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WIENER POLYNOMIALS AND WIENER INDICES OF THE TRANSFORMATION GRAPHS

S. B. CHANDRAKALA, B. SOORYANARAYANA, K. MANJULA, AND ISMAIL NACI CANGUL

Abstract. Topological graph indices are proven to be one of the most useful mathematical tools in the study of graphs, especially molecular graphs. Probably the most famous one is the Wiener index which was primarily used in determining the boiling points of the isomers of alkane molecules. The Wiener polynomial of G, denoted by $W(G;q)$ is defined by $W(G;q) = \sum_{\{u,v\}\subseteq V(G)} q^{d(u,v)}$ where $d(u,v)$ is the distance between the vertices u and v. The Wiener index $W(G)$, a distance based graph invariant, is given by $W(G) = \sum_{\{u,v\}\subseteq V(G)} d(u,v)$. In this paper, motivated by the above facts, the Wiener polynomial and Wiener index of transformation graph G^{+++} are determined when G is isomorphic to $P_n,\ C_n,\ K_{1,n},\ K_n,\ W_{n+1}$, the comb and finally the tadpole graph. We have also determined the Wiener polynomial and Wiener index of transformation graphs G^{xyz} of G when G isomorphic to P_n and C_n. Finally, the formula for $W(G^{++-};q)$ is given.

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Dedicated to Professor Gradimir V. Milovanović on the Occasion of his 70th Birthday

1. Introduction

A representation of an object giving information only about the number of elements composing it and their connectivity is named as a topological representation of an object. A topological representation of a molecule is called molecular graph. A molecular graph is a collection of points representing the atoms in the molecule and set of lines representing the covalent bonds. These points are named as vertices and the lines are named as edges in graph theoretical terminology.

A topological index is an invariant number calculated for a graph representing a molecule. In general, topological indices are either related to a vertex adjacency relationship in the graph G or to topological distances in G. The advantage of topological indices is that they may be used directly as simple numerical descriptors in comparison with physical, chemical or biological parameters of molecules in Quantitative Structure Property Relationships (QSPR) and Quantitative Structure Activity Relationships (QSAR). There are more than three thousands topological indices in literature. Some of them has very useful applications in chemistry and physics while some of them entertain us with their mathematical beauty and properties. One of the most widely known topological descriptor having applications is the Wiener index named after the chemist H. Wiener in 1947.

The Wiener index is a distance based topological invariant that has found extensive applications in chemistry. The Wiener index denoted by W was originally termed as the "path number" and is also known as the Wiener number (Plavsic et al. 1993). The Wiener index $W(G)$ of a connected graph G is defined as the sum of the distances between all unordered pairs of vertices of the graph, where the distance between two vertices is the length of the shortest path connecting them in G, [1]. That is

$$W(G) = \sum_{\{u,v\} \subseteq V(G)} d(u,v)$$

where $d(u,v)$ denotes the distance between vertices u and v in G. This index was succesfully used to compare the boiling points of several isomers of alkane molecules.

The Wiener polynomial is a generating function which was first defined by H. Hosoya [2] with this name, in honor of H. Wiener, but this is also known today as Hosoya polynomial, who extends this concept to capture the complete distribution of distances in graph. If q is a parameter, then the Wiener polynomial of G is

$$W(G;q) = \sum_{\{u,v\} \subseteq V(G)} q^{d(u,v)}.$$

Here the degree of the Wiener polynomial is equal to the diameter of G and the coefficient of q is equal to the number of the edges in G. Thus the Wiener polynomial helps us to compute the Wiener index by taking the sum of the product of the coefficient of $q^n (1 \leq n \leq diam(G))$ and the power of q. In 2007 [3], a formula for the Wiener polynomial of the i^{th} power graph was given and used to find the Wiener polynomials and to compute the Wiener indices of the k^{th} power graphs of paths and cycles. M. Rashti and H. Y. Azari [4], calculated the Wiener polynomials and Wiener indices of graph operations called join, cartesian product, composition, disjunction and symmetric difference of n graphs.

So in analogy with these, we have determined the Wiener polynomial and Wiener index of G^{+++} (total graph) when G is isomorphic to path, cycle, star, complete, wheel, comb and tadpole graphs. We also determined the Wiener polynomials and Wiener indices of P_n^{++-}, C_n^{++-} and gave the general formula for G^{++-}.

2. Transformation graphs G^{xyz}

In 2001, Wu and Meng [5] introduced some new graph transformations which generalized the concept of total graph.

Definition 2.1. *Let $G(V,E)$ be a graph on $n \geq 3$ vertices and x, y, z be three variables taking values $+$ or $-$. The transformation graph G^{xyz} is a* *graph having* $V(G) \cup E(G)$ *as the vertex set, and for* $\alpha, \beta \in V(G) \cup E(G)$, α *and* β *are adjacent in* G^{xyz} *if and only if*

- $\alpha, \beta \in V(G)$, α *and* β *are adjacent in* G *if* $x = +$ *and* α *and* β *are not adjacent in* G *if* $x = -$.
- $\alpha, \beta \in E(G)$, α *and* β *are adjacent in* G *if* $y = +$ *and* α *and* β *are not adjacent in* G *if* $y = -$.
- $\alpha \in V(G)$ *and* $\beta \in E(G)$, α *and* β *are adjacent in* G *if* $z = +$ *and* α *and* β *are not adjacent in* G *if* $z = -$.

Since there are eight distinct 3-permutations of $\{+, -\}$, we obtain eight graphical transformations of G. For a given graph G, the graphs G^{+++}, G^{++-}, G^{+-+}, G^{-++} and their complements G^{---}, G^{--+}, G^{-+-}, G^{+--} are the eight transformation graphs. The transformation graphs were investigated in [5, 6, 7, 8].

For convenience, the transformation graph is partitioned into S_x, S_y and S_z as the edge-induced subgraphs. The edge-set of each of them is respectively determined by x, y and z of the permutation of xyz.
$S_x(G) \cong G$ when $x = +$; $S_x(G) \cong \overline{G}$ when $x = -$; $S_y(G) \cong L(G)$ when $y = +$; $S_y(G) \cong \overline{L(G)}$ when $y = -$; when $z = +$, then $\alpha,\ \beta \in V(G^{xyz})$ are adjacent in $S_z(G)$ if they are incident with each other in G. When $z = -$, $\alpha,\ \beta$ are adjacent in $S_z(G)$ if they are not incident in G.

