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SOME RELATIONSHIPS BETWEEN THE NUMBERS OF LYNDON WORDS AND A CERTAIN CLASS OF COMBINATORIAL NUMBERS CONTAINING POWERS OF BINOMIAL COEFFICIENTS

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ABSTRACT. The main aim of this paper is to find out some relationships between the numbers of Lyndon words and a certain class of combinatorial numbers which contains finite sums of powers of binomial coefficients and whose generating functions were constructed and investigated by Simsek in "Generating functions for finite sums involving higher powers of binomial coefficients: Analysis of hypergeometric functions including new families of polynomials and numbers, J. Math. Anal. Appl. 477 (2019), 1328–1352". By applying not only the Dirichlet convolution formula, but also the Mobius inversion formula, we obtain some identities containing the Möbius function, the Euler's totient function, the numbers of Lyndon words, the numbers of necklaces, the Stirling numbers of the second kind, and the aforementioned class of combinatorial numbers. Moreover, a few special cases and consequences of our results are considered. In particular, it should be noted here that in the special case when we take the power of the binomial coefficients to be 1, some of our results are reduced to several results obtained by Kucukoglu and Simsek in "Identities and Derivative Formulas for the Combinatorial and Apostol-Euler Type Numbers by Their Generating Functions, Filomat 32(20) (2018), 6879–6891".

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1. INTRODUCTION

Of late years, by conducting an impressive series of studies (see [29]; and also see [28, 30]), Simsek has made an effort on the construction techniques of generating functions for some families of combinatorial numbers containing different kinds of binomial coefficients to provide convenience to the literature, and for this purpose, Simsek introduced and classified a series of combinatorial number families by indexing them according to their identification order with their generating functions. Among these studies, the article where this classification was initiated is [29], in which, Simsek introduced a class of combinatorial numbers denoted by $y_1(n, k; \lambda)$ and computed with the aid of the following formula:

$$y_1(n, k; \lambda) = \frac{1}{k!} \sum_{j=0}^{k} \binom{k}{j} j^n \lambda^j,$$

together with their exponential generating functions given by

$$\frac{\left(\lambda e^{t}+1\right)^{k}}{k!}=\sum_{n=0}^{\infty} y_1(n,k;\lambda)\frac{t^n}{n!}, \tag{1}$$

(for details see [29]; and also [28, 30]).

Subsequently, as a generalization of the numbers $y_1(n,k;\lambda)$, Simsek [30] constructed the exponential generating functions for another class of combinatorial numbers, denoted by $y_6(n,k;\lambda,v)$ and containing finite sums of powers of binomial coefficients, with the aid of the generalized hypergeometric function as follows:

$$\frac{1}{k!}\ {}_{v}F_{v-1}\left[\begin{array}{c}-k,-k,...,-k\\ 1,1,...,1\end{array};(-1)^{v}\lambda e^{t}\right]=\sum_{n=0}^{\infty} y_6(n,k;\lambda,v)\frac{t^n}{n!}, \tag{2}$$

in which, the numbers $y_6(n,k;\lambda,v)$ are given explicitly by

$$y_6(n,k;\lambda,v)=\frac{1}{k!}\sum_{j=0}^{k}\binom{k}{j}^{v} j^n\lambda^j, \tag{3}$$

where $\lambda\in\mathbb{R}$ (or $\mathbb{C}$) and $n,k,v\in\mathbb{N}_0=\{0,1,2,\dots\}$ (*cf.* [30, p. 1347]).

By the above constructions, Simsek has brought a new perspective to the world of the combinatorial numbers families on how their generating functions can be built. By revealing many properties and relationships regarding these combinatorial numbers, recent studies conducted by Simsek showed that not only combinatorial number families are an indispensable part of the theories of enumerative combinatorics and mathematical physics, but also combinatorial number families are of meaningfull connections with other kinds of special numbers and polynomials. Therefore, it is obvious that any examination to be made on these numbers have potential to affect many researchers.

