy)). \tag{29}$$

Proof. From (25) and (26) of Lemma 1, we have

$$\begin{aligned}\sum_{n=0}^{\infty}G^{*(c,k)}_{n,\lambda}(x,y)\frac{t^n}{n!}&=\frac{2te^x_\lambda(t)\cos^{(y)}_\lambda(t)}{e_\lambda(t)+e^{-1}_\lambda(t)}\frac{1}{2t}\mathrm{Ei}_{k,\lambda}(\log_\lambda(1+2t))\\&=\sum_{j=0}^{\infty}(G^{*}_{j,\lambda}(x+iy)+G^{*}_{j,\lambda}(x-iy))\frac{t^j}{j!}\frac{1}{2t}\sum_{m=1}^{\infty}\left(\sum_{l=1}^{m}\frac{(1)_{l,\lambda}2^m}{l^{k-1}}S_{1,\lambda}(m,l)\right)\frac{t^m}{m!}\\&=\sum_{j=0}^{\infty}(G^{*}_{j,\lambda}(x+iy)+G^{*}_{j,\lambda}(x-iy))\frac{t^j}{j!}\sum_{m=0}^{\infty}\left(\sum_{l=1}^{m+1}\frac{(1)_{l,\lambda}2^m}{(m+1)l^{k-1}}S_{1,\lambda}(m+1,l)\right)\frac{t^m}{m!}\\&=\sum_{n=0}^{\infty}\left(\sum_{m=0}^{n}\sum_{l=1}^{m+1}\binom{n}{m}\frac{(1)_{l,\lambda}2^mS_{1,\lambda}(m+1,l)}{(m+1)l^{k-1}}(G^{*}_{n-m,\lambda}(x+iy)+G^{*}_{n-m,\lambda}(x-iy))\right)\frac{t^n}{n!}.\end{aligned}\tag{30}$$

Therefore, by comparing the coefficients on both sides of (30), we get the desired result.
In addition, we also obtain (29) by the same method as in (30). □

Lemma 3. *For $n,k\geq 0$, we have*

$$G^{*(k)}_{n,\lambda}(x+iy)=\sum_{l=0}^{n}\binom{n}{l}(iy)_{n-l,\lambda}G^{*(k)}_{l,\lambda}(x), \tag{31}$$

and

$$G^{*(k)}_{n,\lambda}(x-iy)=\sum_{l=0}^{n}\binom{n}{l}(-1)^{n-l}<iy>_{n-l,\lambda}G^{*(k)}_{l,\lambda}(x). \tag{32}$$

Proof. From (2) and (21), we observe that

$$\begin{aligned}\sum_{n=0}^{\infty}G^{*(k)}_{n,\lambda}(x+iy)\frac{t^n}{n!}&=\frac{\mathrm{Ei}_{k,\lambda}(\log_\lambda(1+2t))}{e_\lambda(t)+e^{-1}_\lambda(t)}e^x_\lambda(t)e^{iy}_\lambda(t)\\&=\sum_{l=0}^{\infty}G^{*(k)}_{l,\lambda}(x)\frac{t^l}{l!}\sum_{m=0}^{\infty}(iy)_{m,\lambda}\frac{t^m}{m!}\\&=\sum_{n=0}^{\infty}\left(\sum_{l=0}^{n}\binom{n}{l}(iy)_{n-l,\lambda}G^{*(k)}_{l,\lambda}(x)\right)\frac{t^n}{n!}.\end{aligned}\tag{33}$$

Thus, by comparing the coefficients on both sides of (33), we get what we want.
Furthermore, we also obtain (32) by the same method as in (33). □

Corollary 4. *For $n,k\geq 0$, we have*

$$\begin{aligned}G^{*(c,k)}_{n,\lambda}(x,y)=&\sum_{m=0}^{n}\sum_{l=1}^{m+1}\sum_{s=0}^{n-m}\binom{n}{m}\binom{n-m}{s}\frac{(1)_{l,\lambda}2^mS_{1,\lambda}(m+1,l)}{(m+1)l^{k-1}}\\&\times\left((iy)_{n-m-s,\lambda}+(-1)^{n-m-s}<iy>_{n-m-s,\lambda}\right)G^{*}_{s,\lambda}(x).\end{aligned}$$

