

A NOTE ON DEGENERATE TYPE 2 MULTI-POLY-GENOCCHI POLYNOMIALS

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ABSTRACT. In this paper, we introduce the degenerate multiple polylogarithm functions which are multiple versions of the degenerate modified polylogarithm functions. Then we consider the degenerate type 2 multi-poly-Genocchi polynomials which are defined by using these functions and obtain explicit expressions and some properties for these polynomials.

1. INTRODUCTION

As is well known, the polylogarithm function is defined by a power series in x , for all complex arguments x with $|x| < 1$, which is also a Dirichlet series in k for all $k \in \mathbb{Z}$:

$$(1) \quad Li_k(x) = \sum_{n=1}^{\infty} \frac{x^n}{n^k} = x + \frac{x^2}{2^k} + \frac{x^3}{3^k} + \cdots, \quad (\text{see [1, 5, 9, 14, 16]}).$$

The polyexponential functions were studied by Hardy in [4] as follows.

$$(2) \quad e(x, a|s) = \sum_{n=0}^{\infty} \frac{x^n}{(n+a)^s n!}, \quad (Re(a) > 0).$$

. Recently, a slightly different version of these functions, which are called the modified polyexponential functions, are defined as an inverse to polylogarithm function by

$$(3) \quad Ei_k(x) = \sum_{n=1}^{\infty} \frac{x^n}{(n-1)! n^k}, \quad (\text{see [6, 8, 11, 15, 3]}).$$

From (2), we see that

$$(4) \quad Li_1(x) = \sum_{n=1}^{\infty} \frac{x^n}{n} = -\log(1-x).$$

For any nonzero $\lambda \in \mathbb{R}$, the degenerate exponential function is defined by

$$(5) \quad e_{\lambda}^x(t) = (1 + \lambda t)^{\frac{x}{\lambda}}, \quad e_{\lambda}(t) = (1 + \lambda t)^{\frac{1}{\lambda}} = e_{\lambda}^1(t). \quad (\text{see [11, 13]})$$

From (5), it is easy to see that

$$(6) \quad e_{\lambda}^x(t) = \sum_{n=0}^{\infty} (x)_{n,\lambda} \frac{t^n}{n!}, \quad (\text{see [5]}),$$

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where $(x)_{0,\lambda} = 1$ and $(x)_{n,\lambda} = x(x-\lambda)\cdots(x-(n-1)\lambda)$, $(n \geq 1)$. Note that $\lim_{\lambda \rightarrow 0} e_\lambda^x(t) = \sum_{n=0}^{\infty} \frac{x^n t^n}{n!} = e^{xt}$. The degenerate modified polyexponential functions are defined by Kim-Kim to be

$$(7) \quad Ei_{k,\lambda}(x) = \sum_{n=1}^{\infty} \frac{(1)_{n,\lambda}}{(n-1)!n^k} x^n, (\lambda \in \mathbb{R}), \text{ (see [8]).}$$

In [6], Kim-Kim-Kim-Kwon introduced the degenerate poly-Genocchi polynomials defined as

$$(8) \quad \frac{2Ei_{k,\lambda}(\log_\lambda(1+t))}{e_\lambda(t)+1} e_\lambda^x(t) = \sum_{n=0}^{\infty} g_{n,\lambda}^{(k)}(x) \frac{t^n}{n!}.$$

When $x=0$, $g_{n,\lambda}^{(k)} = g_{n,\lambda}^{(k)}(0)$ are called the degenerate poly-Genocchi numbers. For $k_1, \dots, k_r \in \mathbb{Z}$, they also considered the degenerate multi-poly-Genocchi polynomials which are given by

$$(9) \quad \frac{2^r Ei_{k_1, \dots, k_r, \lambda}(\log_\lambda(1+t))}{(e_\lambda(t)+1)^r} e_\lambda^x(t) = \sum_{n=0}^{\infty} g_{n,\lambda}^{(k_1, \dots, k_r)}(x) \frac{t^n}{n!},$$

where $Ei_{k_1, \dots, k_r, \lambda}(x)$ are the degenerate multiple polyexponential functions defined by

$$(10) \quad Ei_{k_1, \dots, k_r, \lambda}(x) = \sum_{0 < n_1 < n_2 < \dots < n_r} \frac{(1)_{n_1, \lambda} \cdots (1)_{n_r, \lambda} x^{n_r}}{(n_1-1)! \cdots (n_r-1)! n_1^{k_1} \cdots n_r^{k_r}}.$$

When $x=0$, $g_{n,\lambda}^{(k_1, \dots, k_r)} = g_{n,\lambda}^{(k_1, \dots, k_r)}(0)$ are called the degenerate multi-poly-Genocchi numbers. We recall that the Euler polynomials are defined by the generating function

$$(11) \quad \frac{2}{e^t+1} e^{xt} = \sum_{n=0}^{\infty} E_n(x) \frac{t^n}{n!}, \text{ (see [5, 2, 16]).}$$

When $x=0$, $E_n = E_n(0)$ are called the Euler numbers.

