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COMBINATORIAL SUMS INVOLVING STIRLING, FUBINI, BERNOULLI NUMBERS AND APPROXIMATE VALUES OF CATALAN NUMBERS

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ABSTRACT. The aim of this paper is to derive some identities and formulas associated with the Bernoulli numbers of negative order, the Stirling numbers of the second kind, the Fubini numbers, and the Catalan numbers. Furthermore, we give some inequalities including the binomial coefficients, the Stirling numbers of the second kind, the Catalan numbers and the Bernoulli numbers of negative order. Finally, we give further remarks and observations for not only on spline curves and the Fubini numbers and polynomials, but also on approximate value of the Catalan numbers.
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1. INTRODUCTION

Generating functions with their PDEs, QDEs, and functional equations are effective techniques and methods in order to obtain formulas, identities, relations for special numbers and polynomials. By using generating functions and their functional equations techniques, some combinatorial sums including the Bernoulli numbers and polynomials of higher order, the Stirling numbers, the Catalan numbers, and the Fubini numbers are given. By using inequalities for the binomial coefficients, we give some inequalities related to the Catalan numbers and the Bernoulli numbers of negative order. Special functions, special numbers and special polynomials studied in this paper have the potential to be used in fields such as physics and engineering, especially in many branches of mathematics.

The notations and definitions helping us to find the main results of this paper are briefly given below.

Let $\mathbb{N}$, $\mathbb{Z}$, $\mathbb{Q}$, $\mathbb{R}$ and $\mathbb{C}$ denote the set of natural numbers, the set of integers, the set of rational numbers, the set of real numbers and the set of complex numbers, respectively. Furthermore

$$\mathbb{N}_0 = \mathbb{N} \cup \{0\}.$$

The binomials coefficients $\binom{n}{k}$ in usual are given by

$$\binom{n}{k} = \begin{cases} \frac{n!}{k!(n-k)!} & \text{for } 0 \leq k \leq n \\ 0 & \text{otherwise,} \end{cases}$$

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with $n, k \in \mathbb{N}_0$.

The falling factorial is given by

$$(x)_n = \begin{cases} x(x-1)(x-2)\ldots(x-n+1) & \text{if} \quad n \in \mathbb{N} \\ 1 & \text{if} \quad n = 0, \end{cases}$$

where $x \in \mathbb{R}$.

The Bernoulli numbers and polynomials of higher order are defined respectively by:

$$F_B(t,n) = \left(\frac{t}{e^t - 1}\right)^n = \sum_{k=0}^{\infty} B_k^{(n)} \frac{t^k}{k!} \tag{1}$$

and

$$G_B(t,x,n) = \left(\frac{t}{e^t - 1}\right)^n e^{tx} = \sum_{k=0}^{\infty} B_k^{(n)}(x) \frac{t^k}{k!}, \tag{2}$$

where $n \in \mathbb{N}_0$. By using (1) and (2), we have

$$B_k^{(n)}(x) = \sum_{v=0}^{k} \binom{k}{v} x^v B_{k-v}^{(n)}, \tag{3}$$

where $n, k \in \mathbb{N}_0$ (*cf.* [2, 5, 9–11, 20, 26, 32, 36, 37]).

Using (1), we also have the following well-known recurrence relation for the numbers $B_k^{(n)}$:

$$B_k^{(n)} = \left(1 - \frac{k}{n-1}\right) B_k^{(n-1)} - k B_{k-1}^{(n-1)}, \tag{4}$$

where $n, k \in \mathbb{N} \setminus \{1\}$ and

$$B_k^{(k+1)} = (-1)^k k!, \tag{5}$$

(*cf.* [2]).

Substituting $n = 1$ into (1) and (2), one have the generating functions for the Bernoulli numbers and polynomials.

The Bernoulli numbers of negative order, $B_k^{(-m)}$, are defined by

$$F_{BN}(t,m) = \left(\frac{e^t - 1}{t}\right)^m = \sum_{k=0}^{\infty} B_k^{(-m)} \frac{t^k}{k!}, \tag{6}$$

where $m \in \mathbb{N}$ (*cf.* [5, 9, 10, 20, 26, 32, 36, 37]).

By using (1) and (6), we have

$$\sum_{j=0}^{k} \binom{k}{j} B_j^{(-n)} B_{k-j}^{(n)} = 0, \tag{7}$$

where $n, k \in \mathbb{N}$ (for details, see also [13, 26, 32, 36, 37]).