By this motivation, to shed light on another side of these combinatorial numbers, we are here mainly concerned with deriving identities which gives us relationships of the combinatorial numbers $y_6(n,k;\lambda,v)$ with not only the numbers of Lyndon words, but also the numbers of necklaces. Indeed, it is aimed to show that the numbers $y_6(n,k;\lambda,v)$ are possessed of close relationships with not only the numbers of Lyndon words, but also the numbers of necklaces. To achieve this aim, the techniques used are based entirely on the techniques used in the work of Kucukoglu and Simsek [17] and these techniques consist of the Dirichlet convolution and the Mobius inversion formula which are explained in detail in the next section. It is worth noting here that some of our results generalize some identities obtained in [17].

The rest of this paper consists of next three sections. As for the summary of these sections: In Section 2, we give some preliminaries regarding the arithmetical functions, the Möbius function, the Euler's totient function, the Dirichlet convolution formula, the Möbius inversion formula, the numbers of Lyndon words and the numbers of necklaces. In section 3, by using not only the Dirichlet convolution formula, but also the Möbius inversion formula, we obtain some identities containing the Möbius function, the Euler's totient function, the numbers of Lyndon words, the numbers of necklaces, the Stirling numbers of the second kind, and the aforementioned class of combinatorial numbers, and in Section 4 we conclude the paper by making a comment on what it reveals.

2. Preliminaries

Before presenting our results to the readers, we here remind the following concepts about the techniques basically used for achieving the results of this study.

An arithmetical function (or a number-theoretic function) is a real (or complex-valued function) whose domain is the set of positive integers, and these type functions allow us to investigate not only the features of divisibility in integers, but also distribution of the prime numbers (see, for details, [1]).

Two of the most frequently used arithmetical functions are the Möbius function $\mu(n)$ and the Euler's totient function $\varphi(n)$, and these arithmetical functions are respectively defined by (*cf.* [1]):

$$\mu(n) = \begin{cases} 1 & \text{if } n = 1, \\ (-1)^m & \text{if } n \text{ is a square-free integer with } m \text{ distinct prime factors}, \\ 0 & \text{if } n \text{ has a squared prime factor.} \end{cases}$$

and

$$\varphi(n) = \sum_{\substack{m=1 \\ (m,n)=1}}^{n} 1,$$

in which, the sum runs over all positive integers $m \leq n$ that are relatively prime to the positive integer n.

The Dirichlet convolution $f * g$ of two arithmetical functions f and g is given by

$$(f * g)(n) = \sum_{d|n} f(d)\, g\left(\frac{n}{d}\right),$$

in which, the sum runs over all divisors d of the positive integer n (*cf.* [1]).

It is also worth mentioning here that the Dirichlet convolution is one of the auxiliary tools that establish the bridge between arithmetical functions, and this tool serves us as the cornerstone of the analytic number theory to reveal relations among the concepts of this theory. In the meanwhile, under the operations of the pointwise addition and the Dirichlet convolution, the set of all arithmetical functions yields a commutative ring with unity (for details, see [9], [19]).

Some of the arithmetic functions arising from the Dirichlet convolution are given as follows:

The arithmetical function $L_k(n)$ that allows us to count how many k-ary Lyndon words of length n are, is given by

$$L_k(n) = \frac{1}{n} \sum_{d|n} \mu\left(\frac{n}{d}\right) k^d, \tag{4}$$

(*cf* [2, 10, 20–24]).

The arithmetical function $N_k(n)$ that allows us to count how many necklaces with n beads of k different colors are, is given by

$$N_k(n) = \frac{1}{n} \sum_{d|n} \varphi\left(\frac{n}{d}\right) k^d \tag{5}$$

(*cf.* [2], [12]).

As mentioned in Kucukoglu and Simsek [19], the k-ary Lyndon words of n length is known to be identified as the lexicographically (i.e. in a dictionary order) smallest element of the set of conjugate classes which are the results of cyclic shifts of k letters in a primitive word of length n.