For $r \in \mathbb{N}$, the Euler polynomials of order r are given by

$$(12) \quad \left(\frac{2}{e^t+1} \right)^r e^{xt} = \sum_{n=0}^{\infty} E_n^{(r)}(x) \frac{t^n}{n!}, \text{ (see [10]).}$$

When $x=0$, $E_n^{(r)} = E_n^{(r)}(0)$ are called the Euler numbers of order r .

Carlitz [9] introduced the degenerate Euler polynomials given by the generating function to be

$$(13) \quad \frac{2}{e_\lambda^x(t)+1} e_\lambda^x(t) = \sum_{n=0}^{\infty} \mathcal{E}_{n,\lambda}(x) \frac{t^n}{n!},$$

When $x=0$, $\mathcal{E}_{n,\lambda} = \mathcal{E}_{n,\lambda}(0)$ are called the degenerate Euler numbers. For $r \in \mathbb{N}$, the degenerate Euler polynomials of order r given by the generating function to be

$$(14) \quad \left(\frac{2}{e_\lambda^x(t)+1} \right)^r e_\lambda^x(t) = \sum_{n=0}^{\infty} \mathcal{E}_{n,\lambda}^{(r)}(x) \frac{t^n}{n!}.$$

When $x=0$, $\mathcal{E}_{n,\lambda}^{(r)} = \mathcal{E}_{n,\lambda}^{(r)}(0)$ are called the degenerate Euler numbers of order r .

For $n \geq 0$, and $\lambda \in \mathbb{R}$, the degenerate Stirling numbers of the second kind are defined as

$$(15) \quad (x)_{n,\lambda} = \sum_{l=0}^n S_{2,\lambda}(n,l)(x)_l, \quad (n \geq 0), \text{ (see [12]),}$$

where $(x)_0 = 1$, $(x)_n = x(x-1)\cdots(x-n+1)$, $(n \geq 1)$.

From (15), we note that

$$(16) \quad \frac{1}{k!}(e_\lambda(t) - 1)^k = \sum_{n=k}^{\infty} S_{2,\lambda}(n, k) \frac{t^n}{n!}, \text{ (see [12]).}$$

In view of (10), we introduce the degenerate multiple polylogarithm functions which are multiple version of the degenerate modified polylogarithm function. Then we consider the degenerate type 2 multi-poly-Genocchi polynomials which are defined by using these functions and obtain explicit expressions and some properties for these polynomials.

2. DEGENERATE TYPE 2 MULTI-POLY-GENOCCHI POLYNOMIALS

In this section, we recall a $Li_k(x)$, for $k \in \mathbb{Z}$, Kim-Kim [7] defined the degenerate polylogarithm function as

$$(17) \quad l_{k,\lambda}(x) = \sum_{n=1}^{\infty} \frac{(-\lambda)^{n-1} (1)_{n, \frac{1}{\lambda}}}{(n-1)! n^k} x^n, \text{ (}|x| < 1).$$

From (17), we note that

$$(18) \quad \frac{d}{dx} l_{k,\lambda}(x) = \frac{1}{x} l_{k-1,\lambda}(x),$$

and

$$(19) \quad l_{1,\lambda}(x) = -\log_\lambda(1-x),$$

where $\log_\lambda x = \frac{1}{\lambda}(t^\lambda - 1)$. For $k_1, k_2, \dots, k_r \in \mathbb{Z}$, we define the degenerate multi-polylogarithm function as

$$(20) \quad l_{k_1, \dots, k_r, \lambda}(x) = \sum_{0 < n_1 < \dots < n_r} \frac{(-\lambda)^{n_1 + \dots + n_r - r} (1)_{n_1, \frac{1}{\lambda}} \dots (1)_{n_r, \frac{1}{\lambda}} x^{n_r}}{(n_1 - 1)! \dots (n_r - 1)! n_1^{k_1} \dots n_r^{k_r}},$$

where the sum is over all integers n_1, \dots, n_r satisfying $0 < n_1 < \dots < n_r$.

