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IDENTITIES FOR SPECIAL NUMBERS AND POLYNOMIALS INVOLVING FIBONACCI-TYPE POLYNOMIALS AND CHEBYSHEV POLYNOMIALS

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Abstract. The main motivation of this paper is to give some identities and series representations for special numbers and polynomials involving the Fibonacci-type polynomials, the Chebyshev polynomials, the cosine-Euler polynomials, the sine-Euler polynomials, the cosine-Bernoulli polynomials, and the sine-Bernoulli polynomials. Moreover, observations and comments on the results of this paper are given.

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1. Introduction

Special numbers and polynomials have been used in many scientific studies recently. In the literature, we see that these are also used in the models used in solving real world problems. Likewise, generating functions for special numbers and polynomials give the same effectiveness on the scientific studies (*cf.* [1]-[25]). In this paper, by using generating functions of the special polynomials involving trigonometric functions and their functional equations, we give some relations, identities, and series representations for the Fibonacci-type polynomials, the Chebyshev polynomials, the Bernoulli numbers and polynomials, the Euler numbers and polynomials, the cosine-Euler polynomials, the sine-Euler polynomials, the cosine-Bernoulli polynomials, and the sine-Bernoulli polynomials.

We use the following notations and definitions throughout of this paper:

Let $\mathbb{N} = \{1, 2, 3, \ldots\}$, $\mathbb{N}_0 = \mathbb{N} \cup \{0\}$, $\mathbb{R}$ denote the set of real numbers and $\mathbb{C}$ denote the set of complex numbers. In addition, let $z = x + iy$, $\bar{z} = x - iy$ and $i^2 = -1$.

The Bernoulli polynomials $B_n(x)$ are defined by

$$F_{Bp}(t, x) = \frac{t}{e^t - 1} e^{xt} = \sum_{n=0}^{\infty} B_n(x) \frac{t^n}{n!}, \tag{1}$$

where $|t| < 2\pi$ (*cf.* [2]-[23]; and references therein).

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When $x = 0$, using (1), we have the Bernoulli numbers B_n, which are defined by

$$\frac{t}{e^t - 1} = \sum_{n=0}^{\infty} B_n \frac{t^n}{n!} \tag{2}$$

(c*f.* [2]-[23]; and references therein).

By combining (1) with (2), we have

$$B_n(x) = \sum_{j=0}^{n} \binom{n}{j} x^j B_{n-j} \tag{3}$$

(c*f.* [2]-[23]; and references therein).

The Euler polynomials $E_n(x)$ are defined by

$$F_{Ep}(t, x) = \frac{2}{e^t + 1} e^{xt} = \sum_{n=0}^{\infty} E_n(x) \frac{t^n}{n!}, \tag{4}$$

where $|t| < \pi$ (c*f.* [2]-[23]; and references therein).

When $x = 0$, using (4), we have the Euler numbers E_n, which are defined by

$$\frac{2}{e^t + 1} = \sum_{n=0}^{\infty} E_n \frac{t^n}{n!} \tag{5}$$

(c*f.* [2]-[23]; and references therein).

By combining (4) with (5), we have

$$E_n(x) = \sum_{j=0}^{n} \binom{n}{j} x^j E_{n-j} \tag{6}$$

(c*f.* [2]-[23]; and references therein).

The Fibonacci polynomials are defined by

$$G_F(t, x) = \frac{t}{1 - xt - t^2} = \sum_{n=0}^{\infty} F_n(x) t^n, \tag{7}$$

where $|t| < 1$ (*cf.* [3], [4], [5], [12], [21]).

The Chebyshev polynomials of the first kind $T_n(x)$ and second kind $U_n(x)$ are defined by means of the following generating functions, respectively:

$$\frac{1 - xt}{1 - 2xt + t^2} = \sum_{n=0}^{\infty} T_n(x) t^n, \tag{8}$$

and

$$\frac{1}{1 - 2xt + t^2} = \sum_{n=0}^{\infty} U_n(x) t^n \tag{9}$$

(*cf.* [1], [2], [6], [13]; and references therein).