3. Wiener polynomials and Wiener indices of P_n^{+++} and C_n^{+++}

Lemma 3.1. *For any connected graph* G,

$$W(G^{+++}; q) = W(G; q) + W(L(G); q) + \sum_{u \in V(G),\ v \in E(G)} q^{d(u,v)}.$$

Proof. In G^{+++}, $S_x = G$ and $S_y = L(G)$. The length of the paths that includes edges of S_y and S_z between any two vertices u, v of $V(G)$ is greater than $d_G(u,v)$ and therefore $d_{G^{+++}}(u,v) = d_G(u,v)$. Similarly, if u, v are two edges of G, since, the length of the paths between u and v, that includes edges of S_x and S_z is greater than $d_{L(G)}(u,v)$, $d_{G^{+++}}(u,v) = d_{L(G)}(u,v)$. Therefore

$$\begin{aligned} W(G^{+++}; q) &= \textstyle\sum_{\{u,v\} \subseteq V(G)} q^{d(u,v)} + \sum_{\{u,v\} \subseteq E(G)} q^{d(u,v)} + \sum_{u \in V(G), v \in E(G)} q^{d(u,v)} \\ &= W(G; q) + W(L(G); q) + \textstyle\sum_{u \in V(G),\ v \in E(G)} q^{d(u,v)}. \end{aligned}$$

□

Theorem 3.2. *For any positive integer* $n \geq 3$, *and a path* P_n, *the Wiener polynomial of the transformation graph* P_n^{+++} *is* $W(P_n^{+++}; q) = \sum_{k=1}^{n-1} (4n - 4k - 1) q^k$ *and its Wiener index is* $W(P_n^{+++}) = \frac{(n-1)n(4n+1)}{6}$.

Proof. Consider the path $P_n :\ v_1 - v_2 - v_3 - \ldots - v_n$, $(n \geq 3)$. Let $e_i = v_i v_{i+1}$, $(1 \leq i \leq n-1)$ be the edges of P_n. If q is a parameter, then from Lemma 3.1, the Wiener polynomial of P_n^{+++} is

$$W(P_n^{+++}; q) = W(P_n; q) + W(L(P_n)); q) + \sum_{u \in V(P_n), v \in E(P_n)} q^{d(u,v)}.$$

In [1], Sagan et al. have obtained $W(P_n;q) = \sum_{k=1}^{n-1}(n-k)q^k$ and since $L(P_n) = P_{n-1}$, $W(L(P_n;q)) = \sum_{k=1}^{n-2}(n-1-k)q^k$. It is evident from the structure of P_n^{+++} that for $u \in V(P_n)$, $v \in E(P_n)$, $1 \le d_{P_n^{+++}}(u,v) \le n-1$. We denote $d_{P_n^{+++}}(u,v) = k$ for $1 \le k \le n-1$. In P_n^{+++}, for each k, there are $2(n-k)$ vertex pairs, of which $(n-k)$ are of the form $\{e_i, v_j\}$ and the other $(n-k)$ are of the form $\{e_i, v_l\}$ where $1 \le i \le n-1,\ j = i+k \le n$ and $l = i+1-k \le n$. Thus $\sum_{u\in V(P_n),v\in E(P_n)} q^{d(u,v)} = 2\sum_{k=1}^{n-1}(n-k)q^k$. Therefore

$$\begin{aligned} W(P_n^{+++};q) &= \sum_{k=1}^{n-1}(n-k)q^k + \sum_{k=1}^{n-2}(n-1-k)q^k + 2\sum_{k=1}^{n-1}(n-k)q^k \\ &= \sum_{k=1}^{n-1}(n-k)q^k + \sum_{k=1}^{n-1}(n-1-k)q^k + 2\sum_{k=1}^{n-1}(n-k)q^k \\ &= \sum_{k=1}^{n-1}(4n-4k-1)q^k. \end{aligned}$$

Hence

$$\begin{aligned} W(P_n^{+++}) &= \sum_{k=1}^{n-1}(4n-4k-1)k = \sum_{k=1}^{n-1}(4n-1)k - \sum_{k=1}^{n-1}4k^2 \\ &= (4n-1)\tfrac{(n-1)n}{2} - 4\tfrac{(n-2)n(2n-1)}{6} \\ &= \tfrac{(n-1)n(4n+1)}{6}. \end{aligned}$$

□

Theorem 3.3. *Let $n \ge 3$ be an integer. Let C_n denote the cycle graph with n vertices. The Wiener polynomial of the graph C_n^{+++} is*

$$W(C_n^{+++};q) = \left(\sum_{k=1}^{\lfloor \frac{n}{2} \rfloor} 4nq^k\right) + (-1)^{n+1}nq^{\lceil \frac{n}{2} \rceil}$$

and its Wiener index is $W(C_n^{+++}) = \frac{n^2(n+1)}{2}$.

Proof. Consider the cycle $C_n : v_1 - v_2 - v_3 - \ldots - v_n - v_1,\ (n \ge 3)$. Let $e_i = v_iv_{i+1},\ (1 \le i \le n-1)$ and $e_n = v_nv_1$ be the edges of C_n. The Wiener polynomial of C_n^{+++} is

$$W(C_n^{+++};q) = W(C_n;q) + W(L(C_n);q) + \sum_{u\in V(C_n),\ v\in E(C_n)} q^{d(u,v)}.$$

We have the following cases:

Case 1: n is even. In [1], Sagan et. al. have obtained $W(C_n;q) = \sum_{k=1}^{\frac{n}{2}-1}(nq^k) + \frac{n}{2}q^{\frac{n}{2}}$. It is evident from the structure of C_n^{+++}, for $u \in V(C_n),\ v \in E(C_n),\ d_{C_n^{+++}}(u,v) = k$, where $1 \le k \le \frac{n}{2}$. For each $i,\ (1 \le i \le n)$, the vertex v_i is at distance k from the vertices e_j and e_l where $j = i+k-1\ (mod\ n),\ l = i+n-k\ (mod\ n)$. For each k, there are $2n$ pairs of vertices in C_n^{+++}. Hence

$$\sum_{u\in V(C_n),v\in E(C_n)} q^{d(u,v)} = \sum_{k=1}^{\frac{n}{2}} 2nq^k.$$

Therefore

$$\begin{aligned} W(C_n^{+++};q) &= 2\left(\sum_{k=1}^{\frac{n}{2}-1} nq^k + \tfrac{n}{2}q^{\frac{n}{2}}\right) + \sum_{k=1}^{\frac{n}{2}} 2nq^k \\ &= 2\left(\sum_{k=1}^{\frac{n}{2}} nq^k - nq^{\frac{n}{2}} + \tfrac{n}{2}q^{\frac{n}{2}}\right) + \sum_{k=1}^{\frac{n}{2}} 2nq^k \\ &= \sum_{k=1}^{\frac{n}{2}} 4nq^k - nq^{\frac{n}{2}} \\ &= \sum_{k=1}^{\frac{n}{2}} 4nk - n\tfrac{n}{2} = 4n\frac{\frac{n}{2}(\frac{n}{2}+1)}{2} - \frac{n^2}{2} \\ &= \frac{n^2(n+1)}{2}. \end{aligned}$$