The arithmetical function $L_k(n)$ that provides the counting these words, are possessed of close relationships with the arithmetical function $N_k(n)$ due to the superposition of the Lyndon words with aperiodic necklace class representatives (see, for setails, [2, 15, 20, 21, 24]). In the solution of some problems in algebraic combinatorics, lie algebra and analytic number theory, it is encountered frequently with these arithmetical functions and it may be seen to appears to be studied with various aspects by many reseracheres (*cf.* [2, 6, 13–15, 17–27]; and also see the references cited therein).

Note that one of the important formulas used in conjunction with the Dirichlet convolution is the Möbius inversion formula which is used for converting the aritmetical function f given by

$$f(n) = \sum_{d|n} g(d)$$

into the arithmetical function g below

$$g(n) = \sum_{d|n} \mu\left(\frac{n}{d}\right) f(d),$$

and vice versa (see, for details, [1, p. 32]).

3. Main Results

In this section, by blending the techniques of the Dirichlet convolution of arithmetic functions and the Möbius inversion formula, we get identities containing the Möbius function, the Euler's totient function, the numbers of Lyndon words, the numbers of necklaces, the Stirling numbers of the second kind, the numbers $y_6(m,k;\lambda,v)$ and other kinds of combinatorial numbers. The results obtained here shows us what kind of relationships exists among the numbers $y_6(m,k;\lambda,v)$, the numbers of Lyndon words and the numbers of necklaces.

Theorem 3.1. *Let $n \in \mathbb{N}$. Then we have*

$$\sum_{d|n} \mu\left(\frac{n}{d}\right) y_6(d,k;\lambda,v) = \frac{n}{k!}\sum_{j=0}^{k}\binom{k}{j}^{v}\lambda^j L_j(n). \tag{6}$$

Proof. In order to prove this theorem, we first consider the Dirichlet convolution of the numbers $y_6(n,k;\lambda,v)$ and the Möbius function $\mu(n)$ as follows:

$$(y_6 * \mu)(n) = \sum_{d|n} \mu\left(\frac{n}{d}\right) y_6(d,k;\lambda,v),$$

which, by using (3), yields

$$(y_6 * \mu)(n) = \frac{1}{k!}\sum_{j=0}^{k}\binom{k}{j}^{v}\lambda^{j}\sum_{d|n}\mu\left(\frac{n}{d}\right)j^{d}.$$

Then, by blending (4) with the equation just above, we get

$$(y_6 * \mu)(n) = \frac{n}{k!}\sum_{j=0}^{k}\binom{k}{j}^{v}\lambda^{j}L_j(n),$$

which yields the desired result. □

Remark 3.1. *The special case of* (6) *when* $v = 1$ *is reduced to the following identity:*

$$\sum_{d|n}\mu\left(\frac{n}{d}\right)y_1(d,k;\lambda) = \frac{n}{k!}\sum_{j=0}^{k}\binom{k}{j}\lambda^{j}L_j(n),$$

which was given by Kucukoglu and Simsek in [17, p. 6887, Theorem 3.8]. *Furthermore, when* $\lambda = v = 1$, *the equation* (6) *also yields*

(7) $$\sum_{d|n}\mu\left(\frac{n}{d}\right)B(d,k) = n\sum_{j=0}^{k}\binom{k}{j}L_j(n),$$

where

$$B(d,k) = k!y_1(d,k;1),$$

which was given by Golombek [11] *with the formula below:*

$$B(d,k) = \frac{\mathrm{d}^{d}}{\mathrm{d}t^{d}}\left(e^{t}+1\right)^{k}\bigg|_{t=0},$$

(see, for details, [11]; *and see also* [16,28–30]*). In addition to above investigations, when* $\lambda = -1$ *and* $v = 1$, *the equation* (6) *yields*