A computation formula for the numbers $B_k^{(-n)}$ is given as follows:

$$B_k^{(-n)} = \frac{\binom{k+n}{n}^{-1}}{n!} \sum_{j=0}^{k+n-1} \sum_{v=0}^{j} (-1)^v \binom{j}{k} \binom{k+n+1}{v} (j+1-v)^{n+k},$$

($cf.$ [13]).

The Stirling numbers of the second kind, $S_2(n,k)$, are defined by

$$F_S(t,k) = \frac{\left(e^t-1\right)^k}{k!} = \sum_{n=0}^{\infty} S_2(n,k)\frac{t^n}{n!}, \tag{8}$$

and

$$x^n = \sum_{k=0}^{n} S_2(n,k)(x)_k, \tag{9}$$

where $k \in \mathbb{N}_0$ ($cf.$ [5, 9, 10, 20, 26, 32, 36, 37]).

By using (8), we have

$$S_2(n,k) = \frac{1}{k!}\sum_{j=0}^{k}(-1)^{k-j}\binom{k}{j}j^n, \tag{10}$$

where $n,k \in \mathbb{N}_0$. If $k > n$ or $k < 0$, we have $\binom{n}{k} = 0$, so

$$S_2(n,k) = 0,$$

($cf.$ [5, 9, 10, 20, 26, 32, 36, 37]).

By using (10), few values of the numbers $S_2(n,k)$ are given by the following table:

$n\backslash k$	0	1	2	3	4	5
0	1	0	0	0	0	0
1	0	1	0	0	0	0
2	0	1	1	0	0	0
3	0	1	3	1	0	0
4	0	1	7	6	1	0
5	0	1	15	25	10	1

($cf.$ [1–26, 28–37]).

By using (6) and (8), we also have the following well-known identity:

$$S_2\left(k+n,n\right) = \binom{k+n}{n}B_k^{(-n)}, \tag{11}$$

where $n \in \mathbb{N}$ ($cf.$ [4, 5, 26, 28, 32, 36, 37]).

Substituting $k = n$ into (11), we have

$$S_2\left(2n,n\right) = \binom{2n}{n}B_n^{(-n)}, \tag{12}$$

(for details, see also [26, 32, 36, 37]).

Using the equation (12) with the values of the numbers $S_2(2n,n)$ given in the table above, several values of the numbers $B_n^{(-n)}$ are calculated as follows:

$$\begin{aligned} B_1^{(-1)} &= \frac{1}{2}, \\ B_2^{(-2)} &= \frac{7}{6}, \\ B_3^{(-3)} &= \frac{9}{2}, \\ B_4^{(-4)} &= \frac{243}{10}, \\ B_5^{(-5)} &= \frac{6075}{36}, \end{aligned}$$

and so on (*cf.* [26]).

By using (1) and (8), we have another well-known identity as follows:

$$\sum_{j=0}^{k} \binom{k}{j} S_2(j,n)\, B_{k-j}^{(n)} = 0, \tag{13}$$

(*cf.* [26]).

The Catalan numbers, C_n, are defined by

$$C_n = \frac{1}{n+1}\binom{2n}{n}, \tag{14}$$

for $n \in \mathbb{N}$, and various representations of these numbers are given as follows:

$$C_n = \frac{1}{n+1}\sum_{k=0}^{n}\binom{n}{k}^2,$$

$$C_n = \frac{(2n)!}{(n+1)!n!},$$

$$C_n = \frac{1}{(n+1)!}\prod_{j=1}^{n}(4j-2),$$

$$C_n = \prod_{j=2}^{n}\frac{n+j}{j},$$

and these numbers have a recurrence relation given by

$$\frac{C_{n+1}}{C_n} = \frac{2(2n+1)}{n+2}. \tag{15}$$

(*cf.* [5–9, 13, 14, 18, 21–23, 25, 29, 31, 33, 37]).

By using (14), first eight values of the numbers C_n are given as follows:

$$\begin{aligned} C_0 &= 1, \quad C_1 = 1, \quad C_2 = 2, \quad C_3 = 5, \quad C_4 = 14, \\ C_5 &= 42, \quad C_6 = 132, \quad C_7 = 429, \end{aligned}$$

and so on (*cf.* [5–9, 13, 14, 18, 21–23, 25, 29, 31, 33, 37]).

By combining (12) with (14) yields the relation among the Catalan numbers, the Stirling numbers of the second kind, and the Bernoulli numbers of negative order given by

$$S_2\left(2n, n\right) = (n+1)C_n B_n^{(-n)}, \tag{16}$$

where $n \in \mathbb{N}$ (*cf.* [13]).