Case 2: n is odd. In [1], Sagan et. al. noted that $W(C_n;q) = \sum_{k=1}^{(n-1)/2} nq^k$. For $u \in V(C_n)$, $v \in E(C_n)$, $d(u,v) = k$ in C_n^{+++} where $1 \le k \le (n-1)/2$. For each i, $(1 \le i \le n-1)$ and k, the vertex v_i is at distance k from the vertices e_j and e_l where $j = i+k-1\ (mod\ n)$, $l = i+n-k\ (mod\ n)$. For each k, there are $2n$ pairs of vertices in C_n^{+++}. Also the vertex v_i is at distance $k = (n+1)/2$ from the vertex e_m, where $m = i+k-1\ (mod\ n)$ and there are n pairs of vertices in C_n^{+++}. Thus,

$$\sum_{u\in V(C_n),\ v\in E(C_n)} q^{d(u,v)} = \sum_{k=1}^{(n-1)/2} 2nq^k + nq^{(n+1)/2}.$$

Therefore

$$\begin{aligned} W(C_n^{+++};q) &= 2\left(\sum_{k=1}^{(n-1)/2} nq^k\right) + \sum_{k=1}^{(n-1)/2} 2nq^k + nq^{(n+1)/2} \\ &= \sum_{k=1}^{(n-1)/2} 4nq^k + nq^{(n+1)/2} \\ &= \sum_{k=1}^{(n-1)/2} 4nk + n\frac{n+1}{2} \\ &= 4n\frac{\frac{n-1}{2}\left(\frac{n(n-1)}{2}+1\right)}{2} + \frac{n(n+1)}{2} \\ &= \frac{n^2(n+1)}{2} \\ &= \left(\sum_{k=1}^{\lfloor\frac{n}{2}\rfloor} 4nq^k\right) + (-1)^{n+1}nq^{\lceil\frac{n}{2}\rceil} \end{aligned}$$

and its Wiener index is $W(C_n^{+++}) = \frac{n^2(n+1)}{2}$. □

4. Wiener polynomials and Wiener indices of $K_{1,n}^{+++}$, K_n^{+++} and W_{n+1}^{+++}

Theorem 4.1. *For any positive integer $n \ge 3$, the Wiener polynomial of the transformation graph $K_{1,n}^{+++}$ is $W(K_{1,n}^{+++};q) = \left(\frac{n^2+5n}{2}\right)q^1 + \frac{3n(n-1)}{2}q^2$ and its Wiener index is $W(K_{1,n}^{+++}) = \frac{n(7n-1)}{2}$.*

Proof. Let v_0 be the central vertex and v_i, $(1 \leq i \leq n)$ be the pendant vertices of the star graph $K_{1,n}$ and let $e_i = v_0v_i$ represent the edges of $K_{1,n}$. Also $diam(K_{1,n}^{+++}) = 2$. Now the Wiener polynomial of $K_{1,n}^{+++}$ is

$$W(K_{1,n}^{+++}; q) = W(K_{1,n}; q) + W(L(K_{1,n}); q) + \sum_{u \in V(K_{1,n}),\ v \in E(K_{1,n})} q^{d(u,v)}.$$

In [1], Sagan et. al. gave that $W(K_{1,n}; q) = nq + \binom{n}{2} q^2$ and $W(L(K_{1,n}); q) = W(K_n; q) = \binom{n}{2} q$. For any edge $e_i \in E(K_{1,n})$, $d_{K_{1,n}^{+++}}(v_0, e_i) = 1$, $d_{K_{1,n}^{+++}}(v_i, e_i) = 1$ and $d_{K_{1,n}^{+++}}(v_j, e_i) = 2$, $(j \neq i)$. In S_z, there are $2n$ pairs of vertices having distance 1 and $n(n-1)$ pairs of vertices with distance 2. Thus

$$\sum_{u \in V,\ v \in E} q^{d(u,v)} = 2nq + n(n-1)q^2.$$

Therefore $W(K_{1,n}^{+++}; q) = nq + \binom{n}{2} q^2 + \binom{n}{2} q + 2nq + n(n-1)q^2 = \frac{n^2+5n}{2} q^1 + \frac{3n(n-1)}{2} q^2$ and its Wiener index is $W(K_{1,n}^{+++}) = \frac{n^2+5n}{2} + \frac{3n(n-1)}{2} 2 = \frac{n(7n-1)}{2}$. □

Theorem 4.2. *For any positive integer $n \geq 3$, the Wiener polynomial of the transformation graph K_n^{+++} is $W(K_n^{+++}; q) = \left(\frac{n(n^2-1)}{2}\right) q^1 + \left(\frac{n(n^2-1)(n-2)}{8}\right) q^2$ and its Wiener index is $W(K_n^{+++}) = \frac{n^2(n^2-1)}{4}$.*

Proof. For the complete graph, we have $diam(K_n^{+++}) = 2$. The Wiener polynomial of K_n^{+++} is

$$W(K_n^{+++}; q) = W(K_n; q) + W(L(K_n)); q) + \sum_{u \in V(K_n),\ v \in E(K_n)} q^{d(u,v)}.$$

We also have from [1] that $W(K_n; q) = \binom{n}{2} q$. For $u, v \in V(S_y)$, we know that $diam(L(K_n)) = 2$. Now in S_y, there are $\binom{n(n-1)/2}{2}$ vertex pairs and among these, $\frac{n(n-1)(n-2)}{2}$ pairs are adjacent. The distance between the remaining $\frac{n(n-1)(n-2(n-3))}{8}$ pairs of vertices is 2. Therefore, $W(S_y; q) = \frac{n(n-1)(n-2)}{2} q + \frac{\binom{n-2}{2}\binom{n}{2}}{2} q^2$. Now, let $u \in V(K_n)$ and $v \in E(K_n)$. In S_z, corresponding to each edge $e = uv \in K_n$, we have $d_{K_n^{+++}}(e, u) = 1$ and $d_{K_n^{+++}}(e, v) = 1$, hence there are $2\binom{n}{2}$ adjacent vertices and the distance between the remaining $\frac{n(n-1)(n-2)}{2}$ pairs is 2. Thus

$$\sum_{\{u,v\} \subseteq V(S_z)} q^{d(u,v)} = n(n-1)q + \frac{n(n-1)(n-2)}{2} q^2.$$

Therefore $W(K_n^{+++}; q) = \binom{n}{c} q + \frac{n(n-1)(n-2)}{2} q + \frac{\binom{n-2}{2}\binom{n}{2}}{2} q^2 + n(n-1)q + \frac{n(n-1)(n-2)}{2} q^2$, i.e., $W(K_n^{+++}; q) = \left(\frac{n(n^2-1)}{2}\right) q^1 + \left(\frac{n(n^2-1)(n-2)}{8}\right) q^2$ and hence its Wiener index is $W(K_n^{+++}; q) = \frac{n(n^2-1)}{2} + \left(\frac{n(n-1)(n-2)}{2}\right) \times 2 = \frac{n^2(n^2-1)}{4}$. □

Theorem 4.3. *For any positive integer $n \geq 7$, the Wiener polynomial of the transformation graph of wheel graph $W_{n+1} = C_n + K_1$ is*

$$W(W_{n+1}^{+++};q) = \frac{n^2+17n}{2}q + \frac{5n^2-n}{2}q^2 + \frac{3n^2-13n}{2}q^3$$

and its Wiener index is $W(W_{n+1}^{+++}) = 2n(5n-6)$.

