Advanced Studies in Contemporary Mathematics
30 (2020), No. 4, pp. 547 - 558

www.jangjeon.or.kr
http://dx.doi.org/10.17777/ascm2020.30.4.547

NOTE ON EXTENDED LAH-BELL POLYNOMIALS AND DEGENERATE EXTENDED LAH-BELL POLYNOMIALS

HYE KYUNG KIM[1] AND DAE SIK LEE[2,*]

ABSTRACT. Kim-Kim(Proc. Jangjeon Math. Soc. 23(4) 2020) studied the Lah-Bell numbers and polynomials. Furthermore, many mathematicians have recently been studying various degenerate versions of special polynomials and numbers in some arithmetic and combinatorial aspects. From this point of view, we are interested in a new type of extended Lah-Bell polynomials and numbers, and the the degenerate extended Lah-Bell polynomials and numbers. The paper is divided in two parts. In the first part, we introduce a new type of the extended Lah-Bell polynomials and numbers, and derive several combinatorial identities related to thoese polynomials and numbers. Some of them include the degenerate and other special polynomials and numbers such as the Stirling number of the second kind, the derangement numbers, the Bell polynomials, the Changhee polynomials of order r, etc. In the second part, we also consider the degenerate extended Lah-Bell polynomials and numbers, and give some new explicit expressions and identities involving those polynomials and numbers.

1. INTRODUCTION

The Lah numbers, studied by Ivo Lah in 1955, have many other interesting applications in analysis and combinatorics (see [3, 5, 18-21]). In combinatorics, the Stirling numbers have various generalizations and arise in a variety of combinatorics problems(see [2-5, 8-9, 17-21]). Recently, Kim et al. also introduced new Jindalrae and Gaenari numbers and polynomials in connection with Jindalrae-Stirling numbers, and developed special polynomials and numbers related to Jindalrae and Gaenari numbers and polynomials [15]. In addition, many mathematicians have recently been studying various degenerate versions of special polynomials and numbers in some arithmetic and combinatorial aspects [2, 7-12, 15-17]. These degenerate versions began when Carlitz introduced the degenerate Bernoulli polynomials and the degenerate Euler polynomials [2]. Motivated by their importance and potential for applications in number theory, combinatorics and other fields of applied mathematics as well as Kim-Kim's papers [5, 14], we are interested in the degenerate extended Lah-numbers and the degenerate extended Lah-Bell polynomials unlike Kim-Kim's extended version.

The paper is divided in two parts. In section 2, we introduce a new type of extended Lah-Bell polynomials and numbers. We give several combinatorial identities related to the extended Lah-numbers and the extended Lah-Bell polynomials. Some of them include the degenerate and other special polynomials and numbers such as the Stirling number of the second kind, the derangement numbers, the Bell polynomials, the Changhee polynomials of order s, etc. In section 3, we also consider the degenerate extended Lah-Bell polynomials and numbers, and give some new explicit expressions and identities involving those polynomials and numbers.

Now, we provide some definitions and properties needed for this paper.

2010 *Mathematics Subject Classification.* 11B73; 11B83; 05A19.
Key words and phrases. Lah numbers; Lah-Bell numbers and polynomials; Extended Lah-Bell numbers and polynomials; degenerate extended Lah-Bell numbers and polynomials; the derangement numbers; the Changhee polynomials.

* is corresponding author.

The unsigned Lah number $L(n,k)$ counts the number of ways of all distributions of n balls, labelled $1,\cdots,n$, among k unlabelled, contents-ordered boxes, with no box left empty and have an explicit formula

$$L(n,k)=\binom{n-1}{k-1}\frac{n!}{k!}. \tag{1}$$

From (1), the generating fucntion of $L(n,k)$ is given by

$$\frac{1}{k!}\left(\frac{t}{1-t}\right)^k=\sum_{n=k}^{\infty}L(n,k)\frac{t^n}{n!},\ (k\geq 0)\qquad \text{(see [3, 5, 18])}. \tag{2}$$

Naturally, Kim-Kim [5] introduced the n-th Lah-Bell number B_n^L, $(n\geq 0)$ given by

$$B_n^L=\sum_{k=0}^{n}L(n,k),\ \ (n\geq 0),$$

and the generating function of the Lah-Bell polynomials $B_n^L(x)$ given by

$$e^{x(\frac{1}{1-t}-1)}=\sum_{n=0}^{\infty}B_n^L(x)\frac{t^n}{n!},\ (k\geq 0)\qquad \text{(see [5])}. \tag{3}$$

When $x=1$, the generating function of Lah-Bell numbers is

$$e^{(\frac{1}{1-t}-1)}=\sum_{n=0}^{\infty}B_n^L\frac{t^n}{n!},\qquad \text{(see [5])}. \tag{4}$$

The Bell polynomials (also called Touchard polynomials or exponential polynomials) are defined by the generating function

$$e^{x(e^t-1)}=\sum_{n=0}^{\infty}Bel_n(x)\frac{t^n}{n!},\qquad \text{(see [10, 12, 16])}. \tag{5}$$

For any nonzero $\lambda\in\mathbb{R}$, the degenerate exponential function is defined by

$$e_\lambda^x(t)=(1+\lambda t)^{\frac{x}{\lambda}},\quad e_\lambda(t)=(1+\lambda t)^{\frac{1}{\lambda}},\qquad \text{(see [2, 7-12, 15-17])}. \tag{6}$$

By Taylor expansion, we get

$$e_\lambda^x(t)=\sum_{n=0}^{\infty}(x)_{n,\lambda}\frac{t^n}{n!},\qquad \text{(see [7-12, 15-17])}, \tag{7}$$

where $(x)_{0,\lambda}=1,\ (x)_{n,\lambda}=x(x-\lambda)(x-2\lambda)\cdots(x-(n-1)\lambda),\ (n\geq 1)$.