The Fubini numbers are defined by

$$F_w\left(t\right) = \frac{1}{2-e^t} = \sum_{n=0}^{\infty} w_g(n)\frac{t^n}{n!}, \tag{17}$$

(*cf.* [5, 12, 16, 17]).

In [5] and [15], we see that

$$w_g(n) = \sum_{j=0}^{n} j! S_2(n,j). \tag{18}$$

2. Identities and formulas related to the numbers $B_k^{(-m)}$, $S_2(n,k)$, and $w_g(n)$

In this section, by combining equations (6), (8), and (17), we give some formulas and identities including the numbers $B_k^{(-m)}$, $S_2(n,k)$, and $w_g(n)$.

By using (6), (8), and (17), the following functional equations are obtained:

$$\sum_{j=0}^{m-1} t^j F_{BN}(t,j) = F_w\left(t\right)\left(1 - t^m F_{BN}(t,m)\right), \tag{19}$$

$$\sum_{j=0}^{m-1} t^j F_{BN}(t,j) = F_w\left(t\right)\left(1 - m! F_S(t,m)\right), \tag{20}$$

$$\sum_{j=0}^{m-1} j! F_S(t,j) = F_w\left(t\right)\left(1 - t^m F_{BN}(t,m)\right), \tag{21}$$

and

$$\sum_{j=0}^{m-1} j! F_S(t,j) = F_w\left(t\right)\left(1 - m! F_S(t,m)\right). \tag{22}$$

Theorem 2.1. *Let $m, k \in \mathbb{N}$ with $k \geq m$. Then we have*

$$w_g(k) = (k)_m \sum_{v=0}^{k-m} \binom{k-m}{v} B_{k-m-v}^{(-m)} w_g(v) + \sum_{j=0}^{m-1} (k)_j B_{k-j}^{(-j)}. \tag{23}$$

Proof. Combining (19), with (6) and (17), we get

$$\begin{aligned}\sum_{j=0}^{m-1}\sum_{k=0}^{\infty}(k)_j B_{k-j}^{(-j)}\frac{t^k}{k!} &= \sum_{k=0}^{\infty} w_g(k)\frac{t^k}{k!}\\ &\quad -\sum_{k=0}^{\infty}(k)_m \sum_{v=0}^{k-m}\binom{k-m}{v} B_{k-m-v}^{(-m)} w_g(v)\frac{t^k}{k!}.\end{aligned}$$

Comparing the coefficients of $\frac{t^k}{k!}$ on both sides of the above equation, we arrive at the desired result. □

Theorem 2.2. *Let $k \in \mathbb{N}_0$ and $m \in \mathbb{N}$. Then we have*

$$w_g(k) = m!\sum_{v=0}^{k}\binom{k}{v} S_2(k-v,m) w_g(v) + \sum_{j=0}^{m-1}(k)_j B_{k-j}^{(-j)}. \tag{24}$$

Proof. Combining (20), with (6), (8) and (17), we get

$$\sum_{j=0}^{m-1}\sum_{k=0}^{\infty}(k)_j B_{k-j}^{(-j)}\frac{t^k}{k!} = \sum_{k=0}^{\infty} w_g(k)\frac{t^k}{k!} - m!\sum_{k=0}^{\infty}\sum_{v=0}^{k}\binom{k}{v} S_2(k-v,m) w_g(v)\frac{t^k}{k!}.$$

Comparing the coefficients of $\frac{t^k}{k!}$ on both sides of the above equation, we arrive at the desired result. □

Theorem 2.3. *Let $m, k \in \mathbb{N}$ with $k \geq m$. Then we have*

$$\sum_{j=0}^{m-1} j! S_2(k,j) = w_g(k) - (k)_m \sum_{v=0}^{k-m}\binom{k-m}{v} w_g(v) B_{k-m-v}^{(-m)}. \tag{25}$$

Proof. Combining (21), with (6), (8) and (17), we get

$$\sum_{k=0}^{\infty}\sum_{j=0}^{m-1} j! S_2(k,j)\frac{t^k}{k!} = \sum_{k=0}^{\infty} w_g(k)\frac{t^k}{k!} - \sum_{k=0}^{\infty}(k)_m \sum_{v=0}^{k-m}\binom{k-m}{v} w_g(v) B_{k-m-v}^{(-m)}\frac{t^k}{k!}.$$

Comparing the coefficients of $\frac{t^k}{k!}$ on both sides of the above equation, the proof of theorem is completed. □