Proof. Let v_0 be central vertex and $v_i (1 \leq i \leq n)$ be the vertices adjacent to v_0 in W_{n+1}. Let $e_i' = v_0 v_i$ represents the spokes of W_{n+1}, $e_i = v_i v_{i+1},\ (1 \leq i \leq n-1)$ and $e_n = v_n v_1$ be the hubs of the wheel. $diam(W_{n+1}^{+++}) = 3$. The Wiener polynomial of W_{n+1}^{+++} is

$$W(W_{n+1}^{+++};q) = W(S_x;q) + W(S_y;q) + \sum_{u \in V(W_{n+1}), v \in E(W_{n+1})} q^{d(u,v)}.$$

We have by Sagan et al. [1] that $W(S_x;q) = W(W_{n+1};q) = 2nq + \frac{n(n-3)}{2}q^2$. For $u, v \in V(S_y)$, we have the following possibilities:

- The n spokes of W_{n+1} are adjacent to each other, induces a complete graph K_n in S_y and there are $n(n-1)/2$ pairs of adjacent vertices. Each vertex $v_i \in W_{n+1}$ is at the same time the central vertex of a $K_{1,3}$. Each of these $K1,3$ induces a K_3 in S_y. Each K_3 corresponds to three pairs of adjacent vertices. Therefore in S_y, the number of pairs of adjacent vertices is $\frac{n(n-1)}{2} + 3n = \frac{n(n+5)}{2}$.
- In S_y, $d(e_i, e_j) = 2$ where $1 \leq i \leq n,\ j = i+2\ (mod\ n))$. Their number is n. For each $1 \leq i \leq n$, $d(e_i', e_j) = 2$ where $1 \leq j \leq n$, but $j \neq i,\ i-1\ (mod\ n))$ and the number of these pairs is $n(n-2)$.
- In S_y, for each $1 \leq i \leq n$, $d(e_i, e_j) = 3$ where $1 \leq j \leq n$, but $j \neq i+1,\ i+2,\ i-1,\ i-2\ (mod\ n)$ and these are $\frac{n(n-5)}{2}$ pairs. Thus $W(S_y;q) = \frac{n(n+5)}{2}q + n(n-1)q^2 + \frac{n(n-1)}{2}q^3$.

In S_z, we have the following possibilities

- $d(v_0, e_i') = 1$ and the distance between v_i and each of the three edges incident at v_i in W_{n+1} is 1. Total number of pairs of adjacent vertices is $n + 3n = 4n$.
- $d(v_0, e_i) = d(v_i, e_j) = d(v_i, e_k') = 2$ where $(1 \leq i \leq n)$, $(j = i+1, i-2(mod\ n))$, $(k \neq i,\ 1 \leq k \leq n)$. Number of such pairs of vertices are $n + 2n + n(n-1) = n(n+2)$.
- $d(v_i, e_j) = 3$ for each $(1 \leq i \leq n)$; $1 \leq j \leq n$ but $j \neq i,\ i+1,\ i-1,\ i-2\ (mod\ n)$ and the number of these pairs is $n(n-4)$.

Thus

$$\sum_{\{u,v\} \subseteq V(S_z)} q^{d(u,v)} = 4nq + n(n+2)q^2 + n(n-4)q^3.$$

Therefore for $n \geq 5$, we have

$$W(W_{n+1}^{+++};q) = \frac{n^2+17n}{2}q + \frac{5n^2-n}{2}q^2 + \frac{3n^2-13n}{2}q^3$$

and hence its Wiener index is

$$W(W_{n+1}^{+++}) = \frac{n(n+17)}{2} + \frac{n(5n-1)}{+}\frac{3n(3n-13)}{2}q^3 = 2n(5n-6).$$

For example, $W(W_5^{+++}; q) = 42q + 36q^2$ and its Wiener index is $W(W_5^{+++}) = 78$. □

5. Wiener index of G^{+++} when G is isomorphic to comb and tadpole graphs

Theorem 5.1. *The Wiener polynomial of the transformation graph of the comb graph $G \cong P_n \odot K_1$, $(n \geq 3)$ is*

$$W(G^{+++}; q) = q^n + 3q^{n+1} - (7n+3)q - 3nq^2 + \sum_{k=1}^{n}(16n - 16k + 12)q^k$$

and its Wiener index is $W(G^{+++}) = 2n(n+1)(4n+5)3 - 9n$.

Proof. Consider $G \cong P_n \odot K_1 (n \geq 3)$, a comb graph with $2n$ vertices. Let v_i, $(1 \leq i \leq n)$ be the vertices of P_n. Let v_i' $(1 \leq i \leq n)$ be the vertices adjacent to v_i, $(1 \leq i \leq n)$, respectively. Let $e_i = v_i v_{i+1}$, $(1 \leq i \leq n-1)$ and $e_i' = v_i v_i'$ denote the edges of G. Note that $diam(G) = n+1 = diam(G^{+++})$. The Wiener polynomial of G^{+++} is

$$W(G^{+++}; q) = W(G; q) + W(L(G); q) + \sum_{u \in V, v \in E} q^{d(u,v)}.$$

Let u and v be two vertices in G^{+++}. Then $d(u,v) = k$, $(1 \leq k \leq n+1)$.

In S_x, we have the following possibilities:

- $d(v_i, v_j) = k$ where $1 \leq k \leq n-1$ for each $1 \leq i \leq n-1$; $1 \leq j = i + k \leq n$.
- $d(v_i', v_j) = k$ where $1 \leq k \leq n$ for each $1 \leq i \leq n-1$, ; $(1 \leq j = i + k - 1 \leq n$. Similarly $d(v_i', v_l) = k$ where $2 \leq k \leq n$ for each $2 \leq i \leq n-1$; $1 \leq l = i - k + 1 \leq n-1$.
- $d(v_i', v_j') = k$ where $3 \leq k \leq n+1$ for each $1 \leq i \leq n-1$; $(1 \leq j = i + k - 2 \leq n$.

Hence

$$W(S_x; q) = q^{n+1} - (2n+1)q - nq^2 + \sum_{k=1}^{n}(4n - 4k + 4)q^k.$$

In S_y, we have the folllowing possibilities:

- $d(e_i, e_j) = k$ where $1 \leq k \leq n-2$ for each $1 \leq i \leq n-2$, $(2 \leq j = i + k \leq n-1)$.
- $d(e_i, e_j') = k$ where $1 \leq k \leq n-1$ for each $1 \leq i \leq n-1$, $(2 \leq j = i + k \leq n)$. Similarly $d(e_i, e_l) = k$ where $1 \leq k \leq n-1$ for each $1 \leq i \leq n-1$, $(1 \leq l = i - k + 1 \leq n-1)$.