Now, we consider the degenerate type 2 multi-poly-Genocchi polynomials which are given by

$$(21) \quad \frac{2^r l_{k_1, \dots, k_r, \lambda}(1 - e_\lambda(-t))}{(e_\lambda(t) + 1)^r} e_\lambda^x(t) = \sum_{n=0}^{\infty} G_{n,\lambda}^{(k_1, \dots, k_r)}(x) \frac{t^n}{n!}.$$

When $x = 0$, $G_{n,\lambda}^{(k_1, \dots, k_r)} = G_{n,\lambda}^{(k_1, \dots, k_r)}(0)$ are called the degenerate type 2 multi-poly-Genocchi numbers. When $r = 1$ and $k_1 = 1$,

$$(22) \quad l_{1,\lambda}(1 - e_\lambda(-t)) = -\log_\lambda(1 - 1 + e_\lambda(-t)) = t$$

and

$$(23) \quad \lim_{\lambda \rightarrow 0} \frac{2 l_{1,\lambda}(1 - e_\lambda(-t))}{e_\lambda(t) + 1} e_\lambda^x(t) = \frac{2t}{e^t + 1} e^{xt} = \sum_{n=0}^{\infty} G_n(x) \frac{t^n}{n!},$$

where $G_n(x)$ are Genocchi polynomials. From (23), we note that $\lim_{\lambda \rightarrow 0} G_{n,\lambda}^{(1)}(x) = G_n(x)$, ($n \geq 0$). From (21) and (6), we get

$$\begin{aligned}
 & \frac{2^r I_{k_1, \dots, k_r, \lambda}(1 - e_\lambda(-t))}{(e_\lambda(t) + 1)^r} e_\lambda^x(t) \\
 &= \left(\sum_{m=0}^{\infty} G_{m,\lambda}^{(k_1, \dots, k_r)} \frac{t^m}{m!} \right) \left(\sum_{k=0}^{\infty} (x)_{k,\lambda} \frac{t^k}{k!} \right) \\
 (24) \quad &= \sum_{n=0}^{\infty} \sum_{m=0}^n \binom{n}{m} G_{m,\lambda}^{(k_1, \dots, k_r)}(x)_{n-m,\lambda} \frac{t^n}{n!}.
 \end{aligned}$$

From (24), we obtain the following theorem.

Theorem 2.1. For $n \in \mathbb{N} \cup \{0\}$, we have

$$(25) \quad G_{n,\lambda}^{(k_1, \dots, k_r)}(x) = \sum_{m=0}^n \binom{n}{m} G_{n,\lambda}^{(k_1, \dots, k_r)}(x)_{n-m,\lambda}.$$

From (21), we observe

$$\begin{aligned}
 & \sum_{n=0}^{\infty} G_{n,\lambda}^{(k_1, \dots, k_r)}(x) \frac{t^n}{n!} \\
 &= \left(\frac{2}{e_\lambda(t) + 1} \right)^r e_\lambda^x(t) \sum_{0 < n_1 < \dots < n_r} \frac{(-\lambda)^{n_1 + \dots + n_r - r} (1)_{n_1, \frac{1}{\lambda}} \dots (1)_{n_r, \frac{1}{\lambda}} (1 - e_\lambda(-t))^{n_r}}{(n_1 - 1)! \dots (n_r - 1)! n_1^{k_1} \dots n_r^{k_r}} \\
 &= \left(\sum_{l=0}^{\infty} \mathcal{E}_{l,\lambda}^{(r)}(x) \frac{t^l}{l!} \right) \sum_{m=n_r}^{\infty} \\
 & \quad \times \left(\sum_{0 < n_1 < \dots < n_r \leq m} \frac{(-\lambda)^{n_1 + \dots + n_r - r} (1)_{n_1, \frac{1}{\lambda}} \dots (1)_{n_r, \frac{1}{\lambda}} (-1)^{n_r + m} S_{2,\lambda}(m, n_r) t^m}{(n_1 - 1)! \dots (n_{r-1} - 1)! n_1^{k_1} \dots n_{r-1}^{k_{r-1}} n_r^{k_r - 1} m!} \right) \\
 &= \sum_{n=r}^{\infty} \left(\sum_{l=0}^{n-r} \binom{n}{l} \mathcal{E}_{l,\lambda}^{(r)}(x) \right. \\
 (26) \quad & \quad \times \left. \sum_{0 < n_1 < \dots < n_r \leq n-l} \frac{(-\lambda)^{n_1 + \dots + n_r - r} (1)_{n_1, \frac{1}{\lambda}} \dots (1)_{n_r, \frac{1}{\lambda}} (-1)^{n_r + n - l} S_{2,\lambda}(n - l, n_r)}{(n_1 - 1)! \dots (n_{r-1} - 1)! n_1^{k_1} \dots n_{r-1}^{k_{r-1}} n_r^{k_r - 1}} \right) \frac{t^n}{n!}.
 \end{aligned}$$