For $n\geq 0$, the Stirling numbers of the second kind are defined by

$$x^n=\sum_{l=0}^{n}S_2(n,l)(x)_l,\qquad \text{(see [3, 6-8])}. \tag{8}$$

where $(x)_0=1,\ (x)_n=x(x-1)\dots(x-n+1)$, $(n\geq 1)$.
From (7), we have

$$\frac{1}{k!}(e^t-1)^k=\sum_{n=k}^{\infty}S_2(n,k)\frac{t^n}{n!},\qquad \text{(see [3, 6-8])}. \tag{9}$$

A derangement is a permutation with no fixed points. The number of derangements of an n-element set is called the n-th derangement number and denoted by d_n. This number satisfies the following recurrences:

$$d_n = n \cdot d_{n-1} + (-1)^n, \ \ n \geq 1.$$

By above recurrences of the n-th derangement number, we get

$$d_n = n! \sum_{k=0}^{n} \frac{(-1)^k}{k!}, \ \ (n \geq 0) \qquad \text{(see [3, 19])}.$$

Thus, the following generating function of derangement numbers is given by

$$\frac{1}{1-t} e^{-t} = \sum_{n=0}^{\infty} \left(n! \sum_{k=0}^{n} \frac{(-1)^k}{k!} \right) \frac{t^n}{n!} = \sum_{n=0}^{\infty} d_n \frac{t^n}{n!}, \ \ (n \geq 0) \qquad \text{(see [19])}. \tag{10}$$

Now, the Changhee polynomial of order r is given by

$$\left(\frac{2}{2+t} \right)^r = \sum_{n=0}^{\infty} Ch_n^{(r)} \frac{t^n}{n!}, \qquad \text{(see [6])}. \tag{11}$$

Replacing t by $-2t$ in (11), we get

$$\left(\frac{1}{1-t} \right)^r = \sum_{n=0}^{\infty} (-2)^n Ch_n^{(r)} \frac{t^n}{n!}. \tag{12}$$

It was known ([3] p.37)

$$(1-t)^{-m} = \sum_{l=0}^{\infty} \binom{-m}{l} (-1)^l t^l = \sum_{l=0}^{\infty} < m >_l \frac{t^l}{l!}, \tag{13}$$

where

$$< x >_n = \begin{cases} x(x+1)(x+2)\cdots(x+n-1), & n \geq 1, \\ 1, & n = 0. \end{cases}$$

For $k \geq 0$, Kim-Kim studied the extended Stirling polynomials of the second kind given by the generating function

$$\frac{1}{k!} e^{xt} (e^t - 1 + rt)^k = \sum_{n=k}^{\infty} S_{2,r}(n,k|x) \frac{t^n}{n!} \ (k \geq 0) \qquad \text{(see [14])}.$$

where $x, r \in \mathbb{R}$.

They also defined the extended Bell polynomials given by the generating function as follows:

$$e^{x(e^t - 1 + rt)} = \sum_{n=0}^{\infty} Bel_{n,r}(x) \frac{t^n}{n!} \ (k \geq 0) \qquad \text{(see [14])}.$$

2. The extended Lah-numbers and extended Lah-Bell polynomials

In this section, we introduce a new type of the extended Lah numbers and the extended Lah-Bell polynomials. We derive several combinatorial identities related to the extended Lah-Bell polynomials and numbers.

First of all, we observe that

$$(14)\quad \begin{aligned}\frac{1}{k!}\left(\frac{t}{1-t}\right)^k e^{st} &= \sum_{l=k}^{\infty} L(l,k)\frac{t^l}{l!}\sum_{m=0}^{\infty} s^m \frac{t^m}{m!} \\ &= \sum_{n=k}^{\infty}\left(\sum_{l=0}^{n}\binom{n}{l}L(l,k)s^{n-l}\right)\frac{t^n}{n!},\end{aligned}$$

when $s=0$,

$$\frac{1}{k!}\left(\frac{t}{1-t}\right)^k = \sum_{n=k}^{\infty} L(n,k)\frac{t^n}{n!}.$$

Naturally, from (14), we can consider the generating function of extended Lah-numbers $L_s(n,k)$ given by

$$(15)\quad \frac{1}{k!}\left(\frac{t}{1-t}\right)^k e^{st} = \sum_{n=k}^{\infty} L_s(n,k)\frac{t^n}{n!}$$

where $L_s(n,k) = \sum_{l=0}^{n}\binom{n}{l}L(l,k)s^{n-l}$ are called the extended Lah-numbers.
When $s=0$, $L_0(n,k) := L(n,k)$ are the Lah-numbers.

Below are the few values of the extended Lah-numbers $L_s(n,k)$ with $s=1,2$:

$n\backslash k$	1	2	3	4	5
1	1	0	0	0	0
2	4	1	0	0	0
3	15	9	1	0	0
4	64	66	16	1	0
5	295	490	190	25	1

Table1. Extended Lah-number $L_1(n,k)$

$n\backslash k$	1	2	3	4	5
1	1	0	0	0	0
2	6	1	0	0	0
3	30	12	1	0	0
4	152	108	20	1	0
5	840	920	280	30	1

Table2. Extended Lah-number for $L_2(n,k)$

In view of the relationship between the Lah-numbers and the Lah-Bell numbers, we define the extended Lah-Bell polynomials by

$$(16)\quad B^L_{n,s}(x) = \sum_{k=0}^{n} L_s(n,k)x^k,\ (n \geq 0),$$

when $x=1$,

$$(17)\quad B^L_{n,s}(1) := \sum_{k=0}^{n} L_s(n,k) = B^L_{n,s}$$

are called the extended Lah-Bell numbers.