(8) $$\sum_{d|n}\mu\left(\frac{n}{d}\right)S_2(d,k) = \frac{n}{k!}\sum_{j=0}^{k}(-1)^{k-j}\binom{k}{j}L_j(n),$$

where $S_2(d,k)$ *denotes the Stirling numbers of the second kind defined by the followings:*

$$S_2(d,k) = \frac{1}{k!}\sum_{j=0}^{k}(-1)^{j}\binom{k}{j}(k-j)^{d},$$

$$S_2(d,k) = S_2(d-1,k-1) + kS_2(d-1,k),$$

$$x^{d} = \sum_{k=0}^{d}S_2(d,k)(x)_k,$$

and

$$\frac{(e^{t}-1)^{k}}{k!} = \sum_{d=0}^{\infty}S_2(d,k)\frac{t^{d}}{d!},$$

such that $S_2(0,0) = 1$, $S_2(d,k) = 0$ *if* $k > d$; $S_2(d,0) = 0$ *if* $d > 0$ *(cf.* [3,5,7,8,32]*).*

Let p be a prime number. Since the divisors of the prime number p are 1 and p, setting $n = p$ into Theorem 3.1 yields the following corollary:

Corollary 3.1. *Let p be prime number. Then we have*

$$y_6(p, k; \lambda, v) - y_6(1, k; \lambda, v) = \frac{p}{k!} \sum_{j=0}^{k} \binom{k}{j}^{v} \lambda^j L_j(p). \tag{9}$$

With the displacement of k by n, Theorem 3.1 are reduced to the following corollary:

Corollary 3.2. *Let $n \in \mathbb{N}$. Then we have*

$$\sum_{d|n} \mu\left(\frac{n}{d}\right) y_6(d, n; \lambda, v) = \frac{1}{(n-1)!} \sum_{j=0}^{n} \binom{n}{j}^{v} \lambda^j L_j(n).$$

Theorem 3.2. *Let $n \in \mathbb{N}$. Then we have*

$$y_6(n, k; \lambda, v) = \sum_{d|n} \sum_{j=0}^{k} \binom{k}{j}^{v} \frac{d}{k!} \lambda^j L_j(d). \tag{10}$$

Proof. By applying the Möbius inversion formula to (6), we convert the left-hand side of the equation (6) into

$$y_6(n, k; \lambda, v) = \sum_{d|n} \frac{d}{k!} \sum_{j=0}^{k} \binom{k}{j}^{v} \lambda^j L_j(d),$$

which yields the assertion of Theorem 3.2. □

Remark 3.2. *By Theorem 3.2, we have another relationship between the numbers $y_6(n, k; \lambda, v)$ and the numbers of Lyndon words. Substantially, Theorem 3.2 shows us that the numbers $y_6(n, k; \lambda, v)$ can be expressed in terms of the numbers of Lyndon words.*

The special case of (10) for varying values of λ and v yields the following corollaries: In the special case when $\lambda = 1$, (10) is reduced to the following corollary:

Corollary 3.3. *Let $n \in \mathbb{N}$. Then we have*

$$y_6(n, k; 1, v) = \sum_{d|n} \sum_{j=0}^{k} \binom{k}{j}^{v} \frac{d}{k!} L_j(d). \tag{11}$$

In the special case when $v = 1$, (10) is also reduced to the following corollary:

Corollary 3.4. *Let $n \in \mathbb{N}$. Then we have*

$$y_1(n, k; \lambda) = \sum_{d|n} \sum_{j=0}^{k} \binom{k}{j} \frac{d}{k!} \lambda^j L_j(d). \tag{12}$$

Remark 3.3. *Different proof of* (12) *was given by Kucukoglu and Simsek in* [17, p. 6887, Theorem 3.9].

It is known from the work of Boyadzhiev [4, p.4, Eq-(7)] that

$$\sum_{j=0}^{k}\binom{k}{j}j^{n}x^{j}=\sum_{j=0}^{n}\binom{k}{j}j!S_{2}\left(n,j\right)x^{j}\left(1+x\right)^{k-j}, \tag{13}$$

(*cf.* [4, p.4, Eq-(7)]).