- $d(e'_i, e'_j) = k$ where $2 \leq k \leq n$ for each $1 \leq i \leq n-1$, $(2 \leq j = i+k-1 \leq n)$.

Hence

$$W(S_y; q) = q^n - nq + \sum_{k=1}^{n}(4n-4k)q^k.$$

In S_z, we have the possibilities below:

- $d(v_i, e_j) = k$ where $1 \leq k \leq n-1$ for each $1 \leq i \leq n-1$, $(1 \leq j = i+k \leq n-1)$ and $d(v_i, e_l) = k$ where $1 \leq k \leq n-1$ for each $1 \leq i \leq n-1$, $(1 \leq l = i-k \leq n-1)$.

- $d(v_i, e'_j) = k$ where $1 \leq k \leq n$ for each $1 \leq i \leq n$, $(1 \leq j = i+k-1 \leq n)$ and $d(v'_i, e'_l) = k$ where $2 \leq k \leq n$ for each $2 \leq i \leq n$, $(1 \leq l = i-k+1 \leq n-1)$.

- $d(v'_i, e_j) = k$ where $2 \leq k \leq n+1$ for each $1 \leq i \leq n-1$, $(1 \leq j = i+k-1 \leq n-1)$ and $d(v'_i, e_l) = k$ where $2 \leq k \leq n$ for each $2 \leq i \leq n$, $1 \leq l = i-k+1 \leq n-2$.

- $d(v'_i, e'_j) = k$ where $1 \leq k \leq n+1$, but $k \neq 2$ for each $1 \leq i \leq n$, $(2 \leq j = i+k-1 \leq n)$ and $d(v'_i, e'_l) = k$ where $3 \leq k \leq n+1$ for each $2 \leq i \leq n$, $(1 \leq l = i-k+2 \leq n-1)$.

Hence

$$W(S_z; q) = 2q^{n+1} - (4n+2)q - 2nq^2 + \sum_{k=1}^{n}(8n-8k+8)q^k.$$

Therefore

$$W(G^{+++}; q) = q^n + 3q^{n+1} - (7n+3)q - 3nq^2 + \sum_{k=1}^{n}(16n-16k+12)q^k$$

and hence its Wiener index is $W(G^{+++}) = 2n(n+1)(4n+5)3 - 9n$. □

Theorem 5.2. *The Wiener index of the transformation graph of a tadpole graph is*

$$W(T^{+++}_{n,m}) = \begin{cases} \frac{n^2(n+1)}{2} + \frac{(m-1)m(4m+1)}{6} + (m-1)(6n + m(2n-1) - 4) & if\ n\ is\ even \\ \frac{n^2(n+1)}{2} + \frac{(m-1)m(4m+1)}{6} + \frac{1}{2}(m^2(4n-2) + 8m(n-1) - 13n + 9) & o/w \end{cases}$$

Proof. Let $C_n : v_1 - v_2 - v_3 - ... - v_n - v_1$ be a cycle on n vertices, $P_m : v'_1(= v_1) - v'_2 - v'_3 - ... - v'_m$ be a path on $m \geq 2$ vertices such that $e_i = v_iv_{i+1}(1 \leq i \leq n-1)$, $e_n = v_{n-1}v_n$ be the edges of C_n and $e'_j = v'_jv'_{j+1}$, $(1 \leq j \leq m-1)$ be the edges of P_m. So that the tadpole $T_{n,m} = C_n \cup P_m$ such that $V(C_n) \cap (VP_m) = \{v_1\}$. Now we have $V(T^{+++}_{n,m}) = V(T_{n,m}) \cup E(T_{n,m})$. Therefore $W(T^{+++}_{n,m}) = W(C^{+++}_n) + W(P^{+++}_m) + \sum d(u,v)$, where $u, v \neq v_1$ and $u \in V(C^{+++}_n)$, $V(P^{+++}_m)$. We have from Theorem 3.2 that $W(C^{+++}_n) =$

$\frac{n^2(n+1)}{2}$ and from Theorem 3.3 that $W(P_m^{+++}) = \frac{(m-1)m(4m+1)}{6}$. We can find

$$\sum_{u\in V(C_n^{+++})v\in(P_m^{+++})} d(u,v)$$

as below:

- When $u = v_i$ $(2 \le i \le n)$ and $v = v'_j$ $(2 \le j \le m)$, v_i $(2 \le i \le \lceil \frac{n}{2} \rceil)$ is at distance $k = j + i - 2$ to v'_j $(2 \le j \le m)$; v_i $(\lfloor \frac{n}{2} \rfloor + 2 \le i \le n)$ is at distance $k = j + n - i$ to v'_j $(2 \le j \le m)$. When n is even,

$$\sum_{\{u,v\}} d(u,v) = 2\sum_{i=2}^{\frac{n}{2}}\sum_{j=2}^{m}(i+j-2)+\sum_{j=2}^{m}(\frac{n}{2}+1+j-2) = \frac{1}{2}(m-1)m(n-1)+3n-4.$$

Similarly when n is odd,

$$\sum_{\{u,v\}} d(u,v) = 2\sum_{i=2}^{(n+1)/2}\sum_{j=2}^{m}(i+j-2) = \frac{1}{2}(m-1)(m+2)(n-1).$$

- When $u = e_i$ $(1 \le i \le n)$ and $v = e'_j$ $(1 \le j \le m-1)$, e_i $(1 \le i \le \lceil \frac{n}{2} \rceil)$ is at distance $k = j + i - 1$ from e'_j $(1 \le j \le m-1)$; e_i $(\lfloor \frac{n}{2} \rfloor + 2 \le i \le n)$ is at distance $k = n-i+j$ from e'_j $(1 \le j \le m-1)$.

When n is even,

$$\sum_{\{u,v\}} d(u,v) = 2\sum_{i=1}^{\frac{n}{2}}\sum_{j=1}^{m-1}(i+j-1) = \frac{1}{2}n(m^2+m-2).$$

When n is odd,

$$\sum_{\{u,v\}} d(u,v) = 2\sum_{i=1}^{\lfloor\frac{n}{2}\rfloor}\sum_{j=1}^{m-1}(i+j-1)+\sum_{j=2}^{m-1}(\lceil\frac{n}{2}\rceil+j-1) = \frac{1}{2}(m^2n+m(2n-3)-4n+2).$$

- When $u = v_i$ $(2 \le i \le n)$ and $v = e'_j$ $(1 \le j \le m-1)$, v_i $(2 \le i \le \lceil \frac{n}{2} \rceil)$ is at distance $k = j + i - 1$ from e'_j, $(1 \le j \le m-1)$; v_i $(\lfloor \frac{n}{2} \rfloor + 2 \le i \le n)$ is at distance $k = j + n - i + 1$ from e'_j, $(1 \le j \le m-1)$.