Theorem 1. *For $n \geq 0$, $k,s \in \mathbb{Z}$, The generating function of extended Lah-Bell polynomials is*

$$(18)\quad \sum_{n=0}^{\infty} B^L_{n,s}(x)\frac{t^n}{n!} = e^{x(\frac{t}{1-t})}e^{st}.$$

Proof. From (15) and (16), we note that

$$\sum_{n=0}^{\infty} B_{n,s}^{L}(x)\frac{t^n}{n!} = \sum_{n=0}^{\infty}\left(\sum_{k=0}^{n} L_s(n,k)x^k\right)\frac{t^n}{n!}$$
$$= \sum_{k=0}^{\infty} x^k\left(\sum_{n=k}^{\infty} L_s(n,k)\frac{t^n}{n!}\right)$$
$$= \sum_{k=0}^{\infty} x^k\frac{1}{k!}\left(\frac{t}{1-t}\right)^k e^{st} = e^{x(\frac{t}{1-t})}e^{st}.$$

Thus, we have the generating function of extended Lah-Bell polynomials. □

Theorem 2. *For $n \geq 0$ and $s \in \mathbb{Z}$, we have*

$$\sum_{l=k}^{n}(-1)^{n+l}L_s(n,k)S_2(n,l) = \sum_{l=k}^{n}\binom{n}{l}(-1)^{n-l}Bel_{n-l}(-s)S_2(l,k).$$

Proof. Replace t to $1-e^{-t}$ in (15). From (5), the left side of (15) is

$$\tag{19} \begin{aligned}\frac{1}{k!}(e^t-1)^k e^{s(1-e^{-t})} &= \sum_{l=k}^{\infty} S_2(l,k)\frac{t^l}{l!}\sum_{m=0}^{\infty} Bel_m(-s)(-1)^m\frac{t^m}{m!}\\ &= \sum_{n=k}^{\infty}\left(\sum_{l=k}^{n}\binom{n}{l}(-1)^{n-l}Bel_{n-l}(-s)S_2(l,k)\right)\frac{t^n}{n!}.\end{aligned}$$

On the other hand, from (9), the right side of (15) is

$$\tag{20} \begin{aligned}\sum_{l=k}^{\infty} L_s(l,k)\frac{(1-e^{-t})^l}{l!} &= \sum_{l=k}^{\infty} L_s(n,k)(-1)^l\frac{(e^{-t}-1)^l}{l!}\\ &= \sum_{l=k}^{\infty} L_s(n,k)(-1)^l\sum_{n=l}^{\infty} S_2(n,l)(-1)^n\frac{t^n}{n!}\\ &= \sum_{n=k}^{\infty}\left(\sum_{l=k}^{n}(-1)^{n+l}L_s(n,k)S_2(n,l)\right)\frac{t^n}{n!}.\end{aligned}$$

Therefore, by comparing the coefficients of (19) and (20), we get the desired result.

□

We obtain the following Dobinski-like formula for extended Lah-Bell numbers.

Theorem 3. *For $n \geq 0$ and $s \in \mathbb{Z}$,*

$$B_{n,s}^{L}(x) = e^{-x}\sum_{l=0}^{n}\sum_{k=0}^{\infty}\binom{n}{l}\frac{<k>_l}{k!}s^{n-l}x^k.$$

Proof. From (18), we observe

$$\begin{aligned}
\sum_{n=0}^{\infty} B_{n,s}^{L}(x)\frac{t^n}{n!} = e^{-x}e^{x(\frac{1}{1-t})}e^{st} &= e^{-x}\sum_{k=0}^{\infty}\frac{x^k}{k!}(1-t)^{-k}\sum_{m=0}^{\infty}s^m\frac{t^m}{m!} \\
&= e^{-x}\sum_{k=0}^{\infty}\frac{x^k}{k!}\left(\sum_{l=0}^{\infty}<k>_l\frac{t^l}{l!}\right)\sum_{m=0}^{\infty}s^m\frac{t^m}{m!} \\
&= e^{-x}\sum_{l=0}^{\infty}\left(\sum_{k=0}^{\infty}\frac{<k>_l}{k!}x^k\right)\frac{t^l}{l!}\sum_{m=0}^{\infty}s^m\frac{t^m}{m!} \\
&= e^{-x}\sum_{n=0}^{\infty}\left(\sum_{l=0}^{n}\sum_{k=0}^{\infty}\binom{n}{l}\frac{<k>_l}{k!}x^k s^{n-l}\right)\frac{t^n}{n!}.
\end{aligned} \tag{21}$$

Therefore, by comparing the coefficients on both sides of (21), we have what we want. □

Theorem 4. *For $n \geq 0$ and $s \in \mathbb{Z}$, we get*

$$\sum_{l=0}^{n}(-1)^{n+l}B_{l,s}^{L}(x)S_2(n,l) = \sum_{l=0}^{n}\binom{n}{l}(-1)^l Bel_{n-l}(x)Bel_l(-s),$$

where $Bel_l(x)$ is the Bell-polynomial.