By comparing the coefficients on the both sides of (26), we have the following theorem.

Theorem 2.2. For $k_1, \dots, k_r \in \mathbb{Z}$, and $n, r \in \mathbb{N}$ with $n \geq r$, we have

$$\begin{aligned}
 & G_{n,\lambda}^{(k_1, \dots, k_r)}(x) = \sum_{l=0}^{n-r} \binom{n}{l} \mathcal{E}_{l,\lambda}^{(r)}(x) \\
 (27) \quad & \quad \times \sum_{0 < n_1 < \dots < n_r \leq n-l} \frac{(-\lambda)^{n_1 + \dots + n_r - r} (1)_{n_1, \frac{1}{\lambda}} \dots (1)_{n_r, \frac{1}{\lambda}} (-1)^{n_r + n - l} S_{2,\lambda}(n - l, n_r)}{(n_1 - 1)! \dots (n_{r-1} - 1)! n_1^{k_1} \dots n_{r-1}^{k_{r-1}} n_r^{k_r - 1}}.
 \end{aligned}$$

Further, we have $G_{n,\lambda}^{(k_1, \dots, k_r)}(x) = 0$, for $0 \leq n \leq r - 1$.

It is well known that

$$(28) \quad \mathcal{E}_{l,\lambda}^{(r)}(x) = \frac{1}{r! \binom{n+m}{n}} G_{n+r,\lambda}^{(r)}(x), \quad (n, r \geq 0), \quad (\text{see [6]})$$

where $G_{n,\lambda}^{(r)}(x)$ is the degenerate Genocchi polynomials of order r defined by

$$(29) \quad \left(\frac{2t}{e_\lambda(t)+1}\right)^r e_\lambda^x(t) = \sum_{n=0}^\infty G_{n,\lambda}^{(r)}(x) \frac{t^n}{n!}, \quad (\text{see [6]}).$$

Theorem 2.3. For $k_1, \dots, k_r \in \mathbb{Z}$ and $n, r \in \mathbb{N}$ with $n \geq r$, we have

$$(30) \quad G_{n,\lambda}^{(k_1, \dots, k_r)}(x) = \sum_{l=0}^{n-r} \binom{n}{l} \frac{1}{r! \binom{n+m}{n}} G_{n+r,\lambda}^{(r)}(x) \\ \times \sum_{0 < n_1 < \dots < n_r \leq n-l} \frac{(-\lambda)^{n_1 + \dots + n_r - r} (1)_{n_1, \frac{1}{\lambda}} \dots (1)_{n_r, \frac{1}{\lambda}} (-1)^{n_r + n - l} S_{2,\lambda}(n-l, n_r)}{(n_1 - 1)! \dots (n_{r-1} - 1)! n_1^{k_1} \dots n_{r-1}^{k_{r-1}} n_r^{k_r - 1}}.$$

