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A SURVEY ON SOME OLD AND NEW IDENTITIES ASSOCIATED WITH LAPLACE DISTRIBUTION AND BERNOULLI NUMBERS

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Abstract. The purpose of this paper is to give some survey on old and new identities related to the characteristic function, the Laplace distribution and special numbers and polynomials with comparative results and observations. Additionally, we give some computation formulas for the higher-order moments of some kinds of random variables with the Laplace distribution in terms of the Bernoulli numbers of the first kind, the Euler numbers of the second kind and Riemann zeta function by using the techniques of generating functions and characteristic function of the aforementioned random variables. Finally, with the aid of the Hankel determinants formed by the moments corresponding to the weight function that reveals the orthogonality feature of the orthogonal polynomials, we give futher remarks and observations on not only orthogonality properties of some orthogonal polynomials such as the Hermite polynomials, but also construction methods of the three-term recurrence relations for the orthogonal polynomials.

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1. Introduction

In recent years, many studies have been conducted by researchers about the relations of some special numbers and polynomials with probability theory and their applications (see [6, 9, 11, 13, 14, 16, 23, 27]). Among these studies, the papers [6, 9, 11, 16, 27] focused on the moments of random variables arising from Laplace distribution. In addition, Simsek and Simsek [23] gave a computation of expected values and moments of special polynomials via characteristic and generating functions. Furthermore, Kucukoglu et al. [13] presented an approach to negative hypergeometric distribution by generating function for special numbers and polynomials. Over and above, by generating functions for families of combinatorial numbers and polynomials, Kucukoglu et al. [14] gave some identities related to a discrete probability

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distribution related to the binomial distribution and the Poisson distribution. Especially, for the moments of two kinds of random variables arising from Laplace distribution, Kim et al. [11] provided formulas in terms of the type 2 Bernoulli numbers and the Euler numbers of the second kind with the help of their generating functions. All these studies in recent years have shown that combining the generating function techniques with the concepts of the probability theory provides us to obtain elegant identities and formulas.

With this motivation, in this study, we not only give some survey on old and new identities related to the characteristic function, the Laplace distribution and special numbers and polynomials, but also provide some comparative results and observations. In addition, by using the techniques of generating functions and characteristic function of the aforementioned random variables, we give some computation formulas for the higher-order moments of some kinds of random variables with the Laplace distribution in terms of the Bernoulli numbers of the first kind, the Euler numbers of the second kind and Riemann zeta function.

In the rest of this section, we recall some definitions and notations of generating functions for some special numbers and polynomials, trigonometric functions, characteristic functions, the Laplace distribution, which are used in a wide range of branches of mathematics such as mathematical statistics, probability theory. Throughout this paper, we use the following standards notations:

$\mathbb{N} = \{1, 2, 3, \dots\}$, $\mathbb{N}_0 = \{0, 1, 2, 3, \dots\}$, $\mathbb{Z}$ denotes the set of integers, $\mathbb{R}$ denotes the set of real numbers and $\mathbb{C}$ denotes the set of complex numbers. In addition, we assume that $i^2 = -1$.

The Bernoulli polynomials $B_n(x)$ of the first kind are defined by means of the following generating function:

$$\frac{t}{e^t - 1} e^{xt} = \sum_{n=0}^{\infty} B_n(x) \frac{t^n}{n!}, \tag{1}$$

where $|t| < 2\pi$. Substituting $x = 0$ or $x = 1$ into (1), we have

$$B_n(0) = B_n(1) = B_n,$$

where B_n denotes the Bernoulli numbers of the first kind defined as follows:

$$\frac{t}{e^t - 1} = \sum_{n=0}^{\infty} B_n \frac{t^n}{n!} \tag{2}$$

(*cf.* [1]-[27]).

Using (2), a computation formula for the Bernoulli numbers of the first kind is given as follows:

$$B_n = \sum_{k=0}^{n} \binom{n}{k} B_k \tag{3}$$

with $B_0 = 1$ and $n \in \mathbb{N}$ (*cf.* [1]-[27]).

Using (1) and (2), we have

$$B_n\left(\frac{1}{2}\right) = (2^{1-n} - 1) B_n \tag{4}$$

(*cf.* [1]-[27]).