Proof. Replacing t to $1-e^{-t}$ in (18), from (5), the right side of (18) is

$$\begin{aligned}
e^{x(e^t-1)}e^{s(1-e^{-t})} &= \sum_{j=0}^{\infty}Bel_j(x)\frac{t^j}{j!}\sum_{l=0}^{\infty}Bel_l(-s)(-1)^l\frac{t^l}{l!} \\
&= \sum_{n=0}^{\infty}\left(\sum_{l=0}^{n}\binom{n}{l}(-1)^l Bel_{n-l}(x)Bel_l(-s)\right)\frac{t^n}{n!}.
\end{aligned} \tag{22}$$

On the other hand, the left side of (18) is

$$\begin{aligned}
\sum_{l=0}^{\infty}B_{l,s}^{L}(x)\frac{(-1)^l(e^{-t}-1)^l}{l!} &= \sum_{l=0}^{\infty}B_{l,s}^{L}(x)(-1)^l\sum_{n=l}^{\infty}S_2(n,l)\frac{(-t)^n}{n!} \\
&= \sum_{n=0}^{\infty}\left(\sum_{l=0}^{n}(-1)^{n+l}B_{l,s}^{L}(x)S_2(n,l)\right)\frac{t^n}{n!}.
\end{aligned} \tag{23}$$

Therefore, by comparing the coefficients of (22) and (23), we get the desired result. □

Theorem 5. *For $n \geq 0$ and $s \in \mathbb{Z}$, we have*

$$B_{n,s}^{L} = \frac{1}{e}\sum_{m=0}^{n}\sum_{l=0}^{m}\sum_{k=0}^{\infty}\binom{n}{m}\binom{m}{l}\frac{l!}{k!}k^{m-l}s^{n-m}\sum_{l_1+\cdots+l_k=l}\frac{d_{l_1}d_{l_2}\cdots d_{l_k}}{l_1!l_2!\cdots l_k!},$$

where d_{l_i} are the derangement numbers.

Proof. By using (10) and (18), we observe that

$$
\begin{aligned}
\sum_{n=0}^{\infty} B_{n,s}^{L}\frac{t^n}{n!} &= e^{\frac{1}{1-t}}e^{-1}e^{st} = \frac{1}{e}e^{st}\sum_{k=0}^{\infty}\left(\frac{1}{1-t}\right)^{k}\frac{1}{k!} \\
&= \frac{1}{e}e^{st}\sum_{k=0}^{\infty}\left(\frac{1}{1-t}e^{-t}\right)^{k}\frac{e^{kt}}{k!} \\
&= \frac{1}{e}e^{st}\sum_{k=0}^{\infty}\sum_{l=0}^{\infty}\frac{l!}{k!}\left(\sum_{l_1+l_2+\cdots+l_k=l}\frac{d_{l_1}d_{l_2}\cdots d_{l_k}}{l_1!l_2!\cdots l_k!}\right)\frac{t^l}{l!}\sum_{i=0}^{\infty}k^i\frac{t^i}{i!} \\
&= \frac{1}{e}\sum_{j=0}^{\infty}s^j\frac{t^j}{j!}\sum_{m=0}^{\infty}\left(\sum_{l=0}^{m}\sum_{k=0}^{\infty}\binom{m}{l}\frac{l!k^{m-l}}{k!}\sum_{l_1+\cdots+l_k=l}\frac{d_{l_1}d_{l_2}\cdots d_{l_k}}{l_1!l_2!\cdots l_k!}\right)\frac{t^m}{m!} \\
&= \frac{1}{e}\sum_{n=0}^{\infty}\left(\sum_{m=0}^{n}\sum_{l=0}^{m}\sum_{k=0}^{\infty}\binom{n}{m}\binom{m}{l}\frac{l!}{k!}k^{m-l}s^{n-m}\sum_{l_1+\cdots+l_k=l}\frac{d_{l_1}d_{l_2}\cdots d_{l_k}}{l_1!l_2!\cdots l_k!}\right)\frac{t^n}{n!}.
\end{aligned}
\tag{24}
$$

Therefore, by comparing the coefficients on both sides of (24), we have what we want. □

Theorem 6. *For $n \geq 0$ and $s \in \mathbb{Z}$, we have*

$$B_{n,s}^{L} = \frac{1}{e}\sum_{m=0}^{n}\sum_{l=0}^{\infty}\binom{n}{m}\frac{(-2)^m}{l!}Ch_m^{(l)}s^{n-m},$$

where $Ch_m^{(l)}$ are the Changhee numbers of higher order l.

Proof. From (12) and (18), we have

$$
\begin{aligned}
\sum_{n=0}^{\infty} B_{n,s}^{L}\frac{t^n}{n!} &= e^{\frac{1}{1-t}}e^{-1}e^{st} = \frac{1}{e}e^{st}\sum_{l=0}^{\infty}\left(\frac{1}{1-t}\right)^{l}\frac{1}{l!} \\
&= \frac{1}{e}e^{st}\sum_{l=0}^{\infty}\frac{1}{l!}\sum_{m=0}^{\infty}(-2)^m Ch_m^{(l)}\frac{t^m}{m!} \\
&= \frac{1}{e}\sum_{j=0}^{\infty}s^j\frac{t^j}{j!}\sum_{m=0}^{\infty}\left(\sum_{l=0}^{\infty}\frac{(-2)^m}{l!}Ch_m^{(l)}\right)\frac{t^m}{m!} \\
&= \frac{1}{e}\sum_{n=0}^{\infty}\left(\sum_{m=0}^{n}\sum_{l=0}^{\infty}\binom{n}{m}\frac{(-2)^m}{l!}Ch_m^{(l)}s^{n-m}\right)\frac{t^n}{n!}.
\end{aligned}
\tag{25}
$$