By combining (13) with (12), we get the following corollary:

Corollary 3.5. *Let* $n\in\mathbb{N}$. *Then we have*

$$\sum_{j=0}^{n}\binom{k}{j}j!S_{2}\left(n,j\right)\lambda^{j}\left(1+\lambda\right)^{k-j}=\sum_{d|n}\sum_{j=0}^{k}\binom{k}{j}\lambda^{j}dL_{j}\left(d\right). \tag{14}$$

In the special case when $\lambda=1$ and $v=1$, (10) is also reduced to the following corollary:

Corollary 3.6. *Let* $n\in\mathbb{N}$. *Then we have*

$$B(n,k)=\sum_{d|n}\sum_{j=0}^{k}\binom{k}{j}dL_{j}\left(d\right). \tag{15}$$

It is known from the work of Spivey [31, Identity 12] that

$$B\left(m,n\right)=\sum_{j=0}^{m}\binom{n}{j}j!2^{n-j}S_{2}\left(m,j\right), \tag{16}$$

(*cf.* [31, Identity 12]).

By combining (16) with (15), we get the following corollary:

Corollary 3.7. *Let* $n\in\mathbb{N}$. *Then we have*

$$\sum_{j=0}^{n}\binom{k}{j}j!2^{k-j}S_{2}\left(n,j\right)=\sum_{d|n}\sum_{j=0}^{k}\binom{k}{j}dL_{j}\left(d\right). \tag{17}$$

In the special case when $\lambda=-1$ and $v=1$, (10) is also reduced to the following corollary:

Corollary 3.8. *Let* $n\in\mathbb{N}$. *Then we have*

$$S_{2}(n,k)=\sum_{d|n}\sum_{j=0}^{k}(-1)^{k-j}\binom{k}{j}\frac{d}{k!}L_{j}\left(d\right). \tag{18}$$

Remark 3.4. *Corollary* 3.8 *shows how the Stirling numbers of the second kind* $S_2(n,k)$ *are expressed in terms of the numbers of Lyndon words.*

Theorem 3.3. *Let* $n\in\mathbb{N}$. *Then we have*

$$\sum_{d|n}\varphi\left(\frac{n}{d}\right)y_{6}(d,k;\lambda,v)=\frac{n}{k!}\sum_{j=0}^{k}\binom{k}{j}^{v}\lambda^{j}N_{j}\left(n\right). \tag{19}$$

Proof. In order to prove this theorem, we first consider the Dirichlet convolution of the numbers $y_6(n,k;\lambda,v)$ and the Euler's totient function $\varphi(n)$ as follows:

$$(y_6 * \varphi)(n) = \sum_{d|n} \varphi\left(\frac{n}{d}\right) y_6(d,k;\lambda,v),$$

which, by using (3), yields

$$(y_6 * \varphi)(n) = \frac{1}{k!}\sum_{j=0}^{k} \binom{k}{j}^{v} \lambda^j \sum_{d|n} \varphi\left(\frac{n}{d}\right) j^d.$$

Then, by blending (5) with the equation just above, we get

$$(y_6 * \varphi)(n) = \frac{n}{k!}\sum_{j=0}^{k} \binom{k}{j}^{v} \lambda^j N_j(n),$$

which is the desired result. □

In the special case when $v = 1$, (19) is reduced to the following corollary:

Corollary 3.9. *Let $n \in \mathbb{N}$. Then we have*

$$\sum_{d|n} \varphi\left(\frac{n}{d}\right) y_1(d,k;\lambda) = \frac{n}{k!}\sum_{j=0}^{k} \binom{k}{j} \lambda^j N_j(n).$$

Moreover, when $\lambda = v = 1$, the equation (19) also yields the following corollary:

Corollary 3.10. *Let $n \in \mathbb{N}$. Then we have*

$$\sum_{d|n} \varphi\left(\frac{n}{d}\right) B(d,k) = n\sum_{j=0}^{k} \binom{k}{j} N_j(n). \tag{20}$$