In addition,

$$G_{n,\lambda}^{*(s,k)}(x,y)=(-i)\sum_{m=0}^{n}\sum_{l=1}^{m+1}\sum_{s=0}^{n-m}\binom{n}{m}\binom{n-m}{s}\frac{(1)_{l,\lambda}2^{m}S_{1,\lambda}(m+1,l)}{(m+1)l^{k-1}}\times\Big((iy)_{n-m-s,\lambda}-(-1)^{n-m-s}<iy>_{n-m-s,\lambda}\Big)G_{s,\lambda}^{*}(x).$$

Proof. By using (3), Theorem 2 and Lemma 3 when $k=1$, we get the desired result. □

Lemma 5. *For $n,k\geq 0$, we have*

$$G_{n,\lambda}^{*(k)}(x)=\sum_{m=0}^{n}\sum_{j=1}^{m+1}\binom{n}{m}\frac{(1)_{j,\lambda}2^{m}}{(m+1)j^{k-1}}S_{1,\lambda}(m+1,j)G_{n-m,\lambda}^{*}(x).$$

Proof. From (21) and (24), we get

$$\begin{aligned}\sum_{n=0}^{\infty}G_{n,\lambda}^{*(k)}(x)\frac{t^{n}}{n!}&=\frac{2t}{e_{\lambda}(t)+e_{\lambda}^{-1}(t)}e_{\lambda}^{x}(t)\frac{1}{2t}\mathrm{Ei}_{k,\lambda}(\log_{\lambda}(1+2t))\\&=\sum_{l=0}^{\infty}G_{l,\lambda}^{*}(x)\frac{t^{l}}{l!}\sum_{m=0}^{\infty}\sum_{j=1}^{m+1}\frac{(1)_{j,\lambda}2^{m}}{(m+1)j^{k-1}}S_{1,\lambda}(m+1,j)\frac{t^{m}}{m!}\\&=\sum_{n=0}^{\infty}\left(\sum_{m=0}^{n}\sum_{j=1}^{m+1}\binom{n}{m}\frac{(1)_{j,\lambda}2^{m}}{(m+1)j^{k-1}}S_{1,\lambda}(m+1,j)G_{n-m,\lambda}^{*}(x)\right)\frac{t^{n}}{n!}.\end{aligned}\tag{34}$$

Therefore, by comparing the coefficients on both sides of (34), we have what we want. □

Theorem 6. *For $n,k\geq 0$, we have*

$$G_{n,\lambda}^{*(c,k)}(x,y)=\sum_{l=0}^{n}\sum_{m=0}^{[\frac{l}{2}]}\sum_{\alpha=0}^{n-l}\sum_{j=1}^{\alpha+1}\binom{n}{l}\binom{n-l}{\alpha}\frac{\lambda^{l-2m}(-1)^{m}y^{2m}(1)_{j,\lambda}2^{\alpha}}{(\alpha+1)j^{k-1}}S_{1}(l,2m)S_{1,\lambda}(\alpha+1,j)G_{n-l-\alpha,\lambda}^{*}(x).$$

In addition, for $n\geq 1$, we have

$$\begin{aligned}G_{n,\lambda}^{*(s,k)}(x,y)&=\sum_{l=1}^{n}\sum_{m=0}^{[\frac{n-1}{2}]}\sum_{\alpha=0}^{n-l}\sum_{j=1}^{\alpha+1}\binom{n}{l}\binom{n-l}{\alpha}\frac{\lambda^{l-2m-1}(-1)^{m}y^{2m+1}(1)_{j,\lambda}2^{\alpha}}{(\alpha+1)j^{k-1}}\\&\quad\times S_{1}(l,2m+1)S_{1,\lambda}(\alpha+1,j)G_{n-l-\alpha,\lambda}^{*}(x),\end{aligned}\tag{35}$$

and $G_{0,\lambda}^{*(s,k)}(x,y)=0$.