Therefore, by comparing the coefficients on both sides of (25), we have what we want. □

Theorem 7. *For $n \geq 0$ and $s \in \mathbb{Z}$, we have*

$$B_{n+1,s}^{L}(x) = sB_{n,s}^{L}(x) + x\sum_{l=0}^{n}\binom{n}{l} < 2 >_{n-l} B_{l,s}^{L}(x).$$

Proof. The derivative of the right side of (18) is

$$
\begin{aligned}
\frac{d}{dt}\left(e^{x(\frac{1}{1-t}-1)}e^{st}\right) &= \frac{x}{(1-t)^2}e^{x(\frac{1}{1-t}-1)}e^{st}+se^{x(\frac{1}{1-t}-1)}e^{st} \\
&= \left(x(1-t)^{-2}+s\right)\sum_{l=0}^{\infty}B_{l,s}^{L}(x)\frac{t^l}{l!} \\
&= s\sum_{n=0}^{\infty}B_{n,s}^{L}(x)\frac{t^n}{n!}+x\left(\sum_{m=0}^{\infty}<2>_m\frac{t^m}{m!}\right)\sum_{l=0}^{\infty}B_{l,s}^{L}(x)\frac{t^l}{l!} \\
&= s\sum_{n=0}^{\infty}B_{n,s}^{L}(x)\frac{t^n}{n!}+x\sum_{n=0}^{\infty}\left(\sum_{l=0}^{n}\binom{n}{l}<2>_{n-l}B_{l,s}^{L}(x)\right)\frac{t^n}{n!} \\
&= \sum_{n=0}^{\infty}\left(sB_{n,s}^{L}(x)+x\sum_{l=0}^{n}\binom{n}{l}<2>_{n-l}B_{l,s}^{L}(x)\right)\frac{t^n}{n!}.
\end{aligned} \tag{26}
$$

On the other hand, the derivative of the left side of (18) is

$$
\frac{d}{dt}\sum_{n=0}^{\infty}B_{n,s}^{L}(x)\frac{t^n}{n!}=\sum_{n=1}^{\infty}B_{n,s}^{L}(x)\frac{nt^{n-1}}{n!}=\sum_{n=0}^{\infty}B_{n+1,s}^{L}(x)\frac{t^n}{n!}. \tag{27}
$$

Therefore, by comparing the coefficients of (26) and (27), we have the desired result. □

Theorem 8. *For $n \geq 1$ and $s \in \mathbb{Z}$, we have*

$$
\frac{d}{dx}B_{n,s}^{L}(x)=\sum_{m=0}^{n-1}\binom{n}{m}(n-m)!B_{m,s}^{L}(x).
$$

Proof. From (18), we observe that

$$
\begin{aligned}
\sum_{n=1}^{\infty}\frac{d}{dx}B_{n,s}^{L}(x)\frac{t^n}{n!} &= \frac{d}{dx}e^{x(\frac{1}{1-t}-1)}e^{st} \\
&= \left(\frac{1}{1-t}-1\right)e^{x(\frac{1}{1-t}-1)}e^{st} \\
&= \sum_{l=1}^{\infty}l!\frac{t^l}{l!}\sum_{m=0}^{\infty}B_{m,s}^{L}(x)\frac{t^m}{m!} \\
&= \sum_{n=1}^{\infty}\left(\sum_{m=0}^{n-1}\binom{n}{m}(n-m)!B_{m,s}^{L}(x)\right)\frac{t^n}{n!}
\end{aligned} \tag{28}
$$

Thus, by comparing the coefficients on both sides of (28), we have the desired result. □

3. The degenerate extended Lah-Bell polynomials and numbers

In this section, we introduce the degenerate extended Lah-Bell polynomials and numbers. We also derive the several combinatorial identities involving those polynomials and numbers.

We define the degenerate extended Lah-Bell polynomials by

$$
e_{\lambda}^{x}\left(\frac{1}{1-t}-1\right)e_{\lambda}^{s}(t)=\sum_{n=0}^{\infty}B_{n,s,\lambda}^{L}(x)\frac{t^n}{n!}. \tag{29}
$$

When $x=1$, $B_{n,s,\lambda}^{L}=B_{n,s,\lambda}^{L}(1)$ are called the degenerate extended Lah-bell numbers.
As $\lambda \to 0$, $\lim_{\lambda\to 0}B_{n,s,\lambda}^{L}=B_{n,s}^{L}$ are the extended Lah-Bell numbers.