Using (3), a few values of the numbers B_n are given as follows:

$$B_0 = 1, \quad B_1 = -\frac{1}{2}, \quad B_2 = \frac{1}{6}, \quad B_3 = 0, \quad B_4 = -\frac{1}{30}, \quad B_5 = 0,$$

and so on (*cf.* [1]-[27]).

The Euler polynomials of the second kind, $E_n(x)$, are defined by means of the following generating function:

$$\frac{2}{e^t + e^{-t}} e^{xt} = \operatorname{sech}(t) e^{xt} = \sum_{n=0}^{\infty} E_n(x) \frac{t^n}{n!}, \tag{5}$$

(*cf.* [11]).

Substituting $x = 0$ into (5), we have

$$E_n(0) = E_n,$$

where E_n denotes the Euler numbers of the second kind defined as follows:

$$\frac{1}{\cosh t} = \frac{2}{e^t + e^{-t}} = \sum_{n=0}^{\infty} E_n \frac{t^n}{n!}, \tag{6}$$

(*cf.* [1], [11], [18], [21], [22]).

Note that the odd-indexed Euler numbers of the second kind are all zero. That is, for $n \in \mathbb{N}_0$, we have

$$E_{2n+1} = 0.$$

On the other hand, using (6), a computation formula for the even-indexed Euler numbers of the second kind is given as follows:

$$E_{2n} = -\sum_{k=0}^{n-1} \binom{2n}{2k} E_{2k}, \tag{7}$$

where $n \in \mathbb{N}$ (*cf.* [1], [11], [18], [21]). With the help of the formula given by (7), a few values of the even-indexed Euler numbers of the second kind are computed as follows:

$$E_0 = 1, \quad E_2 = -1, \quad E_4 = 5, \quad E_6 = -61, \quad E_8 = 1385,$$

and so on (*cf.* [1], [11], [18], [21]).

The infinite product formulas for the hyperbolic sine function $\sinh(z)$, the cosine function $\cos(z)$ and the sine function $\sin(z)$ are given respectively as follows:

$$\sinh(z) = z \prod_{n=1}^{\infty} \left(1 + \left(\frac{z}{n\pi}\right)^2\right), \tag{8}$$

$$\cos(z) = \prod_{n=1}^{\infty} \left(1 - \left(\frac{2z}{(2n-1)\pi}\right)^2\right) \tag{9}$$

and

$$\sin(z) = z \prod_{n=1}^{\infty} \left(1 - \left(\frac{z}{n\pi}\right)^2\right) \tag{10}$$

(*cf.* [3]).

Let X be a continuous random variable and f be the probability density function of X. Then, the characteristic function $\phi_x(t)$ of the probability density function f is defined by

$$\phi_x(t) = \int_{-\infty}^{\infty} e^{itx} f(x)dx \tag{11}$$

(*cf.* [4], [16]).

Let X be a continuous random variable and f be the probability density function of X. Then, the expected value of X^k, which is also called kth moment of the random variable X, is defined by

$$m_k = E\left[X^k\right] = \int_{-\infty}^{\infty} x^k f(x)dx$$

(*cf.* [4], [16]).

Let X be a continuous random variable and f be the probability density function of X. Then, the moment generating function $m_x(t)$ of the random variable X is defined by

$$m_x(t) = E\left[e^{tX}\right] = \int_{-\infty}^{\infty} e^{tx} f(x)dx \tag{12}$$

(*cf.* [4], [16]).

The *classical Laplace distribution* (also known as *first law of Laplace*) is a probability distribution on $(-\infty,\infty)$ with parameters θ and s and this distribution is given by the following probability density function:

$$f(x;\theta,s) = \frac{1}{2s}e^{-\frac{|x-\theta|}{s}}, \tag{13}$$

where $(-\infty < x < \infty)$, $\theta \in (-\infty,\infty)$ and $s > 0$ (*cf.* [9], [12], [16]).

Substituting $\theta = 0$ and $s = 1$ into (13), we have the probability density function for the standard classical Laplace distribution with parameters 0 and 1 as follows:

$$f(x;0,1) = \frac{1}{2}e^{-|x|} \tag{14}$$

(*cf.* [9], [12], [16]) and its characteristic function is as follows:

$$\phi_x(t) = \frac{1}{t^2+1} \tag{15}$$

(see, for detail, [6], [27], [16]).