For $\lambda = -1$ and $v = 1$, the equation (19) yields the following corollary:

Corollary 3.11. *Let $n \in \mathbb{N}$. Then we have*

$$\sum_{d|n} \varphi\left(\frac{n}{d}\right) S_2(d,k) = \frac{n}{k!}\sum_{j=0}^{k} (-1)^{k-j} \binom{k}{j} N_j(n). \tag{21}$$

Remark 3.5. *Corollary* 3.11 *shows that the Dirichlet convolution of the Euler's totient function and the Stirling numbers of the second kind $S_2(n,k)$ can be calculated by a combinatorial sum including the binomials coeficients and the numbers of necklaces.*

Let p be a prime number. Since the divisors of the prime number p are 1 and p, setting $n = p$ into Theorem 3.3 yields the following corollary:

Corollary 3.12. *Let p be prime number. Then we have*

$$(p-1)\, y_6(1,k;\lambda,v) + y_6(p,k;\lambda,v) = \frac{p}{k!}\sum_{j=0}^{k} \binom{k}{j}^{v} \lambda^j N_j(p). \tag{22}$$

With the displacement of k by n, (3.3) is reduced to the following corollary:

Corollary 3.13. *Let $n \in \mathbb{N}$. Then we have*

$$\sum_{d|n} \varphi\left(\frac{n}{d}\right) y_6(d, n; \lambda, v) = \frac{1}{(n-1)!} \sum_{j=0}^{n} \binom{n}{j}^v \lambda^j N_j(n).$$

By subtracting (9) from (22), we get the following corollary:

Corollary 3.14.

$$y_6(1, k; \lambda, v) = \frac{1}{k!} \sum_{j=0}^{k} \binom{k}{j}^v \lambda^j \left(N_j(p) - L_j(p)\right). \tag{23}$$

Applying the Euler operator $\lambda \frac{d}{d\lambda}$ to $y_6(0, k; \lambda, v)$ and using (23), we arrive at the following theorem:

Theorem 3.4.

$$\sum_{j=0}^{k} \binom{k}{j}^v \lambda^j \left(N_j(p) - L_j(p)\right) = k! \lambda \frac{d}{d\lambda} \{y_6(0, k; \lambda, v)\}. \tag{24}$$

4. Conclusion

In conclusion, this paper reveals that applying the Dirichlet convolution formula and the Möbius inversion formula to the combinatorial numbers yields miscellaneous relations of the numbers of Lyndon words with a certain class of combinatorial numbers containing powers of binomial coefficients.

References

[1] T. M. Apostol, Introduction to Analytic Number Theory, *Springer-Verlag*, New York-Heidelberg-Berlin, 1976.

[2] J. Berstel and D. Perrin, *The origins of combinatorics on words*, European J. Combin. **28** (2007), 996–1022.

[3] M. Bona, Introduction to Enumerative Combinatorics, *The McGraw-Hill Companies Inc.*, New York, 2007.

[4] K. N. Boyadzhiev, Binomial transform and the backward difference, ArXiv:1410.3014v2.

[5] N. P. Cakic and G. V. Milovanovic, *On generalized Stirling numbers and polynomials*, Mathematica Balkanica **18** (2004), 241–248.

[6] K. Cattell, F. Ruskey, J. Sawada, M. Serra, C. R. Miers, *Fast algorithms to generate necklaces, unlabeled necklaces, and irreducible polynomials over $GF(2)$*, J. Algorithm **37**(2) (2000), 267–282.

[7] C. A. Charalambides, Enumerative Combinatorics, *Chapman&Hall/Crc Press Company*, London, New York, 2002.

[8] L. Comtet, Advanced combinatorics, *D. Reidel*, Dordrecht, 1974.

[9] J. E. Delany, *Groups of Arithmetical Functions*, Math. Mag., **78**(2) (2005), 83–97.

[10] T. W. Cusick and P. Stanica, Cryptographic Boolean Functions and Applications, *Academic Press, Elsevier*, 2009.