Theorem 9. *For $n \geq 0$ and $s \in \mathbb{Z}$, we have*

$$B^L_{n,s,\lambda}(x) = \sum_{l=0}^{n}\sum_{k=0}^{l}\binom{n}{l}L(l,k)(x)_{k,\lambda}(s)_{n-l,\lambda}.$$

Proof. From (2), (7) and (29), we observe

$$\begin{aligned}
\sum_{n=0}^{\infty} B^L_{n,s,\lambda}(x)\frac{t^n}{n!} &= e^x_\lambda\left(\frac{1}{1-t}-1\right)e^s_\lambda(t)\\
&= \sum_{k=0}^{\infty}(x)_{k,\lambda}\frac{1}{k!}\left(\frac{1}{1-t}-1\right)^k\sum_{m=0}^{\infty}(s)_{m,\lambda}\frac{t^m}{m!}\\
&= \sum_{k=0}^{\infty}(x)_{k,\lambda}\left(\sum_{l=k}^{\infty}L(l,k)\frac{t^l}{l!}\right)\sum_{m=0}^{\infty}(s)_{m,\lambda}\frac{t^m}{m!} \qquad (30)\\
&= \sum_{l=0}^{\infty}\left(\sum_{k=0}^{l}L(l,k)(x)_{k,\lambda}\right)\frac{t^l}{l!}\sum_{m=0}^{\infty}(s)_{m,\lambda}\frac{t^m}{m!}\\
&= \sum_{n=0}^{\infty}\left(\sum_{l=0}^{n}\sum_{k=0}^{l}\binom{n}{l}L(l,k)(x)_{k,\lambda}(s)_{n-l,\lambda}\right)\frac{t^n}{n!}.
\end{aligned}$$

Therefore, by comparing the coefficients on both sides of (30), we have what we want. □

Theorem 10. *For $n \geq 0$ and $s \in \mathbb{Z}$, we have*

$$B^L_{n,s,\lambda}(x) = \sum_{l=0}^{n}\sum_{m=0}^{\infty}\sum_{i=0}^{m}\binom{n}{l}\binom{m}{i}\frac{(-1)^{m-i} <i>_l (x)_{m,\lambda}(s)_{n-l,\lambda}}{m!}.$$

Proof. From (7), (13) and (29), we have

$$\begin{aligned}
\sum_{n=0}^{\infty} B^L_{n,s,\lambda}(x)\frac{t^n}{n!} &= e^x_\lambda\left(\frac{1}{1-t}-1\right)e^s_\lambda(t)\\
&= \sum_{m=0}^{\infty}\frac{(x)_{m,\lambda}}{m!}\sum_{i=0}^{m}\binom{m}{i}(-1)^{m-i}\left(\frac{1}{1-t}\right)^i\sum_{j=0}^{\infty}(s)_{j,\lambda}\frac{t^j}{j!}\\
&= \sum_{m=0}^{\infty}\frac{(x)_{m,\lambda}}{m!}\sum_{i=0}^{m}\binom{m}{i}(-1)^{m-i}\sum_{l=0}^{\infty}<i>_l\frac{t^l}{l!}\sum_{j=0}^{\infty}(s)_{j,\lambda}\frac{t^j}{j!} \qquad (31)\\
&= \sum_{l=0}^{\infty}\left(\sum_{m=0}^{\infty}\sum_{i=0}^{m}\binom{m}{i}\frac{(-1)^{m-i}<i>_l(x)_{m,\lambda}}{m!}\right)\frac{t^l}{l!}\sum_{j=0}^{\infty}(s)_{j,\lambda}\frac{t^j}{j!}\\
&= \sum_{n=0}^{\infty}\left(\sum_{l=0}^{n}\sum_{m=0}^{\infty}\sum_{i=0}^{m}\binom{n}{l}\binom{m}{i}\frac{(-1)^{m-i}<i>_l(x)_{m,\lambda}(s)_{n-l,\lambda}}{m!}\right)\frac{t^n}{n!}.
\end{aligned}$$

Therefore, by comparing the coefficients on both sides of (31), we get what we want. □

Theorem 11. *For $n, s \geq 0$, we have*

$$\sum_{l=0}^{n}\binom{n}{l}(-s)_{n-l,\lambda}B^L_{l,s,\lambda}(x) = \sum_{m=0}^{\infty}\sum_{i=0}^{m}\binom{m}{i}\frac{(-1)^{m-i+n}2^n Ch^{(i)}_n(x)_{m,\lambda}}{m!},$$

where $Ch^{(i)}_m$ are the Changhee numbers of order i.

Proof. From (7) and (29), we observe

$$
\begin{aligned}
e_\lambda^x(\frac{1}{1-t}-1) &= e_\lambda^{-s}(t)\sum_{l=0}^{\infty} B_{l,s,\lambda}^L(x)\frac{t^l}{l!}\\
&= \sum_{m=0}^{\infty}(-s)_{m,\lambda}\frac{t^m}{m!}\sum_{l=0}^{\infty} B_{l,s,\lambda}^L(x)\frac{t^l}{l!}\\
&= \sum_{n=0}^{\infty}\left(\sum_{l=0}^{n}\binom{n}{l}(-s)_{n-l,\lambda}B_{l,s,\lambda}^L(x)\right)\frac{t^n}{n!}.
\end{aligned}
\tag{32}
$$

On the other hand, by using (7) and (12), we get

$$
\begin{aligned}
e_\lambda^x(\frac{1}{1-t}-1) &= \sum_{m=0}^{\infty}\frac{(x)_{m,\lambda}}{m!}\sum_{i=0}^{m}\binom{m}{i}(-1)^{m-i}(\frac{1}{1-t})^i\\
&= \sum_{m=0}^{\infty}\frac{(x)_{m,\lambda}}{m!}\sum_{i=0}^{m}\binom{m}{i}(-1)^{m-i}\sum_{n=0}^{\infty}(-2)^n Ch_n^{(i)}\frac{t^n}{n!}\\
&= \sum_{n=0}^{\infty}\left(\sum_{m=0}^{\infty}\sum_{i=0}^{m}\binom{m}{i}\frac{(-1)^{m-i+n}2^n Ch_n^{(i)}(x)_{m,\lambda}}{m!}\right)\frac{t^n}{n!}.
\end{aligned}
\tag{33}
$$