When n is even,

$$\sum_{\{u,v\}} d(u,v) = 2\sum_{i=2}^{\frac{n}{2}}\sum_{j=1}^{m-1}(i+j-1)+\sum_{j=1}^{m-1}(\frac{n}{2}+1+j-1) = \frac{1}{2}(m-1)(3n+m(n-1)-4.$$

When n is odd,

$$\sum_{\{u,v\}} d(u,v) = 2\sum_{i=2}^{((n+1)/2)}\sum_{j=1}^{m-1}(i+j-1) = \frac{1}{2}(n-1)(m^2+m-2).$$

- When $u = e_i$ $(1 \le i \le n)$ and $v = v'_j$ $(2 \le j \le m)$, e_i $(1 \le i \le \lceil \frac{n}{2} \rceil)$ is at distance $k = j+i-1$ from v'_j $(2 \le j \le m)$; v_i $(\lfloor \frac{n}{2} \rfloor + 1 \le i \le n)$

is at distance $k = j + n - i$ from v'_j $(2 \le j \le m)$.

When n is even,

$$\sum_{\{u,v\}} d(u,v) = 2\sum_{i=1}^{\frac{n}{2}}\sum_{j=2}^{m}(i+j-1) = \frac{1}{2}n(m^2+3m-4).$$

When n is odd,

$$\sum_{\{u,v\}} d(u,v) = 2\sum_{i=1}^{(n-1)/2}\sum_{j=2}^{m}(i+j-1)+\sum_{j=2}^{m}(\lfloor\frac{n}{2}\rfloor+j) = \frac{1}{2}(m-1)(n(m+5)-3.$$

Therefore

$$\sum_{u\in V(C_n^{+++})v\in(P_m^{+++})} d(u,v) = \begin{cases} (m-1)(6n+m(2n-1)-4) & \textit{when } n \textit{ is even} \\ +\frac{1}{2}(m^2(4n-2)+8m(n-1)-13n+9 & \textit{when } n \textit{ is odd} \end{cases}$$

and therefore we get

$$W(T_{n,m}^{+++}) = \begin{cases} \frac{n^2(n+1)}{2} + \frac{(m-1)m(4m+1)}{6} + (m-1)(6n+m(2n-1)-4) & \textit{when } n \textit{ is even} \\ \frac{n^2(n+1)}{2} + \frac{(m-1)m(4m+1)}{6} + \frac{1}{2}(m^2(4n-2)+8m(n-1)-13n+9 & o/w. \end{cases}$$

□

6. Wiener polynomials and Wiener indices of the transformation graphs of path and cycle graphs

Remark 6.1. ***(i)*** *$diam(P_n^{++-}) = diam(P_n^{--+}) = diam(P_n^{---}) = diam(P_n^{+--}) = diam(P_n^{-+-}) = 2$.*

(ii) *$diam(P_n^{+-+}) = diam(P_n^{-++}) = 3$.*

Remark 6.2. ***(i)*** *$diam(C_n^{++-}) = diam(C_n^{--+}) = diam(C_n^{---}) = diam(C_n^{+--}) = diam(C_n^{-+-}) = 2$.*

(ii) *$diam(C_n^{+-+}) = diam(C_n^{-++}) = 3$.*

Let $N_G^k = \{\{u,v\}|u,v \in V(G) \textit{ and } d(u,v) = k\}$ be the set of vertex pairs with distance between each pair is k.

Theorem 6.1. *For any positive integer $n \ge 4$, the Wiener polynomial of the transformation graphs of the path P_n are*

(i) *$W(P_n^{++-};q) = a_1q + a_2q^2$; $W(P_n^{--+};q) = a_2q + a_1q^2$ where $a_1 = n^2 - n - 1 = a_2 = a_1 - (n-3)$.*

(ii) *$W(P_n^{-++};q) = a_1q + a_2q^2 + a_3q^3$; $W(P_n^{+--};q) = (a_2 + a_3)q + a_1q^2$ where $a_1 = n(n+1)/2$ $a_2 = n^2 - n - 2$ $a_3 = (n-4)(n-3)/2$.*

(iii) *$W(P_n^{-++};q) = a_1q + a_2q^2 + a_3q^3$; $W(P_n^{+--};q) = (a_2 + a_3)q + a_1q^2$ where $a_1 = (n^2 + 3n - 6)/2$; $a_2 = n^2 - n - 2$; $a_3 = (n-4)(n-3)/2$.*

(iv) $W(P_n^{---};q) = (n-2)(2n-3)q + (4n-5)q^2$.

Hence their Wiener indices are related as below: $W(P_n^{++-}) = W(P_n^{--+}) - k$; $W(P_n^{-+-}) = W(P_n^{+--}) - k$; $W(P_n^{-++}) = W(P_n^{+-+}) - 2k$; $W(P_n^{---}) = 2n^2 - n - 4$ *where* $k = n-3$.

Proof. **(i)** We have $diam(P_n^{++-}) = 2$. Thus any two vertices in P_n^{++-} are either at distance 1 or at distance 2. The number of pairs of vertices at distance 1 in P_n^{++-} is $|E(P_n^{++-})|$. Therefore $|E(P_n^{++-})| = |E(S_x)| + |E(S_y)| + |E(S_z)| = |E(P_n)| + |E(P_{n-1})| + |E(S_z)| = (n-1) + (n-2) + (n-1)(n-2) = n^2 - n - 1 = a_1$. Since $diam(P_n^{++-}) = 2$, the number of vertex pairs u and v in P_n^{++-} such that $d(u,v) = 2$ is ${}^{(2n-1)}C_2 - |E(P_n^{++-})| = n^2 - 2n + 2 = a_1 - (n-3) = a_2$. Therefore $W(P_n^{++-};q) = a_1 q + a_2 q^2$.

(ii) We have $diam(P_n^{--+}) = 2$ and P_n^{--+} is the complement of P_n^{++-}. Then the vertices which are adjacent in P_n^{++-} are at distance 2 from each other in P_n^{--+} and the vertices which are not adjacent in P_n^{++-} are adjacent in P_n^{--+}. Hence the number of pairs of vertices in P_n^{--+} which are adjacent is a_2 and at distance 2 from each other is a_1. Therefore $W(P_n^{--+};q) = a_2 q + q_1 q^2$.

(iii) We have $diam(P_n^{+-+}) = 3$. The vertex pairs which are adjacent in P_n^{+-+} are

- $N_{S_x}^1 = \{\{v_i, v_{i+1}\} : 1 \le i \le n-1\}$ and $|N_{S_x}^1| = n-1$.
- $N_{S_y}^1 = \{\{e_i, e_j\} : 1 \le i \le n-3$ and $j = i+2, i+3, \ldots, (n-1)\}$ and $|N_{S_y}^1| = \sum_{i=1}^{n-3}(n-2-i) = (n-2)(n-3)/2$.
- $N_{S_z}^1 = \{\{e_i, v_i\} : 1 \le i \le n-1\}$ and $|N_{S_z}^1| = 2(n-1)$.