2. Computation formulas for the higher-order moments of some kinds of random variables with the Laplace distribution

In this section, by using the techniques of generating functions and characteristic function of the aforementioned random variables, we give not only some computation formulas, but also explicit formulas for the higher-order moments of some kinds of random variables with the Laplace distribution in terms of the Bernoulli numbers of the first kind and Riemann zeta function.

Next, in same manner with the works of [6], [9], [11] and [16], [27], let us start with assuming that the independent random variables $X_1, X_2, X_3, ...$ have the standard classical Laplace distribution with parameters 0 and 1. Then, the following series becomes convergent

$$H = \sum_{k=1}^{\infty} \frac{X_k}{k}. \tag{16}$$

Observe that the above convergent series H is a random variable derived from the independent random variables $X_1, X_2, X_3, \ldots$ with the Laplace distribution.

Thus, the characteristic function of the random variable H is obtained as follows:

$$E\left[e^{itH}\right] = E\left[e^{it\left(\sum_{k=1}^{\infty}\frac{X_k}{k}\right)}\right] = \prod_{k=1}^{\infty} E\left[e^{\left(\frac{X_k}{k}\right)it}\right], \tag{17}$$

where

$$E\left[e^{\left(\frac{X_k}{k}\right)it}\right] = \frac{1}{1+\left(\frac{t}{k}\right)^2}. \tag{18}$$

Combining (17) with (18), we get

$$E\left[e^{itH}\right] = \prod_{k=1}^{\infty}\left(1+\left(\frac{t}{k}\right)^2\right)^{-1}. \tag{19}$$

By using (8) we get

$$\frac{\pi t}{\sinh \pi t} = \prod_{n=1}^{\infty}\left(1+\left(\frac{t}{n}\right)^2\right)^{-1}. \tag{20}$$

Thus, by using (19) and (20) we have

$$E\left[e^{itH}\right] = \frac{\pi t}{\sinh \pi t}. \tag{21}$$

By using (17), we have

$$\begin{aligned} E\left[e^{itH}\right] &= \sum_{n=0}^{\infty}\frac{(it)^n}{n!}E\left[\left(\sum_{k=1}^{\infty}\frac{X_k}{k}\right)^n\right] \\ &= \sum_{n=0}^{\infty}\frac{(it)^n}{n!}\mu_n, \end{aligned} \tag{22}$$

where

$$\mu_n = E\left[H^n\right]$$

which is called the nth moment of the random variable H relative to 0.

On the other hand, by using (21), we obtain

$$E\left[e^{itH}\right] = \frac{2\pi t}{e^{2\pi t}-1}e^{\pi t}. \tag{23}$$

Replacing t by $2\pi t$ in (1) and when $x=\frac{1}{2}$, we have

$$\frac{2\pi t}{e^{2\pi t}-1}e^{\pi t}=\sum_{n=0}^{\infty}B_n\left(\frac{1}{2}\right)(2\pi)^n\frac{t^n}{n!}. \tag{24}$$

By using (22) in (23), and then combining the final equation with (24), we get

$$\sum_{n=0}^{\infty}i^n\mu_n\frac{t^n}{n!}=\sum_{n=0}^{\infty}B_n\left(\frac{1}{2}\right)(2\pi)^n\frac{t^n}{n!}.$$

Comparing the coefficients of $\frac{t^n}{n!}$ on both sides of the above equation, we get the following theorem:

Theorem 2.1. *Let $n\in\mathbb{N}_0$. Then we have*

$$\mu_n=i^{-n}2^n\pi^nB_n\left(\frac{1}{2}\right). \tag{25}$$

By using (3) and (4) in (25), we compute a few value of the moments μ_n as follows:

$$\mu_0=1,\quad \mu_1=0,\quad \mu_2=\frac{\pi^2}{3},\quad \mu_3=0,\quad \mu_4=\frac{7\pi^4}{15},\quad \mu_5=0,\dots$$

Combining (4) with (25) yields the following corollary:

Corollary 2.1. *Let $n\in\mathbb{N}_0$. Then we have*

$$\mu_n=2i^{-n}\pi^n\left(1-2^{n-1}\right)B_n.$$

Remark 2.1. *It is well-known that the odd indexed Bernoulli numbers of the first kind are equal to zero. Therefore, we may conclude from Corollary* 2.1 *that the odd-order moments of the random variable H are zero. That is, we have*

$$\mu_{2n+1}=0. \tag{26}$$

Thus, only the even-order moments of the random variable Y need to be calculated. That is, replacing n by $2n$ in Corollary 2.1 *yields the even-order moments of the random variable H as follows:*

$$\mu_{2n}=2(-1)^n\pi^{2n}\left(1-2^{2n-1}\right)B_{2n}. \tag{27}$$

Recall that the relation between the Bernoulli numbers of the first kind and the Riemann zeta function is given as follows:

$$B_{2n}=(-1)^{n-1}\frac{2(2n)!}{(2\pi)^{2n}}\zeta(2n);\qquad (n\in\mathbb{N}_0) \tag{28}$$

where ζ denotes the Riemann zeta function defined by

$$\zeta(s)=\sum_{n=1}^{\infty}\frac{1}{n^s};\qquad (s\in\mathbb{C};\mathrm{Re}(s)>1) \tag{29}$$

(*cf.* [3], [24], [25], [26]).

By substituting (28) into (27), we get

$$\mu_{2n} = 2\left(-1\right)^n \pi^{2n} \left(1-2^{2n-1}\right) \left(\left(-1\right)^{n-1} \frac{2\left(2n\right)!}{\left(2\pi\right)^{2n}} \zeta\left(2n\right)\right).$$

After some elementary simplifications in the above equation, we arrive at a relation between the Riemann zeta function and the even-order moments of the random variable H by the following corollary:

Corollary 2.2. *Let* $n \in \mathbb{N}_0$. *Then we have*

$$\mu_{2n} = 2\left(1-2^{1-2n}\right)(2n)!\zeta\left(2n\right).$$

Also, recall that Srivastava [24, Eq.-(2.8)] gave the following recurrence relation for the Riemann zeta function:

$$\zeta\left(2n\right) = \frac{2}{2n+1} \sum_{k=1}^{n-1} \zeta\left(2k\right)\zeta\left(2n-2k\right); \qquad \left(n \in \mathbb{N} \setminus \{1\}\right). \tag{30}$$

By combining (30) with Corollary 2.2, we get

$$\mu_{2n} = \frac{4\left(1-2^{1-2n}\right)(2n)!}{2n+1} \sum_{k=1}^{n-1} \zeta\left(2k\right)\zeta\left(2n-2k\right); \qquad \left(n \in \mathbb{N} \setminus \{1\}\right)$$

which, by using (28), yields another computation formula for the even-order moments of the random variable H by the following corollary:

Corollary 2.3. *Let* $n \in \mathbb{N} \setminus \{1\}$. *Then we have*

$$\mu_{2n} = \frac{2\left(-1\right)^n \pi^{2n}\left(2^{2n-1}-1\right)}{2n+1} \sum_{k=1}^{n-1} \binom{2n}{2k} B_{2k} B_{2n-2k}.$$

It should be also recall that an explicit formula for the Bernoulli numbers of the first kind is given as follows:

$$B_n = \sum_{k=0}^{n} \sum_{j=0}^{k} \frac{(-1)^j}{k+1} \binom{k}{j} j^n \tag{31}$$

where $n \in \mathbb{N}$ (*cf.* [3], [24, 25, 26]). Combining (31) with Corollary 2.1 yields an explicit formula for the higher-order moments of the random variable H by the following corollary:

Corollary 2.4. *Let* $n \in \mathbb{N}_0$. *Then we have*

$$\mu_{2n} = 2\left(-1\right)^n \pi^{2n}\left(1-2^{2n-1}\right) \sum_{k=0}^{2n} \sum_{j=0}^{k} \frac{(-1)^j}{k+1} \binom{k}{j} j^{2n}. \tag{32}$$

Remark 2.2. *By supposing that the independent random variables* $X_1, X_2, X_3, \ldots$ *have the standard classical Laplace distribution with parameters* 0 *and* 1, *Kim et al.* [11, Eq.-(66)] *considered the characteristic function of*

$$Z = \sum_{k=1}^{\infty} \frac{X_k}{2k\pi},$$

and then proved that the following equality holds true

$$E\left[Z^{n}\right]=i^{-n}\left(\frac{1}{2}\right)^{n-1}b_{n}, \tag{33}$$

where b_n denotes the type 2 Bernoulli numbers which is different from the classical Bernoulli numbers B_n.