[11] R. Golombek, *Aufgabe 1088*, Elem. Math. **49** (1994), 126–127.

[12] R. L. Graham, D. E. Knuth and O. Pataschnik, Concrete Mathematics, *Addison Wesley*, 1988.

[13] I. Kucukoglu and Y. Simsek, *Identities on the k-ary Lyndon words related to a family of zeta functions*, ArXiv: 1606.03837v1.

[14] I. Kucukoglu and Y. Simsek, *On k-ary Lyndon words and their generating functions*, AIP Conference Proceedings **1863**(300004) (2017), 1–4.

[15] I. Kucukoglu, A. Bayad and Y. Simsek, *k-ary Lyndon Words and Necklaces Arising as Rational Arguments of Hurwitz-Lerch Zeta Function and Apostol-Bernoulli Polynomials*, Mediterr. J. Math. **14** (2017), Article number: 223.

[16] I. Kucukoglu and Y. Simsek, Observation on identities and relations for interpolation functions and special numbers, Adv. Stud. Contemp. Math. (Kyungshang) **28**(1) (2018), 41–56

[17] I. Kucukoglu and Y. Simsek, *Identities and Derivative Formulas for the Combinatorial and Apostol-Euler Type Numbers by Their Generating Functions*, Filomat **32**(20) (2018), 6879–6891.

[18] I. Kucukoglu and Y. Simsek, *Numerical evaluation of special power series including the numbers of Lyndon words: An approach to interpolation functions for Apostol-type numbers amd polynomials*, Electron. Trans. Numer. Anal. **50**(20) (2018), 98–108.

[19] I. Kucukoglu and Y. Simsek, *Identities for Dirichlet and Lambert-type series arising from the numbers of a certain special word*, Appl. Anal. Discrete Math. **13** (2019), 787–804.

[20] M. Lothaire, Combinatorics on Words, *Cambridge University Press*, Cambridge, 1997.

[21] R. Lyndon, *On Burnside problem I*, Trans. Amer. Math. Soc. **77** (1954), 202–215.

[22] N. Metropolis and G. C. Rota, *Witt Vectors and the Algebra of Necklaces*, Adv. Math. **50** (1983), 95–125.

[23] V. M. Petrogradsky, *Witts formula for restricted Lie algebras*, Adv. Appl. Math. **30** (2003), 219–227.

[24] N. Rebenich, Counting Prime Polynomials and Measuring Complexity and Similarity of Information, *PhD Thesis, University of Victoria*, Canada, 2016.

[25] F. Ruskey and J. Sawada, *An efficient algorithm for generating necklaces with fixed density*, SIAM J. Comput. **29**(2) (1999), 671–684.

[26] F. Ruskey, Information on necklaces, unlabelled necklaces, Lyndon words, De Bruijn sequences, `http://theory.cs.uvic.ca/inf/neck/NecklaceInfo.html`

[27] F. Ruskey, *Generating necklaces*, J. Algorithms **13**(3) (1992), 414–430.

[28] Y. Simsek, *Computation methods for combinatorial sums and Euler type numbers related to new families of numbers*, Math. Methods Appl. Sci. **40**(7) (2017), 2347–2361.

[29] Y. Simsek, *New families of special numbers for computing negative order Euler numbers and related numbers and polynomials*, Appl. Anal. Discrete Math. **12** (2018), 1–35.

[30] Y. Simsek, *Generating functions for finite sums involving higher powers of binomial coefficients: Analysis of hypergeometric functions including new families of polynomials and numbers*, J. Math. Anal. Appl. **477** (2019), 1328–1352.

[31] M. Z. Spivey, *Combinatorial sums and finite differences*, Discrete Math. **307**(24) (2007), 3130–3146.

[32] H. M. Srivastava and J. Choi, Zeta and q-Zeta functions and associated series and integrals, Elsevier Science Publishers: Amsterdam, London and New York, 2012.

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