Combining (7) with (8) and (9), we have

$$\sum_{n=0}^{\infty} T_n(ix) t^n = \sum_{n=0}^{\infty} F_{n+1}(2x) i^n t^n - \sum_{n=0}^{\infty} x F_n(2x) i^n t^n$$

and

$$\sum_{n=0}^{\infty} F_{n+1}(2x)\, i^n t^n = \sum_{n=0}^{\infty} U_n(ix)\, t^n,$$

where $i^2 = -1$. By using the above equations, we get the following well-known relation between the Fibonacci polynomials and the Chebyshev polynomials of the first kind and second kind, respectively:

$$T_n(ix) = i^n F_{n+1}(2x) - x i^n F_n(2x)$$

and

$$i^n F_{n+1}(2x) = U_n(ix)$$

for detail, see also [17].

For $x = \frac{1}{2}$, we have the following well-known identities involving the Fibonacci numbers and the Lucas numbers:

$$U_n\left(\frac{i}{2}\right) = i^n F_{n+1}$$

and

$$T_n\left(\frac{i}{2}\right) = \frac{i^n}{2} L_n,$$

where $F_0 = 0$, $F_1 = F_2 = 1$, $F_{n+2} = F_{n+1} + F_n$ and $L_0 = 2$, $L_1 = 1$, $L_{n+2} = L_{n+1} + L_n$ (*cf.* [25]).

The Fibonacci-type polynomials in two variables $\mathcal{G}_n(x, y; k, m, l)$ are defined by

$$\frac{1}{1 - x^k t - y^m t^{m+l}} = \sum_{n=0}^{\infty} \mathcal{G}_n(x, y; k, m, l)\, t^n, \tag{10}$$

where $k, m, l \in \mathbb{N}_0$ (*cf.* [17]). The explicit formula for the polynomials $\mathcal{G}_n(x, y; k, m, l)$ is given as follows:

$$\mathcal{G}_n(x, y; k, m, l) = \sum_{v=0}^{\left[\frac{n}{m+l}\right]} \binom{n - v(m + l - 1)}{v} y^{mv} x^{nk-mvk-lvk},$$

where $[b]$ denotes the largest integer less than or equal to b (*cf.* [17], [18]).

Substituting $y = 1$ and $k = m = l = 1$ into (10), we have

$$F_n(x) = \mathcal{G}_{n-1}(x, 1; 1, 1, 1)$$

for detail, see also [17].

The polynomials $C_n(x, y)$ and $S_n(x, y)$ are defined by means of the following generating functions, respectively:

$$F_C(t, x, y) = e^{xt} \cos(yt) = \sum_{n=0}^{\infty} C_n(x, y) \frac{t^n}{n!}, \tag{11}$$

and

$$F_S(t, x, y) = e^{xt} \sin(yt) = \sum_{n=0}^{\infty} S_n(x, y) \frac{t^n}{n!}. \tag{12}$$

Using (11) and (12), we have the following formulas for the polynomials $C_n(x,y)$ and $S_n(x,y)$, respectively:

$$C_n(x,y)=\sum_{j=0}^{\left[\frac{n}{2}\right]}(-1)^j\binom{n}{2j}x^{n-2j}y^{2j}$$

and

$$S_n(x,y)=\sum_{j=0}^{\left[\frac{n-1}{2}\right]}(-1)^j\binom{n}{2j+1}x^{n-2j-1}y^{2j+1}$$

(*cf.* [7]-[11], [14]-[16], [19], [22], [23]).

By using (8), (9), (11) and (12), the relations among the polynomials $C_n(x,y)$, the polynomials $S_n(x,y)$ and the Chebyshev polynomials are given as follows, respectively:

$$T_n(x)=C_n\left(x,\sqrt{1-x^2}\right) \tag{13}$$

and

$$U_{n-1}(x)=\frac{S_n\left(x,\sqrt{1-x^2}\right)}{\sqrt{1-x^2}} \tag{14}$$

(*cf.* [9], [10]).