3. Further remarks and observations on characteristic function of the random variables with Laplace distribution and known identites related to Bernoulli and Euler numbers

In this section, we give further remarks and observations on characteristic function of the random variables with Laplace distribution and some identities related to the Bernoulli numbers of the first kind and the Euler numbers of the second kind.

We now give modification and unification of the continuous random variable with Laplace distribution, which is derived from (16), as follows:

$$Y=\sum_{k=1}^{\infty}\frac{X_k}{(2k-1)\pi}$$

and

$$Z=\sum_{k=1}^{\infty}\frac{X_k}{2k\pi}.$$

Due to the convergency of the series H defined in (16), the above series are also convergent(see, for details, [6], [9], [11], [16], [27]).

Using the characteristic function of the random variable H, it is easy to compute the characteristic functions of the random variables Y and Z, which are related to the random variable H as follows:

The characteristic functions of the random variable Y is given by

$$E\left[e^{2itY}\right]=\prod_{k=1}^{\infty}E\left[e^{\left(\frac{X_k}{(2k-1)\pi}2it\right)}\right] \tag{34}$$

where

$$\begin{aligned} E\left[e^{\left(\frac{X_k}{(2k-1)\pi}2it\right)}\right] &= \int_{-\infty}^{\infty}\frac{1}{2}e^{\left(\frac{2it}{(2k-1)\pi}\right)x}e^{-|x|}dx \\ &= \left(1+\left(\frac{2t}{(2k-1)\pi}\right)^{2}\right)^{-1}, \end{aligned} \tag{35}$$

(see, for details, [6], [9], [11], [16], [27]).

The characteristic function of Z is given by

$$E\left[e^{itZ}\right]=\prod_{k=1}^{\infty}E\left[e^{\left(\frac{X_k}{2k\pi}\right)it}\right]. \tag{36}$$

where

$$E\left[e^{\left(\frac{X_k}{2k\pi}\right)it}\right]=\int_{-\infty}^{\infty}\frac{1}{2}e^{\left(\frac{it}{2k\pi}\right)x}e^{-|x|}dx=\left(1+\left(\frac{t}{2k\pi}\right)^2\right)^{-1}, \tag{37}$$

(see, for details, [6], [9], [11], [16], [27]).

By (37), we have

$$\prod_{k=1}^{\infty}E\left[e^{\left(\frac{X_k}{2k\pi}\right)it}\right]=\prod_{k=1}^{\infty}\left(1+\left(\frac{t}{2k\pi}\right)^2\right)^{-1}. \tag{38}$$

On the other hand, replacing t by it in (10) yields the following well-known result:

$$2\prod_{n=1}^{\infty}\left(1+\left(\frac{t}{2n\pi}\right)^2\right)^{-1}=\frac{it}{\sin\left(\frac{it}{2}\right)}=\frac{2t}{e^t-1}e^{\frac{t}{2}},$$

(*cf.* [11]).

Observe that the right-hand side of the above equation is the generating function for the Bernoulli polynomials of the first kind. It is not difficult to see that $E\left[Z^n\right]$, which is derived from the characteristic function for the Laplace distribution, is associated with the Bernoulli polynomials of the first kind. For details, see [11].

By using(12) and (35), we also have

$$E\left[e^{2itY}\right]=\sum_{n=0}^{\infty}E\left[Y^n\right]\frac{(2it)^n}{n!}=\prod_{k=1}^{\infty}\left(1+\left(\frac{2t}{(2k-1)\pi}\right)^2\right)^{-1}. \tag{39}$$

Replacing t by it in (9) and combining the final equation with (6), we have

$$\frac{1}{\cos(it)}=\sum_{n=0}^{\infty}E_n\frac{t^n}{n!}, \tag{40}$$

(*cf* [11]).