Proof. From (11), (22) and Lemma 5, we have

$$
\begin{aligned}
\sum_{n=0}^{\infty} G_{n,\lambda}^{*(c,k)}(x,y)\frac{t^n}{n!} &= \frac{\mathrm{Ei}_{k,\lambda}(\log_\lambda(1+2t))e_\lambda^x(t)}{e_\lambda(t)+e_\lambda^{-1}(t)}\cos_\lambda^{(y)}(t) \\
&= \sum_{j=0}^{\infty} G_{j,\lambda}^{*(k)}(x)\frac{t^j}{j!}\sum_{l=0}^{\infty}\left(\sum_{m=0}^{[\frac{l}{2}]}\lambda^{l-2m}(-1)^m y^{2m} S_1(l,2m)\right)\frac{t^l}{l!} \\
&= \sum_{n=0}^{\infty}\left(\sum_{l=0}^{n}\sum_{m=0}^{[\frac{l}{2}]}\binom{n}{l}\lambda^{l-2m}(-1)^m y^{2m} S_1(l,2m) G_{n-l,\lambda}^{*(k)}(x)\right)\frac{t^n}{n!} \\
&= \sum_{n=0}^{\infty}\left(\sum_{l=0}^{n}\sum_{m=0}^{[\frac{l}{2}]}\sum_{\alpha=0}^{n-l}\sum_{j=1}^{\alpha+1}\binom{n}{l}\binom{n-l}{\alpha}\frac{\lambda^{l-2m}(-1)^m y^{2m}(1)_{j,\lambda}2^\alpha}{(\alpha+1)j^{k-1}}S_1(l,2m)\right. \\
&\qquad \left.\times S_{1,\lambda}(\alpha+1,j)G_{n-l-\alpha,\lambda}^{*}(x)\right)\frac{t^n}{n!}.
\end{aligned} \tag{36}
$$

Therefore, by comparing the coefficients on both sides of (36), we get the desired result. Furthermore, we also obtain (35) by the same method as in (36). □

Theorem 7. *For $n,k \geq 0$, we have*

$$
G_{n,\lambda}^{*(c,k)}(x+z,y) = \sum_{l=0}^{n}\binom{n}{l}(z)_{n-l,\lambda}G_{l,\lambda}^{*(c,k)}(x,y),
$$

$$
G_{n,\lambda}^{*(s,k)}(x+z,y) = \sum_{l=0}^{n}\binom{n}{l}(z)_{n-l,\lambda}G_{l,\lambda}^{*(s,k)}(x,y).
$$

Proof. From (2) and (22), we observe that

$$
\begin{aligned}
\sum_{n=0}^{\infty} G_{n,\lambda}^{*(c,k)}(x+z,y)\frac{t^n}{n!} &= \frac{\mathrm{Ei}_{k,\lambda}\left(\log_\lambda(1+2t)\right)}{e_\lambda(t)+e_\lambda^{-1}(t)}e_\lambda^{x+z}(t)\cos_\lambda^{(y)}(t) \\
&= \frac{\mathrm{Ei}_{k,\lambda}\left(\log_\lambda(1+2t)\right)}{e_\lambda(t)+e_\lambda^{-1}(t)}e_\lambda^{x}(t)\cos_\lambda^{(y)}(t)e_\lambda^{z}(t) \\
&= \sum_{l=0}^{\infty} G_{l,\lambda}^{*(c,k)}(x,y)\frac{t^n}{n!}\sum_{m=0}^{\infty}(z)_{m,\lambda}\frac{t^m}{m!} \\
&= \sum_{n=0}^{\infty}\left(\sum_{l=0}^{n}\binom{n}{l}(z)_{n-l,\lambda}G_{l,\lambda}^{*(c,k)}(x,y)\right)\frac{t^n}{n!}.
\end{aligned} \tag{37}
$$