The cosine-Bernoulli polynomials $B_n^{(C)}(x,y)$ and the sine-Bernoulli polynomials $B_n^{(S)}(x,y)$ are defined by means of the following generating functions, respectively:

$$F_{BC}(t,x,y)=\frac{t}{e^t-1}e^{xt}\cos(yt)=\sum_{n=0}^{\infty}B_n^{(C)}(x,y)\frac{t^n}{n!}, \tag{15}$$

and

$$F_{BS}(t,x,y)=\frac{t}{e^t-1}e^{xt}\sin(yt)=\sum_{n=0}^{\infty}B_n^{(S)}(x,y)\frac{t^n}{n!} \tag{16}$$

(*cf.* [11], [16]). Note that so-called cosine-Bernoulli polynomials and sine-Bernoulli polynomials were also studied with the name of the parametric type of Bernoulli polynomials in the literature, see for detail (*cf.* [11], [16], [22], [23]).

By using (11), (12), (15) and (16), we have the following well-known identities, respectively:

$$B_n^{(C)}(x+1,y)-B_n^{(C)}(x,y)=nC_{n-1}(x,y) \tag{17}$$

and

$$B_n^{(S)}(x+1,y)-B_n^{(S)}(x,y)=nS_{n-1}(x,y) \tag{18}$$

(*cf.* [11], [16]).

The cosine-Euler polynomials $E_n^{(C)}(x,y)$ and the sine-Euler polynomials $E_n^{(S)}(x,y)$ are defined by means of the following generating functions, respectively:

$$F_{EC}(t,x,y)=\frac{2}{e^t+1}e^{xt}\cos(yt)=\sum_{n=0}^{\infty}E_n^{(C)}(x,y)\frac{t^n}{n!}, \tag{19}$$

and

$$F_{ES}(t,x,y)=\frac{2}{e^t+1}e^{xt}\sin(yt)=\sum_{n=0}^{\infty}E_n^{(S)}(x,y)\frac{t^n}{n!} \tag{20}$$

(*cf.* [11], [15]). Note that so-called cosine-Euler polynomials and sine-Euler polynomials were also studied with the name of new type of Euler polynomials in the literature, see for detail (*cf.* [11], [15], [22], [23]).

By using (11), (12), (19) and (20), we get the following identities:

$$E_n^{(C)}(x+1,y)+E_n^{(C)}(x,y)=2C_n(x,y) \tag{21}$$

and

$$E_n^{(S)}(x+1,y)+E_n^{(S)}(x,y)=2S_n(x,y) \tag{22}$$

(*cf.* [11], [15]).

2. Identities and relations involving Fibonacci-type polynomials and some special numbers and polynomials

In this section, by using functional equations of the generating functions, we derive some formulas for the Euler numbers and polynomials, the Bernoulli numbers and polynomials. We also derive some identities related to the Fibonacci-type polynomials, the Chebyshev polynomials, the polynomials $C_n(x,y)$ and the polynomials $S_n(x,y)$. Moreover, we give many interesting relations among the cosine-Euler polynomials, the sine-Euler polynomials, the cosine-Bernoulli polynomials, the sine-Bernoulli polynomials and the Fibonacci-type polynomials.

By using (15) and (1), we obtain the following functional equation:

$$F_{Bp}(t,z)+F_{Bp}(t,\bar{z})=2F_{BC}(t,x,y). \tag{23}$$

From (23), we have

$$B_n(z)+B_n(\bar{z})=2B_n^{(C)}(x,y) \tag{24}$$

(*cf.* [11]).

By using (16) and (1), the following functional equation is obtained:

$$F_{Bp}(t,z)-F_{Bp}(t,\bar{z})=2iF_{BS}(t,x,y). \tag{25}$$

From (25), we have

$$B_n(z)-B_n(\bar{z})=2iB_n^{(S)}(x,y) \tag{26}$$

(*cf.* [11]).

After some calculations in the equations (24) and (26), we arrive at the following theorem:

Theorem 2.1. *Let $n\in\mathbb{N}_0$. Then we have*

$$B_n(z)=B_n^{(C)}(x,y)+iB_n^{(S)}(x,y) \tag{27}$$

and

$$B_n(\bar{z})=B_n^{(C)}(x,y)-iB_n^{(S)}(x,y). \tag{28}$$

Combining (27) and (28) with (3), we have the following corollary:

Corollary 2.2. *Let $n \in \mathbb{N}_0$. Then we have*

$$B_n^{(C)}(x,y) + iB_n^{(S)}(x,y) = \sum_{j=0}^{n} \binom{n}{j} (x+iy)^j B_{n-j}$$

and

$$B_n^{(C)}(x,y) - iB_n^{(S)}(x,y) = \sum_{j=0}^{n} \binom{n}{j} (x-iy)^j B_{n-j}.$$

By using (19) and (4), we obtain the following functional equation:

$$F_{Ep}(t,z) + F_{Ep}(t,\bar{z}) = 2F_{EC}(t,x,y). \tag{29}$$

From (29), we have

$$E_n(z) + E_n(\bar{z}) = 2E_n^{(C)}(x,y), \tag{30}$$

(*cf.* [11]).