Similarly, we have

$$E\left[e^{2itY}\right]=\sum_{n=0}^{\infty}E_n\frac{t^n}{n!}, \tag{41}$$

(*cf* [11]). After some elementary calculations, for $n\in\mathbb{N}_0$, we have

$$E\left[Y^n\right]=\frac{1}{2}\int_{-\infty}^{\infty}Y^ne^{-|Y|}dY=2^{-n}i^{-n}E_n \tag{42}$$

which was given by Kim et al. [11].

Remark 3.1. *It is well-known that the odd indexed Euler numbers of the second kind are equal to zero. Therefore, we may conclude from* (42) *that the odd-order moments of the random variable* Y *are zero. That is, we have*

$$E\left[Y^{2n+1}\right]=0. \tag{43}$$

Thus, only the even-order moments of the random variable Y need to be calculated. That is, replacing n by $2n$ in (42) yields the even-order moments of the random variable Y as follows:

$$E\left[Y^{2n}\right]=(-1)^{n}\frac{E_{2n}}{2^{2n}}. \tag{44}$$

By combining (44) with the computation formula for the Euler numbers of the second kind given by (7), we have another computation formula for the even-order moments of the random variable Y in terms of the Euler numbers of the second kind by the following corollary:

Corollary 3.1. *Let $n\in\mathbb{N}\setminus\{1\}$. Then we have*

$$E\left[Y^{2n}\right]=(-1)^{n+1}\frac{1}{2^{2n}}\sum_{k=0}^{n-1}\binom{2n}{2k}E_{2k}.$$

Recall that Wei and Qi [28, Theorem 1.4] gave an explicit formula for the Euler numbers of the second kind as follows:

$$E_{2n}=\sum_{k=1}^{2n}\sum_{j=0}^{2k}\frac{(-1)^{k+j}}{2^{k}}\binom{2k}{j}(k-j)^{2n} \tag{45}$$

where $n\in\mathbb{N}$. Combining (44) with (45) yields an explicit formula for the even-order moments of the random variable Y by the following corollary:

Corollary 3.2. *Let $n\in\mathbb{N}$. Then we have*

$$E\left[Y^{2n}\right]=\frac{(-1)^{n}}{2^{2n}}\sum_{k=1}^{2n}\sum_{j=0}^{2k}\frac{(-1)^{k+j}}{2^{k}}\binom{2k}{j}(k-j)^{2n}. \tag{46}$$

Multiplying the generating functions for the Euler numbers of the second kind and the Bernoulli polynomials of the first kind, we have

$$\left(\frac{1}{\cos(it)}\right)\left(\frac{it}{\sin(it)}\right)=\sum_{n=0}^{\infty}E_{n}\frac{t^{n}}{n!}\sum_{n=0}^{\infty}2^{n}B_{n}\left(\frac{1}{2}\right)\frac{t^{n}}{n!}$$

By using the double-angle formula for sine function on the left-hand side of the above equation and using the Cauchy product on the right-hand side of the above equation, we get

$$\frac{2it}{\sin(2it)}=\sum_{n=0}^{\infty}\left(\sum_{k=0}^{n}\binom{n}{k}2^{k}B_{k}\left(\frac{1}{2}\right)E_{n-k}\right)\frac{t^{n}}{n!}$$

which, by (4), reduces to the following formula:

$$\frac{2it}{\sin(2it)}=\sum_{n=0}^{\infty}\left(2\sum_{k=0}^{n}\binom{n}{k}\left(1-2^{k-1}\right)B_{k}E_{n-k}\right)\frac{t^{n}}{n!}. \tag{47}$$

By (10), we also have

$$\frac{2it}{\sin(2it)}=\prod_{n=1}^{\infty}\left(1+\left(\frac{2t}{n\pi}\right)^{2}\right)^{-1}. \tag{48}$$

On the other hand, observe that the characteristic function of the random variable

$$\frac{2}{\pi}\sum_{k=1}^{\infty}\frac{X_k}{k}$$

is obtained as follows:

$$E\left[e^{it\left(\frac{2}{\pi}\sum_{k=1}^{\infty}\frac{X_k}{k}\right)}\right]=\prod_{k=1}^{\infty}E\left[e^{\left(\frac{2X_k}{\pi k}\right)it}\right], \tag{49}$$

where

$$E\left[e^{\left(\frac{2X_k}{\pi k}\right)it}\right]=\left(1+\left(\frac{2t}{n\pi}\right)^2\right)^{-1}. \tag{50}$$