Remark 2.1. *Substituting $k = m$ into* (25)*, we have* (18)*.*

Theorem 2.4. *Let $k \in \mathbb{N}_0$ and $m \in \mathbb{N}$. Then we have*

$$\sum_{j=0}^{m-1} j! S_2(k,j) = w_g(k) - m!\sum_{v=0}^{k}\binom{k}{v} S_2(k-v,m) w_g(v). \tag{26}$$

Proof. Combining (22) with (8) and (17), we get

$$\sum_{k=0}^{\infty}\sum_{j=0}^{m-1} j! S_2(k,j)\frac{t^k}{k!} = \sum_{k=0}^{\infty} w_g(k)\frac{t^k}{k!} - m!\sum_{k=0}^{\infty}\sum_{v=0}^{k}\binom{k}{v} S_2(k-v,m) w_g(v)\frac{t^k}{k!}.$$

Comparing the coefficients of $\frac{t^k}{k!}$ on both sides of the above equation, we arrive at the desired result. □

2.1. **Further remarks and observations on spline curves and the Fubini numbers and polynomials.** The Fubini numbers and polynomials have close relationships with the Euler numbers and the Bernoulli numbers. Therefore, investigating and studying the relations of these numbers in the future with spline curves including the Fubini numbers and the Fubini polynomials may be an interesting research subject.

The following questions can then be raised:

What kind of relationship can be established between spline curves including Euler (type) numbers and polynomials and Fubini numbers and polynomials?

Can the interpolation functions or series be constructed for the Fubini numbers and polynomials by using the equation between spline curves including Euler (type) numbers and polynomials and the interpolation functions?

3. Inequalities including binomial coefficients, the numbers $S_2(2n,n)$, C_n and $B_n^{(-n)}$

In this section, we investigate, study and survey on the upper bound and the lower bound for the Stirling numbers of the second kind, the Catalan numbers, the Bernoulli numbers of negative order, and also some inequalities including binomial coefficients.

Let m and n be integers with $m \geq 2$ and $n \geq 1$. Then, it is known that

$$\binom{mn}{n} \geq \frac{m^{m(n-1)+1}}{(m-1)^{(m-1)(n-1)}} n^{-\frac{1}{2}}, \tag{27}$$

(*cf.* [24]).

Substituting $m = 2$ into (27), we have

$$\binom{2n}{n} \geq 2^{2n-1} n^{-\frac{1}{2}}, \tag{28}$$

(*cf.* [13]).

Theorem 3.1 (*cf.* [13])**.** *Let $n \in \mathbb{N}$. Then we have*

$$S_2(2n,n) \geq 2^{2n-1} n^{-\frac{1}{2}} B_n^{(-n)}.$$

Comtet [5] gave the upper bound and the lower bound for the Stirling numbers of the second kind $S_2(n,k)$ as follows:

$$S_2(n,k) \leq \binom{n-1}{k-1} k^{n-k} \tag{29}$$

and

$$S_2(n,k) \geq k^{n-k}. \tag{30}$$

3.1. **Inequalities including Catalan numbers and the Bernoulli numbers of negative order.** Here, with the aid of inequalities for the binomial coefficients, we give some inequalities including the Catalan numbers and the Bernoulli numbers of negative order.

Let $k \in \mathbb{N}_0$ and $n \in \mathbb{N}$. Setting the following inequality:

$$\frac{2^{2k}}{nk+1} \leq \binom{2k}{k}. \tag{31}$$

Substituting $n = 1$ into (31), we have the following lemma:

Lemma 3.1 (*cf.* [35]). *Let $k \in \mathbb{N}_0$. Then we have*

$$\frac{2^{2k}}{k+1} \leq \binom{2k}{k}. \tag{32}$$

Proof. In order to prove the inequality given by (32), we use mathematical induction on k. This inequality is frequently used in this section. For detail about this inequality, see the work of Sominskii [35, Problem 49]. Now let's briefly prove, based on the result of Sominskii. For $k = 2$, inequality holds true. Suppose the proposition is true for k ($k \in \mathbb{N}$):

$$\frac{2^{2k}}{k+1} \leq \binom{2k}{k}.$$

In order to prove the proposition for $k + 1$, we need the following well known identity, which holds true for all $k \in \mathbb{N}$:

$$\frac{4(k+1)}{k+2} \leq \frac{(2k+1)(2k+2)}{(k+1)^2},$$

(*cf.* [35, p. 57]). Let's multiply both sides of the above inequality by $\frac{2^{2k}}{k+1}$, we have

$$\frac{4(k+1)}{k+2}\left(\frac{2^{2k}}{k+1}\right) \leq \frac{(2k+1)(2k+2)}{(k+1)^2}\left(\frac{2^{2k}}{k+1}\right).$$