Therefore, by comparing the coefficients on both sides of (37), we have the desired result. Furthermore, by the same method as in (37), we obtain the second equation. □

Theorem 8. *For $n \geq 0$, we have*

$$
\begin{aligned}
&\frac{1}{2(n+1)}\sum_{l=0}^{n}\binom{n+1}{l}\left((1)_{l,\lambda}+(-1)^l <1>_{l,\lambda}\right)G_{n-l,\lambda}^{*(c)}(x,y) \\
&\qquad = \sum_{l=0}^{n}\sum_{k=0}^{[\frac{l}{2}]}\binom{n}{l}(-1)^k\lambda^{l-2k}y^{2k}S_1(l,2k)(x)_{n-l,\lambda}.
\end{aligned}
$$

Furthermore, for $n \geq 1$*, we have*

$$\frac{1}{2(n+1)}\sum_{l=0}^{n+1}\binom{n+1}{l}\Big((1)_{l,\lambda}+(-1)^{l}<1>_{l,\lambda}\Big)G_{n-l,\lambda}^{*(s)}(x,y) = \sum_{l=1}^{n}\sum_{k=0}^{[\frac{l-1}{2}]}\binom{n}{l}(-1)^{k}\lambda^{l-2k-1}y^{2k+1}S_{1}(l,2k+1)(x)_{n-l,\lambda}. \tag{38}$$

Note that $G_{0,\lambda}^{*(c)}=0$.

Proof. From from (2) and (19), we observe that

$$\begin{aligned} e_{\lambda}^{x}(t)\cos_{\lambda}^{(y)}(t) &= \frac{1}{2t}\big(e_{\lambda}(t)+e_{\lambda}^{-1}(t)\big)\sum_{m=0}^{\infty}G_{m,\lambda}^{*(c)}(x,y)\frac{t^{m}}{m!} \\ &= \frac{1}{2t}\sum_{n=1}^{\infty}\left(\sum_{l=0}^{n-1}\binom{n}{l}\Big((1)_{l,\lambda}+(-1)^{l}<1>_{l,\lambda}\Big)G_{n-l,\lambda}^{*(c)}(x,y)\right)\frac{t^{n}}{n!} \\ &= \frac{1}{2}\sum_{n=0}^{\infty}\left\{\frac{1}{n+1}\sum_{l=0}^{n}\binom{n+1}{l}\Big((1)_{l,\lambda}+(-1)^{l}<1>_{l,\lambda}\Big)G_{n-l,\lambda}^{*(c)}(x,y)\right\}\frac{t^{n}}{n!}. \end{aligned} \tag{39}$$

On the other hand, from (2) and (11), we have

$$\begin{aligned} e_{\lambda}^{x}(t)\cos_{\lambda}^{(y)}(t) &= \sum_{l=0}^{\infty}(x)_{l,\lambda}\frac{t^{l}}{l!}\cos_{\lambda}^{(y)}(t) \\ &= \sum_{n=0}^{\infty}\left(\sum_{l=0}^{n}\sum_{k=0}^{[\frac{l}{2}]}\binom{n}{l}(-1)^{k}\lambda^{l-2k}y^{2k}S_{1}(l,2k)(x)_{n-l,\lambda}\right)\frac{t^{n}}{n!}. \end{aligned} \tag{40}$$

Therefore, by comparing the coefficients of (39) and (40), we have the desired result. Furthermore, we also obtain (38) by the same method as in (39) and (40). □

Theorem 9. *For* $n,k \geq 0$*, we have*

$$\sum_{k=0}^{n}\lambda^{n-k}G_{k,\lambda}^{*(c)}(x,y)S_{2}(n,k) = \sum_{m=0}^{n}\sum_{l=0}^{[\frac{m}{2}]}\binom{n}{m}\binom{m}{2l}\frac{\lambda^{n-m}}{n-m+1}(-1)^{l}y^{2l}G_{m-2l}^{*}(x).$$