Thus the number of vertex pairs u and v in P_n^{+-+} such that $d(u,v) = 1$ is $|N_{S_x}^1| + |N_{S_y}^1| + |N_{S_z}^1| = n(n+1)/2 = a_1$. Similarly the vertex pairs u and v in P_n^{+-+} such that $d(u,v) = 2$ are

- $N_{S_x}^2 = \{\{v_i, v_{i+2}\}, 1 \le i \le n-2\}$ and $|N_{S_x}^2| = n-2$.
- $N_{S_y}^2 = \{\{e_i, e_{i+1}\}, 1 \le i \le n-2\}$ and $|N_{S_y}^2| = n-2$.
- $N_{S_z}^2 = \{\{e_i, v_j\} : 1 \le i \le n-1$ and $1 \le j \le n$ but $j \ne i,\ i+1\}$ and $|N_{S_z}^2| = (n-1)(n-2)$.

Thus the number of vertex pairs u and v in P_n^{+-+} such that $d(u,v) = 2$ is $|N_{S_x}^2| + |N_{S_y}^2| + |N_{S_z}^2| = n^2 - n - 2 = a_2$.

The vertex pairs u and v in P_n^{+-+} such that $d(u,v) = 3$ are in the set $N_{S_x}^3 = \{\{v_i, v_j\} : 1 \le i \le n-3$ and $j = i+3,\ i+4,\ n\}$ and $|N_{S_x}^3| = \sum_{i=1}^{n-3}(n-2i) = (n-2)(n-3)/2 = a_3$. Therefore $W(P_n^{+-+};q) = a_1 q + a_2 q^2 + a_3 q^3$ where $a_1 = n(n+1)/2,\ a_2 = n^2 - n - 2,\ a_3 = (n-2)(n-5)/2$. We have $diam(P_n^{-+-}) = 2$ and P_n^{-+-} is the complement of P_n^{+-+}. Then the vertices which are adjacent in P_n^{-+-} are at distance 2 in P_n^{+-+} and the vertices which are not adjacent in P_n^{-+-} are adjacent in P_n^{+-+}. Hence the number of pairs of vertices in P_n^{-+-} which are at distance 1 is $a_2 + a_3$ and at distance 2 is a_1. Therefore $W(P_n^{-+-}; q) = (a_2 + a_3)q + a_1 q^2$.

We have $diam(P_n^{-++}) = 3$. The vertex pairs u and v in P_n^{-++} such that $d(u, v) = 1$ are

- $N_{S_x}^1 = \{\{v_i, v_j\} : 1 \leq i \leq n-2\}$ and $|N_{S_x}^1| = (n-1)(n-2)/2$.
- $N_{S_y}^1 = \{\{e_i, e_{i+1}\} : 1 \leq i \leq n-2\}$ and $|N_{S_y}^1| = \sum_i = 1^{n-3}(n-2-i) = n-2$.
- $N_{S_z}^1 = \{\{e_i, v_i\}, \{e_i, v_{i+1}\} : 1 \leq i \leq n-1\}$ and $|N_{S_x}^1| = 2(n-1)$.

Thus the number of vertex pairs u and v in P_n^{-++} such that $d(u, v) = 1$ is $|N_{S_x}^1| + |N_{S_y}^1| + |N_{S_z}^1| = (n^2 + 3n - 6)/2 = a_1$. The vertex pairs u and v in P_n^{-++} such that $d(u, v) = 2$ are

- $N_{S_x}^2 = \{\{v_i, v_{i+1}\},\ 1 \leq i \leq n-1\}$ and $|N_{S_x}^2| = n-1$.
- $N_{S_y}^2 = \{\{e_i, e_{i+2}\},\ 1 \leq i \leq n-3\}$ and $|N_{S_y}^2| = n-3$.
- $N_{S_z}^2 = \{\{e_i, v_j\} : 1 \leq i \leq n-1$ and $1 \leq j \leq n$ but $j \neq i,\ i+1\}$ and $|N_{S_z}^2| = (n-1)(n-2)$.

Thus the number of vertex pairs u and v in P_n^{-++} such that $d(u, v) = 2$ is $|N_{S_x}^2| + |N_{S_y}^2| + |N_{S_z}^2| = n^2 - n - 2 = a_2$. The vertex pairs u and v in P_n^{-++} such that $d(u, v) = 3$ are

- $N_{S_y}^3 = \{\{e_i, e_j\} : 1 \leq i \leq n-4$ and $j = i+3,\ i+4,\ n-1\}$ and $|N_{S_y}^3| = \sum_{i=1}^{n-3}(n-2-i) = (n-4)(n-3)/2 = a_3$.

Therefore $W(P_n^{+-+}; q) = a_1 q + a_2 q^2 + a_3 q^3$ where $a_1 = n(n+1)/2$, $a_2 = n^2 - n - 2$, $a_3 = (n-2)(n-5)/2$.

(i) We have $diam(P_n^{+--}) = 2$ and P_n^{+--} is the complement of P_n^{-++}. Then the vertices which are adjacent in P_n^{+--} are at distance 2 in P_n^{-++} and the vertices which are not adjacent in P_n^{+--} are adjacent in P_n^{-++}. Hence number of pairs of vertices in P_n^{+--} which are at distance 1 is $(a_2 + a_3)$ and at distance 2 from each other is a_1. Therefore $W(P_n^{++-}; q) = (a_2 + a_3)q + a_1 q^2$.

(ii) We have $diam(P_n^{---}) = 2$. Thus for any two vertices u and v in P_n^{---}, either $d(u, v) = 1$ or 2. The number of pairs of vertices u and v in P_n^{---} such that $d(u, v) = 1$ is $|E(P_n^{---})|$. Hence $|E(P_n^{---})| = |E(S_x)| + |E(S_y)| + |E(S_z)| = |E(P_n)| + |E(\overline{P}_{n-1})| + |E(S_z)| = \sum_{i=1}^{n-2} i + \sum_{i=1}^{n-3} i + (n-1)(n-2) = 2n^2 - 7n + 6$. Since $diam(P_n^{++-}) = 2$, the number of pairs of vertices u and v such that $d(u, v) = 2$ in P_n^{++-} is ${}^{(2n-1)}C_2 - |E(P_n^{---})| = 2n^2 + n - 4$. □

Lemma 6.2. *For any positive integer $n \geq 3$, $C_n^{+-+} \cong C_n^{-++}$.*

Proof. We note that both graphs C_n^{+-+} and C_n^{-++} have the same vertex set

$$V(C_n) \cup E(C_n) = \{v_1,\ v_2,\ \ldots,\ v_n,\ e_1,\ e_2,\ \ldots,\ e_n\}.$$

The function $f : C_n^{+-+} \to C_n^{-++}$ defined by $f(v_i) = e_i$ and $f(e_i) = v_i$ is an isomorphism. □

Similarly

Lemma 6.3. *For any positive integer $n \geq 3$, $C_n^{+--} \cong C_n^{-+-}$.*

Proof. We note that both the graphs C_n^{+--} and C_n^{-+-} have the same vertex set $V(C_n) \cup E(C_n) = \{v_1, v_2, \ldots, v_n, e_1, e_2, \ldots, e_n\}$. The function $f : C_n^{+--} \to C_n^{-+-}$ defined by $f(v_i) = e_i$ and $f(e_i) = v_i$ is an isomorphism. □

Theorem 6.4. *For any positive integer $n \geq 6$, the Wiener polynomial of transformation graphs of path graph C_n are*

(i) $W(C_n^{++-}; q) = a_1q + a_2q^2$; $W(C_n^{--+}; q) = a_2q + a_1q^2$ *where* $a_1 = n^2$ $a_2 = n(n-1)$.