From (21), we note that

$$(31) \quad \sum_{n=0}^\infty G_{n,\lambda}^{(k_1, \dots, k_r)}(r) \frac{t^n}{n!} \\ = 2^r \left(1 - \frac{1}{2} \frac{2}{e_\lambda(t)+1}\right)^r l_{k_1, \dots, k_r, \lambda}(1 - e_\lambda(-t)) \\ = \sum_{m=0}^\infty \left(\sum_{l=0}^r \binom{r}{l} (-1)^l 2^{r-l} \mathcal{G}_{m,\lambda}^{(l)}\right) \frac{t^m}{m!} \sum_{j=n_r}^\infty \\ \times \sum_{0 < n_1 < \dots < n_r \leq j} \frac{(-\lambda)^{n_1 + \dots + n_r - r} (1)_{n_1, \frac{1}{\lambda}} \dots (1)_{n_r, \frac{1}{\lambda}} (-1)^{n_r + n - l} S_{2,\lambda}(j, n_r) t^j}{(n_1 - 1)! \dots (n_{r-1} - 1)! n_1^{k_1} \dots n_{r-1}^{k_{r-1}} n_r^{k_r - 1} j!} \\ = \sum_{n=r}^\infty \left(\sum_{m=0}^{n-r} \sum_{l=0}^r \sum_{0 < n_1 < \dots < n_r \leq n-m} \binom{n}{m}\right) \\ \times \frac{\binom{r}{l} (-1)^l 2^{r-l} \mathcal{G}_{m,\lambda}^{(l)} (-\lambda)^{n_1 + \dots + n_r - r} (1)_{n_1, \frac{1}{\lambda}} \dots (1)_{n_r, \frac{1}{\lambda}} (-1)^{n_r + n - l} S_{2,\lambda}(n-m, n_r)}{(n_1 - 1)! \dots (n_{r-1} - 1)! n_1^{k_1} \dots n_{r-1}^{k_{r-1}} n_r^{k_r - 1}} \frac{t^n}{n!}.$$

Then by (31), the following theorem holds.

Theorem 2.4. For $k_1, \dots, k_r \in \mathbb{Z}$, and $n, r \in \mathbb{N}$ with $n \geq r$, we have

$$(32) \quad G_{n,\lambda}^{(k_1, \dots, k_r)}(r) = \sum_{m=0}^{n-r} \sum_{l=0}^r \sum_{0 < n_1 < \dots < n_r \leq n-m} \binom{n}{m} \\ \times \frac{\binom{r}{l} (-1)^l 2^{r-l} \mathcal{G}_{m,\lambda}^{(l)} (-\lambda)^{n_1 + \dots + n_r - r} (1)_{n_1, \frac{1}{\lambda}} \dots (1)_{n_r, \frac{1}{\lambda}} (-1)^{n_r + n - l} S_{2,\lambda}(n-m, n_r)}{(n_1 - 1)! \dots (n_{r-1} - 1)! n_1^{k_1} \dots n_{r-1}^{k_{r-1}} n_r^{k_r - 1}}.$$

By (21), we observe

$$\begin{aligned}
 \sum_{n=0}^{\infty} G_{n,\lambda}^{(k_1, \dots, k_r)}(x+y) \frac{t^n}{n!} &= \frac{2^r l_{k_1, \dots, k_r, \lambda} (1 - e_{\lambda}(-t))}{(e_{\lambda}(t) + 1)^r} e_{\lambda}^x(t) e_{\lambda}^y(t) \\
 &= \left(\sum_{l=0}^{\infty} G_{l,\lambda}^{(k_1, \dots, k_r)}(x) \frac{t^l}{l!} \right) \left(\sum_{m=0}^{\infty} (y)_{m,\lambda} \frac{t^m}{m!} \right) \\
 (33) \quad &= \sum_{n=0}^{\infty} \left(\sum_{l=0}^n \binom{n}{l} G_{l,\lambda}^{(k_1, \dots, k_r)}(x) (y)_{n-l,\lambda} \right) \frac{t^n}{n!}.
 \end{aligned}$$

Thus, by comparing the coefficients on both sides of (33), we obtain the following result.

Theorem 2.5. For $k_1, \dots, k_r \in \mathbb{Z}$ and any nonnegative integer n , we have

$$(34) \quad G_{n,\lambda}^{(k_1, \dots, k_r)}(x+y) = \sum_{l=0}^n \binom{n}{l} G_{l,\lambda}^{(k_1, \dots, k_r)}(x) (y)_{n-l,\lambda}.$$

3. CONCLUSION

Kim-Kim-Kim-Kwon(2020) considered the degenerate multi-poly-Genocchi polynomials by using the degenerate multiple polyexponential functions. Based on these ideas and similar views, we considered the degenerate type 2 multi-poly-Genocchi polynomials in Eq. (21) by using the degenerate multiple polylogarithm functions. Furthermore we obtained some explicit expressions for the degenerate type 2 multi-poly-Genocchi polynomials in Theorems 2.1, 2.2, and 2.4, Corollary 2.3, and certain property related to these polynomials in Theorem 2.5.

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