By using (20) and (4), the following functional equation is obtained:

$$F_{Ep}(t,z) - F_{Ep}(t,\bar{z}) = 2iF_{ES}(t,x,y). \tag{31}$$

From (31), we have

$$E_n(z) - E_n(\bar{z}) = 2iE_n^{(S)}(x,y), \tag{32}$$

(*cf.* [11]).

After some calculations in the equations (30) and (32), we arrive at the following theorem:

Theorem 2.3. *Let $n \in \mathbb{N}_0$. Then we have*

$$E_n(z) = E_n^{(C)}(x,y) + iE_n^{(S)}(x,y) \tag{33}$$

and

$$E_n(\bar{z}) = E_n^{(C)}(x,y) - iE_n^{(S)}(x,y). \tag{34}$$

Combining (33) and (34) with (6), we get the following corollary:

Corollary 2.4. *Let $n \in \mathbb{N}_0$. Then we have*

$$E_n^{(C)}(x,y) + iE_n^{(S)}(x,y) = \sum_{j=0}^{n} \binom{n}{j} (x+iy)^j E_{n-j}$$

(cf. [11]), and

$$E_n^{(C)}(x,y) - iE_n^{(S)}(x,y) = \sum_{j=0}^{n} \binom{n}{j} (x-iy)^j E_{n-j}.$$

By using (8), (9) and (10), we get the relations among the Chebyshev polynomials of the first kind $T_n(x)$, the Chebyshev polynomials of the second kind $U_n(x)$ and the polynomials $\mathcal{G}_n(x,y;k,m,l)$ by the following corollaries:

Corollary 2.5. *Let $n \in \mathbb{N}$. Then we have*

$$T_n(x) = \mathcal{G}_n(2x,-1;1,1,1) - x\mathcal{G}_{n-1}(2x,-1;1,1,1). \tag{35}$$

Corollary 2.6. *Let $n \in \mathbb{N}_0$. Then we have*

$$U_n(x) = \mathcal{G}_n(2x, -1; 1, 1, 1). \tag{36}$$

Combining (35) with (13), we obtain the following corollary:

Corollary 2.7. *Let $n \in \mathbb{N}$. Then we have*

$$C_n\left(x, \sqrt{1-x^2}\right) = \mathcal{G}_n(2x, -1; 1, 1, 1) - x\mathcal{G}_{n-1}(2x, -1; 1, 1, 1). \tag{37}$$

Combining (36) with (14), we obtain the following corollary:

Corollary 2.8. *Let $n \in \mathbb{N}_0$. Then we have*

$$S_{n+1}\left(x, \sqrt{1-x^2}\right) = \sqrt{1-x^2}\mathcal{G}_n(2x, -1; 1, 1, 1). \tag{38}$$

By combining (17) with (37), we arrive at the following theorem:

Theorem 2.9. *Let $n \in \mathbb{N}$. Then we have*

$$\begin{aligned} &\mathcal{G}_n(2x, -1; 1, 1, 1) - x\mathcal{G}_{n-1}(2x, -1; 1, 1, 1) \\ &= \frac{B_{n+1}^{(C)}\left(x+1, \sqrt{1-x^2}\right) - B_{n+1}^{(C)}\left(x, \sqrt{1-x^2}\right)}{n+1}. \end{aligned}$$

By combining (18) with (38), we arrive at the following theorem:

Theorem 2.10. *Let $x \neq 1$ and $x \neq -1$. Let $n \in \mathbb{N}_0$. Then we have*

$$\mathcal{G}_n(2x, -1; 1, 1, 1) = \frac{B_{n+2}^{(S)}\left(x+1, \sqrt{1-x^2}\right) - B_{n+2}^{(S)}\left(x, \sqrt{1-x^2}\right)}{(n+2)\sqrt{1-x^2}}.$$