In addition, for $n \geq 1$*, we have*

$$\sum_{k=0}^{n}\lambda^{n-k}G_{k,\lambda}^{*(s)}(x,y)S_{2}(n,k) = \sum_{m=1}^{n}\sum_{l=0}^{[\frac{m-1}{2}]}\binom{n}{m}\binom{m}{2l}\frac{\lambda^{n-m}}{n-m+1}(-1)^{l}y^{2l+1}G_{m-2l}^{*}(x). \tag{41}$$

Proof. By replacing t by $\frac{1}{\lambda}(e^{\lambda t}-1)$ in (19) when $k=1$, we get

$$\begin{aligned} \frac{1}{\lambda t}(e^{\lambda t}-1)\left(\frac{2t}{e^{t}+e^{-t}}e^{xt}\cos yt\right) &= \sum_{k=0}^{\infty}G_{k,\lambda}^{*(c)}(x,y)\frac{1}{k!}(e^{\lambda t}-1)^{k}\lambda^{-k} \\ &= \sum_{k=0}^{\infty}G_{k,\lambda}^{*(c)}(x,y)\lambda^{-k}\sum_{n=k}^{\infty}S_{2}(n,k)\lambda^{n}\frac{t^{n}}{n!} \\ &= \sum_{n=0}^{\infty}\left(\sum_{k=0}^{n}\lambda^{n-k}G_{k,\lambda}^{*(c)}(x,y)S_{2}(n,k)\right)\frac{t^{n}}{n!}. \end{aligned} \tag{42}$$

On the other hand, from (1), we have

(43)
$$\frac{1}{\lambda t}(e^{\lambda t}-1)\left(\frac{2t}{e^t+e^{-t}}e^{xt}\cos yt\right)=\sum_{l=0}^{\infty}\frac{\lambda^l}{l+1}\frac{t^l}{l!}\sum_{m=0}^{\infty}\left(\sum_{l=0}^{[\frac{m}{2}]}\binom{m}{2l}(-1)^l y^{2l}G^*_{m-2l}(x)\right)\frac{t^m}{m!}$$
$$=\sum_{n=0}^{\infty}\left(\sum_{m=0}^{n}\sum_{l=0}^{[\frac{m}{2}]}\binom{n}{m}\binom{m}{2l}\frac{\lambda^{n-m}}{n-m+1}(-1)^l y^{2l}G^*_{m-2l}(x)\right)\frac{t^n}{n!}.$$

Therefore, by comparing the coefficients of (42) and (43) respectively, we obtain what we want. Furthermore, we also obtain (41) by the same method as in (42) and (43). □

Theorem 10. *For $n,k\geq 0$, we have*

$$G^{*(c,k)}_{n,\lambda}(x,y)=\sum_{m=0}^{n}\sum_{l=0}^{m}\sum_{k=0}^{[\frac{l}{2}]}\binom{n}{m}\binom{m}{l}(x)_{m-l,\lambda}\lambda^{l-2k}(-1)^k y^{2k}S_{1,\lambda}(l,2k)G^{*(k)}_{n-m,\lambda}.$$