Therefore, by comparing the coefficients of (32) and (33), we get what we want. □

Theorem 12. *For $n \geq 0$ and $s \in \mathbb{Z}$, we have*

$$
\sum_{l=0}^{n}(-1)^{l+n}B_{l,s,\lambda}^L(x)S_2(n,l) = \sum_{m=0}^{n}\sum_{k=0}^{m}\binom{n}{m}(-1)^{k+m}(s)_{k,\lambda}Bel_{n-m,\lambda}(x)S_2(m,k),
\tag{34}
$$

In addition, we have another identity as follows:

$$
\sum_{l=0}^{n}(-1)^{l+n}B_{l,s,\lambda}^L(x)S_2(n,l) = \sum_{m=0}^{n}\binom{n}{m}(-1)^m Bel_{n-m,\lambda}(x)Bel_{m,-\lambda}(-s),
\tag{35}
$$

where $Bel_{n,\lambda}(x)$ is the degenerate Bell polynomial given by

$$
e_\lambda^x(e^t-1) = \sum_{l=0}^{\infty}Bel_{l,\lambda}(x)\frac{t^l}{l!}, \ \text{(see [16])}.
$$

Proof. Replacing t to $1-e^{-t}$ in (29), from (6) and (10), the left side of (29) is

$$
\begin{aligned}
& e_\lambda^x(e^t-1)e_\lambda^s(1-e^{-t})\\
&= \sum_{l=0}^{\infty}Bel_{l,\lambda}(x)\frac{t^l}{l!}\sum_{k=0}^{\infty}(s)_{k,\lambda}\frac{(-1)^k(e^{-t}-1)^k}{k!}\\
&= \sum_{l=0}^{\infty}Bel_{l,\lambda}(x)\frac{t^l}{l!}\sum_{k=0}^{\infty}(s)_{k,\lambda}(-1)^k\sum_{m=k}^{\infty}S_2(m,k)\frac{(-t)^m}{m!}\\
&= \sum_{l=0}^{\infty}Bel_{l,\lambda}(x)\frac{t^l}{l!}\sum_{m=0}^{\infty}\left(\sum_{k=0}^{m}(-1)^{k+m}(s)_{k,\lambda}S_2(m,k)\right)\frac{t^m}{m!}\\
&= \sum_{n=0}^{\infty}\left(\sum_{m=0}^{n}\sum_{k=0}^{m}\binom{n}{m}(-1)^{k+m}(s)_{k,\lambda}S_2(m,k)Bel_{n-m,\lambda}(x)\right)\frac{t^n}{n!}.
\end{aligned}
\tag{36}
$$

On the other hand, from (10), the right side of (29) is

$$\begin{aligned}\sum_{l=0}^{\infty} B^{L}_{l,s,\lambda}(x)\frac{(-1)^l(e^{-t}-1)^l}{l!} &= \sum_{l=0}^{\infty} B^{L}_{l,s,\lambda}(x)\sum_{n=l}^{\infty}(-1)^l S_2(n,l)\frac{(-t)^n}{n!} \\ &= \sum_{n=0}^{\infty}\left(\sum_{l=0}^{n}(-1)^{l+n}B^{L}_{l,s,\lambda}(x)S_2(n,l)\right)\frac{t^n}{n!}.\end{aligned} \tag{37}$$

Therefore, by comparing the coefficients of (36) and (37), we have the first identity (34).

In addition, the left side of (29) is

$$\begin{aligned}&e^x_\lambda(e^t-1)e^s_\lambda(1-e^{-t}) \\ &= \sum_{l=0}^{\infty} Bel_{l,\lambda}(x)\frac{t^l}{l!}\sum_{m=0}^{\infty} Bel_{m,-\lambda}(-s)(-1)^m\frac{t^m}{m!} \\ &= \sum_{n=0}^{\infty}\left(\sum_{m=0}^{n}\binom{n}{m}(-1)^m Bel_{n-m,\lambda}(x)Bel_{m,-\lambda}(-s)\right)\frac{t^n}{n!}.\end{aligned} \tag{38}$$

Thus, by comparing the coefficients of (37) and (38), we have the second identity (35). □

Theorem 13. *For $n \geq 0$ and $s \in \mathbb{Z}$, we have*

$$\sum_{l=0}^{n}(-1)^{l+n}B^{L}_{l,s,\lambda}(x)S_2(n,l) = \sum_{l=0}^{n}\sum_{k=0}^{l}\sum_{m=0}^{n-l}\binom{n}{l}(-1)^{m+n-l}(x)_{k,\lambda}(s)_{m,\lambda}S_2(l,k)S_2(n-l,m).$$

Proof. Replacing t to $1-e^{-t}$ in (29), from (8) and (10), the left side of (29) is

$$\begin{aligned}&e^x_\lambda(e^t-1)e^s_\lambda(1-e^{-t}) \\ &= \sum_{k=0}^{\infty}(x)_{k,\lambda}\frac{(e^t-1)^k}{k!}\sum_{m=0}^{\infty}(s)_{m,\lambda}\frac{(-1)^m(e^{-t}-1)^m}{m!} \\ &= \sum_{l=0}^{\infty}\left(\sum_{k=0}^{l}(x)_{k,\lambda}S_2(l,k)\right)\frac{t^n}{l!}\sum_{j=0}^{\infty}\left(\sum_{m=0}^{j}(s)_{m,\lambda}(-1)^{m+j}S_2(j,m)\right)\frac{t^j}{j!} \\ &= \sum_{n=0}^{\infty}\left(\sum_{l=0}^{n}\sum_{k=0}^{l}\sum_{m=0}^{n-l}\binom{n}{l}(-1)^{m+n-l}(x)_{k,\lambda}(s)_{m,\lambda}S_2(l,k)S_2(n-l,m)\right)\frac{t^n}{n!}.\end{aligned} \tag{39}$$

Therefore, by comparing the coefficients of (37) and (39), we get what we want. □

Acknowledgments

The authors would like to thank the referees for the detailed and valuable comments that helped improve the original manuscript in its present form. This paper is dedicated to professor Seog-Hoon Rim on the occasion of his retirement.