(ii) $W(C_n^{+-+}; q) = W(C_n^{-++}; q) = a_1q + a_2q^2 + a_3q^3$, $W(C_n^{+--}; q) = W(C_n^{-+-}; q) = (a_2 + a_3)q + a_1q^2$ *where* $a_1 = n(n+3)/2$; $a_2 = n^2$; $a_3 = n(n-5)/2$.

(iii) $W(C_n^{---}; q) = n(n-1)q + 4nq^2$.

Hence their Wiener indices are as below:
(i) $W(C_n^{++-}) = n(3n-2)$ *and* $W(C_n^{--+}) = n(3n-1) = W(C_n^{++-}) + n$.

(ii) $W(C_n^{+-+}) = W(C_n^{-++}) - k$ *and* $W(C_n^{+--}) = W(C_n^{-+-}) = n(5n+1)/2$.

(iii) $W(C_n^{---}) = n(2n+3)$.

Proof. It follows as previous ones. □

7. Wiener polynomial and Wiener index of the transformation graph G^{++-}

Theorem 7.1. *The Wiener polynomial and the Wiener index of the transformation graph $K_{1,n}^{++-}$ is $W(K_{1,n}^{++-}; q) = \frac{3n^2-n}{2}q + (n^2+n)q^2$ and $W(K_{1,n}^{++-}) = \frac{7n^2+3n}{2}$.*

Proof. Let v_0 be the central vertex and v_i, $1 \leq i \leq n$ be the pendant vertices of $K_{1,n}$. Let $e_i = v_0v_i$ represents the edges of $K_{1,n}$. The diameter of $K_{1,n}^{++-} = 2$. Following are the pairs of vertices which are adjacent in $K_{1,n}^{++-}$:

(i) $\{v_0, v_i\}$ where $1 \leq i \leq n$ the number of which is n.

(ii) $\{e_i, e_j\}$ where $1 \leq i \leq n$ and for each i, $1 \leq j \neq i \leq n$. There are ${}^nC_2$ such pairs.

(iii) $\{e_i, v_j\}$ where $1 \leq i \leq n$ and for each i, $1 \leq j \neq i \leq n$. The number of such pairs is $n(n-1)$.

The following are the pairs of vertices u and v in $K_{1,n}^{++-}$ such that $d(u, v) = 2$:

(a) $\{v_i, v_j\}$ where $1 \leq i \leq n$ and for each i, $1 \leq i \leq n$. There are $n(n-1)$ such pairs.

(b) $\{v_0, e_i\}$ and $\{v_i, e_i\}$ where $1 \leq i \leq n$ which are $2n$ in number.

Hence $W(K_{1,n}^{++-}; q) = \frac{3n^2-n}{2} q + (n^2+n)q^2$ and $W(K_{1,n}^{++-}) = \frac{7n^2+3n}{2}$. □

Theorem 7.2. *For any graph G of order n and size m, the Wiener polynomial of the transformation graph* $W(G^{++-}; q) = \left\{m + \sum_{v \in V(G)} ({}^{degv}C_2) + m(n-2)\right\} q +$ $\left\{\sum_{i=1}^{n} (n - degv_i - 1) + \sum_{j=1}^{m} (m - dege_j - 1) + 2m\right\} q^2.$

Proof. For a connected graph G of order n and size m, the diameter of $G^{++-} = 2$. Here $V(G^{++-}) = V(G) \cup E(G) = \{v_1, v_2, v_3, \cdots, v_n, e_1, e_2, e_3, \cdots, e_m\}$. Hence the Wiener polynomial of G is $W(G; q) = \sum_{\{u,v\} \subseteq V(G)} q^{d(u,v)}$. Therefore

$$W(G^{++-}; q) = (S_x; q) + (S_y; q) + \sum_{u \in V(G), v \in E(G)} q^{d(u,v)}.$$

By the definition of G^{++-}, we have

(i) If $\{u, v\} \subseteq S_x(G)$, then $W(S_x; q) = mq + \left[\sum_{i=1}^{n} (n - 1 - degv_i)\right] q^2$.

(ii) If $\{u, v\} \subseteq S_y(G)$, then $W(S_y; q) = \left[\sum_{v \in V(G)} ({}^{degv}C_2)\right] q + \left[\sum_{j=1}^{m} (m - 1 - dege_j)\right] q^2$.

(iii) If $\{u, v\} \subseteq V(S_z)$ such that $u \in V(G)$, $v \in E(G)$, such that $W(S_z; q) = m(n-2)q + \left[\sum_{j=1}^{m} (n - dege_i - 1)\right] q^2$.

Therefore

$$\begin{aligned} W(G^{++-}; q) &= \{m + \textstyle\sum_{v \in V(G)} ({}^{degv}C_2) + m(n-2)\} q + \{\sum_{i=1}^{n} (n - degv_i - 1) \\ &\quad + \textstyle\sum_{j=1}^{m} (m - dege_j - 1) + 2m\} q^2. \end{aligned}$$

□

8. Conclusion

Topological indices for the derived graphs such as line graphs, middle graphs, subdivision graphs, r-subdivision graphs are widely studied by different authors. Following this fashion, in this paper, we have obtained the Wiener polynomials and Wiener indices of the transformation graphs of some standard graphs. Different distance based topological indices or even other topological indices can be calculated for all these transformation graphs and their characterizations can be given by similar methods.

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Nitte Meenakshi Institute of Technology, Department of Mathematics, Bengaluru, India
Email address: chandrakalasb14@gmail.com

Dr. Ambedkar Intitute of Technology, Department of Mathematics, Bengaluru, India
Email address: dr_bsnrao@dr-ait.org

Bangalore Institute of Technology, Department of Mathematics, Bengaluru, India
Email address: manju_chandru2005@rediffmail.com

Bursa Uludag University, Department of Mathematics, Gorukle 16059 Bursa, Turkey
Email address: cangul@uludag.edu.tr