Proof. From (2) and (11), we have

(44)
$$\sum_{n=0}^{\infty}G^{*(c,k)}_{n,\lambda}(x,y)\frac{t^n}{n!}=\frac{\mathrm{Ei}_{k,\lambda}(\log_\lambda(1+2t))}{e_\lambda(t)+e_\lambda^{-1}(t)}e^x_\lambda(t)\cos^{(y)}_\lambda(t)$$
$$=\sum_{j=0}^{\infty}G^{*(k)}_{j,\lambda}\frac{t^j}{j!}\sum_{i=0}^{\infty}(x)_{i,\lambda}\frac{t^i}{i!}\sum_{l=0}^{\infty}\left(\sum_{k=0}^{[\frac{l}{2}]}\lambda^{l-2k}(-1)^k y^{2k}S_{1,\lambda}(l,2k)\right)\frac{t^l}{l!}$$
$$=\sum_{j=0}^{\infty}G^{*(k)}_{j,\lambda}\frac{t^j}{j!}\sum_{m=0}^{\infty}\left(\sum_{l=0}^{m}\sum_{k=0}^{[\frac{l}{2}]}\binom{m}{l}(x)_{m-l,\lambda}\lambda^{l-2k}(-1)^k y^{2k}S_{1,\lambda}(l,2k)\right)\frac{t^m}{m!}$$
$$=\sum_{n=0}^{\infty}\left(\sum_{m=0}^{n}\sum_{l=0}^{m}\sum_{k=0}^{[\frac{l}{2}]}\binom{n}{m}\binom{m}{l}(x)_{m-l,\lambda}\lambda^{l-2k}(-1)^k y^{2k}S_{1,\lambda}(l,2k)G^{*(k)}_{n-m,\lambda}\right)\frac{t^n}{n!}.$$

Therefore, by comparing the coefficients on both sides of (44), we get the desired result. □

Theorem 11. *For $n,k\geq 0$, we have*

$$\begin{cases} G^{*(s,k)}_{n,\lambda}(x,y)=\sum_{m=1}^{n}\sum_{l=1}^{m}\sum_{k=0}^{[\frac{l-1}{2}]}\binom{n}{m}\binom{m}{l}(x)_{m-l,\lambda}\lambda^{l=2k-1}(-1)^k y^{2k+1}S_{1,\lambda}(l,2k+1)G^{*(k)}_{n-m,\lambda} \\ \qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad if\ n\geq 1, \\ G^{*(s,k)}_{0,\lambda}(x,y)=0. \end{cases}$$

Proof. From (2) and (12), we have

(45)

$$\sum_{n=0}^{\infty} G_{n,\lambda}^{*(s,k)}(x,y)\frac{t^n}{n!} = \frac{\mathrm{Ei}_{k,\lambda}\big(\log(1+2t)\big)}{e_\lambda(t)+e_\lambda^{-1}(t)} e_\lambda^x(t)\sin_\lambda^{(y)}(t)$$

$$= \sum_{j=0}^{\infty} G_{j,\lambda}^{*(k)}\frac{t^j}{j!}\sum_{i=0}^{\infty}(x)_{i,\lambda}\frac{t^i}{i!}\sum_{l=1}^{\infty}\left(\sum_{k=0}^{[\frac{l-1}{2}]}\lambda^{l-2k-1}(-1)^k y^{2k+1} S_{1,\lambda}(l,2k+1)\right)\frac{t^l}{l!}$$

$$= \sum_{j=0}^{\infty} G_{j,\lambda}^{*(k)}\frac{t^j}{j!}\sum_{m=1}^{\infty}\left(\sum_{l=1}^{m}\sum_{k=0}^{[\frac{l-1}{2}]}\binom{m}{l}(x)_{m-l,\lambda}\lambda^{l-2k-1}(-1)^k y^{2k+1} S_{1,\lambda}(l,2k+1)\right)\frac{t^m}{m!}$$

$$= \sum_{n=1}^{\infty}\left(\sum_{m=1}^{n}\sum_{l=1}^{m}\sum_{k=0}^{[\frac{l-1}{2}]}\binom{n}{m}\binom{m}{l}(x)_{m-l,\lambda}\lambda^{l-2k-1}(-1)^k y^{2k+1} S_{1,\lambda}(l,2k+1) G_{n-m,\lambda}^{*(k)}\right)\frac{t^n}{n!}.$$

Therefore, by comparing the coefficients on both sides of (45), we get the desired result. □

Acknowledgments
The authors would like to thank the referees for the detailed and valuable comments that helped improve the original manuscript in its present form. This paper is dedicated to professor Seog-Hoon Rim on the occasion of his retirement.

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