Combining (49) with (50), we get

$$E\left[e^{it\left(\frac{2}{\pi}\sum_{k=1}^{\infty}\frac{X_k}{k}\right)}\right]=\prod_{k=1}^{\infty}\left(1+\left(\frac{2t}{k\pi}\right)^2\right)^{-1}. \tag{51}$$

By using (49), we have

$$\begin{aligned} E\left[e^{it\left(\frac{2}{\pi}\sum_{k=1}^{\infty}\frac{X_k}{k}\right)}\right] &= \sum_{n=0}^{\infty}\frac{(it)^n}{n!}E\left[\left(\frac{2}{\pi}\sum_{k=1}^{\infty}\frac{X_k}{k}\right)^n\right] \\ &= \sum_{n=0}^{\infty}\frac{(it)^n}{n!}E\left[\left(\frac{2}{\pi}\right)^n\left(\sum_{k=1}^{\infty}\frac{X_k}{k}\right)^n\right] \\ &= \sum_{n=0}^{\infty}\left(\frac{2}{\pi}\right)^n i^n\mu_n\frac{t^n}{n!} \end{aligned} \tag{52}$$

where

$$\mu_n=E\left[H^n\right]$$

which is called the nth moment of the random variable H relative to 0.

Combining Eqs (48), (51), (52) with (47), we get

$$\sum_{n=0}^{\infty}\left(\frac{2}{\pi}\right)^n i^n\mu_n\frac{t^n}{n!}=\sum_{n=0}^{\infty}\left(2\sum_{k=0}^{n}\binom{n}{k}\left(1-2^{k-1}\right)B_kE_{n-k}\right)\frac{t^n}{n!}.$$

Comparing the coefficients of $\frac{t^n}{n!}$ on both sides of the above equation, we get another computation formula for the nth moment of the random variable H including both the Bernoulli numbers of the first kind and the Euler numbers of the second kind by the following theorem:

Theorem 3.1.

$$\mu_n=2\left(\frac{\pi}{2}\right)^n i^n\sum_{k=0}^{n}\binom{n}{k}\left(1-2^{k-1}\right)B_kE_{n-k}.$$

Remark 3.2. *Note that Theorem* 2.1*, Corollary* 2.1*, Corollary* 2.2*, Corollary* 2.3*, Corollary* 3.1*, Corollary* 3.2*, Theorem* 3.1 *can be achieved in two different ways. The first way is using the characteristic function of a random variable with the Laplace distribution. The second way is to take logarithmic derivative of infinite product of trigonometric function. That is, two kinds of proof can be given for the aforementioned results. For further observation, we now write an important theorem from the book of Conway* [3] *explaining here only what we mean:*

The Weierstrass Factorization Theorem (*cf.* [3]): *Let f be an entire function and let $\{a_n\}$ be the non-zero zeros of f repeated according to multiplicity; suppose f has a zero at $z = 0$ of order $m \geq 0$ (a zero of order $m = 0$ at $z = 0$ means $f(0) \neq 0$). Then there is an entire function g and a sequence of integers $\{p_n\}$ such that*

$$f(z) = z^m e^{g(z)} \prod_{n=1}^{\infty} E_{p_n}\left(\frac{z}{a_n}\right)$$

where E denotes the elementary factors defined by

$$\begin{aligned} E_0(z) &= 1 - z, \\ E_p(z) &= (1 - z) e^{\left(\sum\limits_{n=1}^{p} \frac{z^n}{n}\right)}; p \in \mathbb{N}; \end{aligned}$$