By combining (21) with (37), we arrive at the following theorem:

Theorem 2.11. *Let $n \in \mathbb{N}$. Then we have*

$$\begin{aligned} &\mathcal{G}_n(2x, -1; 1, 1, 1) - x\mathcal{G}_{n-1}(2x, -1; 1, 1, 1) \\ &= \frac{E_n^{(C)}\left(x+1, \sqrt{1-x^2}\right) + E_n^{(C)}\left(x, \sqrt{1-x^2}\right)}{2}. \end{aligned}$$

By combining (22) with (38), we arrive at the following theorem:

Theorem 2.12. *Let $x \neq 1$ and $x \neq -1$. Let $n \in \mathbb{N}_0$. Then we have*

$$\mathcal{G}_n(2x, -1; 1, 1, 1) = \frac{E_{n+1}^{(S)}\left(x+1, \sqrt{1-x^2}\right) + E_{n+1}^{(S)}\left(x, \sqrt{1-x^2}\right)}{2\sqrt{1-x^2}}. \tag{39}$$

3. Observations on infinite series representations including special numbers and polynomials

In this section, some infinite series representations including special numbers and polynomials are given, using the works of Ozdemir and Simsek [17] and also Koshy [12].

For $c > 1$, Ozdemir and Simsek [17] gave the following infinite series representation for the polynomials $\mathcal{G}_j(x, y; k, m, n)$:

$$\sum_{j=0}^{\infty} \frac{W_j(x, y; k, m, n)}{c^j} = \frac{c^m}{c^{m+n} - x^k c^{n+m-1} - y^m}, \tag{40}$$

where

$$W_j(x, y; k, m, n) = \mathcal{G}_{j-n}(x, y; k, m, n),$$

where $j \geq n$.

Substituting $c = 10$, $x = y = 1$, $k = m = n = 1$ into (40), we have the following infinite series representation for the Fibonacci numbers, which was proved by Stancliff [24], in 1953:

$$\sum_{n=0}^{\infty} \frac{F_n}{10^{n+1}} = \frac{1}{F_{11}}$$

(*cf.* [12], [17]).

Substituting $c = 2$, $y = 1$, $k = m = n = 1$ into (40), Ozdemir and Simsek [17] gave the following infinite series representation for the Fibonacci polynomials:

$$\sum_{j=0}^{\infty} \frac{F_j(x)}{2^j} = \frac{2}{3 - 2x}.$$

Substituting $x = 1$ into the above equation, we have

$$\sum_{j=0}^{\infty} \frac{F_j}{2^j} = F_3$$

(*cf.* [12], [17]).

For $c > 1$, modification of equation (40) is given as follows:

$$\sum_{j=0}^{\infty} \frac{\mathcal{G}_j(x, y; k, m, n)}{c^j} = \frac{c^{n+m}}{c^{n+m} - x^k c^{n+m-1} - y^m}. \tag{41}$$

Substituting $x = 2\alpha$, $y = -1$, $k = m = n = 1$ into (41) and using (39), we get

$$\begin{aligned} &\sum_{j=0}^{\infty} \frac{\mathcal{G}_j(2\alpha, -1; 1, 1, 1)}{c^j} \\ = &\sum_{j=0}^{\infty} \frac{E_{j+1}^{(S)}\left(\alpha + 1, \sqrt{1 - \alpha^2}\right) + E_{j+1}^{(S)}\left(\alpha, \sqrt{1 - \alpha^2}\right)}{2\sqrt{1 - \alpha^2}\, c^j}. \end{aligned}$$

Now, assuming that $|\alpha| < 1$, we obtain the following theorem:

Theorem 3.1. *Let $|\alpha| < 1$ and $c > 1$. Then we have*

$$\sum_{j=0}^{\infty} \frac{E_{j+1}^{(S)}\left(\alpha + 1, \sqrt{1 - \alpha^2}\right) + E_{j+1}^{(S)}\left(\alpha, \sqrt{1 - \alpha^2}\right)}{c^j} = \frac{2c^2\sqrt{1 - \alpha^2}}{c^2 - 2\alpha c + 1}.$$

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