From the above equation, we get

$$\frac{2^{2k+2}}{(k+1)+1} \leq \frac{(2k)!}{(k!)^2}\frac{(2k+1)(2k+2)}{(k+1)^2} = \binom{2(k+1)}{k+1}.$$

Thus, the proof of lemma is completed. □

Remark 3.1. *Substituting $n = 2$ into (31), we have another interesting inequality in same class with that of (32). In [27, Eq.-(2.8.1)], by considering the relation*

$$(1+1)^{2k},$$

Moll gave the following inequality

$$\binom{2k}{k} \geq \frac{2^{2k}}{2k+1} \tag{33}$$

where $k \in \mathbb{N}_0$.

Using (33), we get the following corollary for the Catalan numbers:

Corollary 3.1. *Let $k \in \mathbb{N}_0$. Then we have*

$$C_k \geq \frac{2^{2k}}{2k+1}.$$

Using (28) and (32), we get the following results for the Catalan numbers, respectively:

Corollary 3.2 (*cf.* [33])**.** *Let $k \in \mathbb{N}$. Then we have*

$$C_k \geq \frac{2^{2k-1}k^{-\frac{1}{2}}}{k+1}. \tag{34}$$

Corollary 3.3. *Let $k \in \mathbb{N}$. Then we have*

$$\frac{2^{2k}}{(k+1)^2} \leq C_k. \tag{35}$$

By combining (16) with (34), we get

$$S_2\left(2k,k\right) \geq 2^{2k-1}k^{-\frac{1}{2}}B_k^{(-k)}, \tag{36}$$

which is a special case of Theorem 3.1.

Using (30) and (36), we obtain the following corollary:

Corollary 3.4. *Let $k \in \mathbb{N}$. Then we have*

$$B_k^{(-k)} \geq \frac{k^{k+\frac{1}{2}}}{2^{2k-1}}.$$

By combining (16) with (35), we also obtain the following theorem:

Theorem 3.2. *Let $k \in \mathbb{N}$. Then we have*

$$\frac{2^{2k}B_k^{(-k)}}{k+1} \leq S_2\left(2k,k\right).$$

4. Further Remarks and Observations for an approximate value of the Catalan numbers

By using Stirling's approximation for factorials and approximate value of the Catalan numbers, many interesting applications for inequalities and approximate value of the special numbers and polynomials including binomial coefficients have recently been studied by some researchers (for details, see [5, 9, 23, 25, 37]).

By using the well-known the Stirling's approximation for factorials:

$$n! \approx \left(\frac{n}{e}\right)^n \sqrt{2n\pi},$$

an approximate value of the numbers C_n was given by Koshy [23, p. 111] as

$$\binom{2n}{n} \approx \frac{2^{2n}}{\sqrt{n\pi}},$$

that is

$$C_n \approx \frac{2^{2n}}{n\sqrt{n\pi}}.$$

Since

$$\lim_{n\to\infty} \frac{C_{n+1}}{C_n} = 4, \tag{37}$$

when n is sufficiently large, we have

$$C_{n+1} \approx 4C_n \tag{38}$$

(*cf.* [23, p. 111]).

By combining (16) with (15), we have

$$\frac{C_{n+1}}{C_n} = \frac{S_2\left(2n+2,n+1\right)(n+1)B_n^{(-n)}}{(n+2)S_2\left(2n,n\right)B_{n+1}^{(-n-1)}}.$$

Thus, we have

$$2(2n+1)S_2\left(2n,n\right)B_{n+1}^{(-n-1)} - (n+1)S_2\left(2n+2,n+1\right)B_n^{(-n)} = 0$$

and

$$\frac{S_2\left(2n,n\right)B_{n+1}^{(-n-1)}}{S_2\left(2n+2,n+1\right)B_n^{(-n)}} = \frac{n+1}{2(2n+1)}.$$

Hence, by (37), we have

$$\lim_{n\to\infty}\frac{S_2\left(2n,n\right)B_{n+1}^{(-n-1)}}{S_2\left(2n+2,n+1\right)B_n^{(-n)}} = \frac{1}{4}.$$

Thus, when n is sufficiently large, we get the following theorem:

Theorem 4.1.

$$S_2\left(2n+2,n+1\right)B_n^{(-n)} \approx 4S_2\left(2n,n\right)B_{n+1}^{(-n-1)}. \tag{39}$$

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