(for details, see [3]). *Let recall applications of the Weierstrass Factorication Theorem to the functions* $\sin \pi z$ *and* $\cos \pi z$*, respectively, as follows (for details see the work of Conway* [3]*): One of these applications is factorization of the sine function given below:*

$$\frac{\sin \pi z}{\pi z} = \prod_{n=1}^{\infty} \left(1 - \frac{z^2}{n^2}\right)$$

which is uniformly convergent over the compact subset of z-plane (*cf.* [3, p. 175])*, taking the reciprocal of the case z of the above factorization gives the well-known Wallis's formula below:*

$$\frac{\pi}{2} = \prod_{n=1}^{\infty} \frac{(2n)^2}{(2n-1)(2n+1)},$$

(*cf.* [3, Exercise 4, p. 176]). *Another application is factorization of the sine function given below:*

$$\cos \pi z = \prod_{n=1}^{\infty} \left(1 - \frac{4z^2}{(2n-1)^2}\right),$$

(*cf.* [3, Exercise 1, p. 176]). *Thus, using Mittag-Leffler expansions of meromorphic functions and the Weierstrass Factorication Theorem gives us not only Eqs.* (8)*,* (9) *and* (10)*, but also another generating functions for the Bernoulli and Euler numbers.*

3.1. **Futher remarks on applications of moments.** Here, we give some further remarks on applications of moments.

Moments and moment generating functions of random variables are frequently used in various fields of mathematics, especially probability. Moments play a key role in modeling, examining the characteristics of a distribution, particularly in the construction of three-term recurrence relations for the orthogonal polynomials.

Let $n \in \mathbb{N}_0$. The moments μ_n of the weight function $w(x)$ are defined as follows:

$$\mu_n = \int_a^b x^n w(x) dx, \tag{53}$$

(*cf.* [2]).

It is well-known that the Hermite polynomials, $H_n(x)$, have an important place in orthogonal polynomial families and these polynomials are orthogonal on the interval $(-\infty, \infty)$ with respect to the weight function:

$$w(x) = e^{-x^2}$$

because these polynormials satisfy the following orthogonality relation:

$$\int_{-\infty}^{\infty} H_n(x) H_m(x) e^{-x^2} dx = 2^n n! \sqrt{\pi} \delta_{nm} \tag{54}$$

where δ_{nm} denotes the Kronecker delta (*cf.* [2]).

Note that the moments μ_n of the weight function $w(x) = e^{-x^2}$ are given as follows:

$$\mu_n = \int_{\infty}^{-\infty} x^n e^{-x^2} dx. \tag{55}$$

The Hankel determinant of the $(n+1) \times (n+1)$ matrix, whose (i, j) entry is μ_{i+j}, is given as follows:

$$\Delta_n = \det(\mu_{i+j})_{i,j=0}^{n} = \begin{vmatrix} \mu_0 & \mu_1 & \cdots & \mu_n \\ \mu_1 & \mu_2 & \cdots & \mu_{n+1} \\ \vdots & \vdots & \cdots & \vdots \\ \mu_n & \mu_{n+1} & \cdots & \mu_{2n} \end{vmatrix} \tag{56}$$

(*cf.* [2], [20]).

Combining (55) with (56) gives us the well-known three-term recurrence relation for the Hermite polynomials as follows:

$$H_{n+1}(x) = 2xH_n(x) - 2nH_{n-1}(x); \quad (n \in \mathbb{N})$$

with the initial conditions $H_0(x) = 1$ and $H_1(x) = 2x$ (*cf.* [2]).

The above type three-term recurrence relations for the special polynomials can be achieved in many different ways such as the generating functions, the umbral calculus, the Hankel determinant with the moments of the weight function of the orthogonal, algebraic manipulations on the coefficients, and the others (for details, see [2]-[29]).

4. Conclusion

In this paper, some old and new identities associated with Laplace distribution and Bernoulli numbers are investigated. Furthermore, various types of computation formulas are provided for the higher-order moments of some kinds of random variables with the Laplace distribution which include the Bernoulli numbers of the first kind, the Euler numbers of the second kind and Riemann zeta function by using the techniques of generating functions and characteristic function of the aforementioned random variables. It should be noted here that the results of the present paper have the potential for attracting attention of researchers working on not only probability theory and number theory, but also other relevant areas. For future studies, it is planned to investigate relations among the higher-order moments of certain random variables with the Laplace distribution and the generating functions for the combinatorial numbers such as $y_1(n,k;\lambda)$ and $y_2(n,k;\lambda)$ recently introduced by Simsek in [22].

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