Funding

This research was supported by the Daegu University Research Grant, 2018.

Ethics approval and consent to participate

All authors declare that there is no ethical problem in the production of this paper.

REFERENCES

[1] S. Araci and Ö. Özer, *Extended q-Dedekind-type Daehee-Changhee sums associated with extended q-Euler polynomials*, Adv. Difference Equ. **2015**, 272(2015).

[2] L. Carlitz, *Degenerate Stirling, Bernoulli and Eulerian numbers,* Utilitas Math. **1979**, *15*, 51-88.

[3] L. Comtet, *Advanced Combinatorics: The Art of Finite and Infinite Expansions* (translated from the French by J.W. Nienhuys), Reidel, Dordrecht and Boston, 1974.

[4] D.V. Dolgy and T. Kim, *Some explicit formulas of degenerate Stirling numbers associated with the degenerate special numbers and polynomials,* Proc. Jangjeon Math. Soc. **2018**, *21(2)*, 309-317.

[5] D.S. Kim and T. Kim, *Lah-Bell numbers and polynomials*, math NT **2020**, *arXiv:2007.13291v1*.

[6] D.S. Kim, T. Kim, J.J. Seo and S. Lee, *Higher-order Changhee numbers and polynomials*, Adv. Studies Theor. Phys. **2014**, *8(8)*, 365 - 373.

[7] D.S. Kim and T. Kim, *A note on a new type of degenerate Bernoulli numbers,* Russ. J. Math. Phys. **2020**, *27(2)*, 227-235.

[8] T. Kim, *A note on degenerate Stirling polynomials of the second kind,* Proc. Jangjeon Math. Soc. **2017**, *20(3)*, 319-331.

[9] T. Kim, λ- *Analouge of Stirling numbers of the first kind*, Advanced Studies in Contemporary mathematics **2017**, *27(3)*, 423-429.

[10] T. Kim and D.S. Kim, *Degenerate polyexponential functions and degenerate Bell polynomials,* J. Math. Anal. Appl. **2020**, *487(2)*, 124017.

[11] T. Kim and D.S. Kim, *Degenerate Laplace transform and degenerate gamma function*, Russ. J. Math. Phys. **2017**, *24(2)*, 241-248.

[12] T. Kim and D.S. Kim, *Degenerate polyexponential functions and degenerate Bell polynomials,* J. Math. Anal. Appl, **2020**, *487*, 124017.

[13] T. Kim and D.S. Kim, *Some relations of two type 2 polynomials and discrete harmonic numbers and polynomials,* Symmetry **2020**, *12(6)*, 905.

[14] T. Kim and D.S. Kim, *Extended Stirling polynomials of the second kind and extended bell polynomials*, math.NT **2017**, arXiv:1705.08375v1.

[15] T. Kim, D.S. Kim, L.C. Jang and H. Lee, *Jindalrae and Gaenari numbers and polynomials in connection with Jindalrae-Stirling numbers,* Adv. Difference Equ. **2020**, *245(2020)*.

[16] T. Kim, D.S. Kim, H.Y. Kim and J. Kwon, *Some identities of degenerate Bell polynomials,* Mathematics **2020**, *8(40)*, 8pp.

[17] T. Kim, D.S. Kim, Y. Yao and H.I. Kwon, *Some identities involving special numbers and moments of random variables*, Rocky Mountain J. Math. **2019**, *487*, 124017.

[18] G. Nyul, and G. Racz, *Sums of r-Lah numbers and r-Lah polynomials*, ARS Math. contemp. **2020**, https://doi.org/10.26493/1855-3974.1793.c4d.

[19] F. Qi and B. Guo, *On the sum of the Lah numbers and zeros of the Kummer confluent hypergeometric function*, Acta Univ. Sapientiae, Mathematica,10,1 **2018**, *50(2)*, 125-133.

[20] F. Qi and B. Guo, *Six proofs for an identity of the Lah numbers*, Online Journal of Analytic Combinatorics **2015**, 5pp.

[21] B.C. Roberto, B.M. Charles and B.M. Maribeth, *The r-Dowling Numbers and Matices Containing r-Whitney Numbers of the Second Kind and Lah Numbers*, European Journal of Pure and Applied Math. **2019**, *12(3)*, 1122-1137.

DEPARTMENT OF MATHEMATICS EDUCATION, DAEGU CATHOLIC UNIVERSITY, GYEONGSAN 38430, REPUBLIC OF KOREA

E-mail address: hkkim@cu.ac.kr

SCHOOL OF ELECTRONIC AND ELECTRIC ENGINEERING, DAEGU UNIVERSITY, GYEONGSAN 38453, REPUBLIC OF KOREA

E-mail address: dslee@daegu.ac.kr