

A NOTE ON DEGENERATE MULTI-POLY-GENOCCHI POLYNOMIALS

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ABSTRACT. In this paper, we introduce the degenerate multiple polyexponential functions which are multiple versions of the degenerate modified polyexponential functions. Then we consider the degenerate multi-poly-Genocchi polynomials which are defined by using those functions and investigate explicit expressions and some properties for those polynomials.

1. Introduction

It was Carlitz who initiated the study of degenerate versions of some special numbers and polynomials, namely the degenerate Bernoulli and Euler polynomials and numbers [1]. In recent years, studying degenerate versions of some special polynomials and numbers regained interests of many mathematicians, and quite a few interesting results were discovered [6,7,9,11-15,17]. The polyexponential functions were first introduced by Hardy in [3,4] and rediscovered by Kim [11,14], as inverses to the polylogarithm functions. Recently, the degenerate polyexponential functions, which are degenerate versions of polyexponential functions, were introduced in [11], and some of their properties were investigated. Furthermore, the so-called new type degenerate Bell polynomials were introduced, and some identities connecting these polynomials to the degenerate polyexponential functions were found in [11]. In [12], the (modified) polyexponential functions were used in order to define the degenerate poly-Bernoulli polynomials, and several explicit expressions about those polynomials and some identities involving them were derived. In [14], the degenerate (modified) polyexponential functions were introduced, and the degenerate type 2 poly-Bernoulli numbers and polynomials were defined by means of those functions. In addition, several explicit expressions and some identities for those numbers and polynomials were deduced.

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In this paper, we introduce the degenerate multiple polyexponential functions. These are multiple versions of the degenerate modified polyexponential functions. Then we define the degenerate multi-poly-Genocchi polynomials by means of those functions. We derive some explicit expressions for the degenerate multi-poly-Genocchi polynomials and certain properties related to those polynomials.

We recall that, for all $k \in \mathbb{Z}$, the polylogarithm functions are defined by

$$Li_k(x) = \sum_{k=1}^{\infty} \frac{x^k}{n^k}, (|x| < 1), \quad (\text{see [16, 18]}). \quad (1.1)$$

The polyexponential functions were studied by Hardy in [3,4].

Recently, a slightly different version of those functions, which are called the modified polyexponential functions, are defined as an inverse to polylogarithm functions by

$$Ei_k(x) = \sum_{n=1}^{\infty} \frac{x^n}{(n-1)!n^k}, (k \in \mathbb{Z}), \quad (\text{see [5, 7 - 9, 11, 13, 14]}). \quad (1.2)$$

When $k = 1$, by (1.2), we get

$$Ei_1(x) = \sum_{n=1}^{\infty} \frac{x^n}{n!} = e^x - 1. \quad (1.3)$$

In [14] (see also [11]), the degenerate modified polyexponential functions are defined by

$$Ei_{k,\lambda}(x) = \sum_{n=1}^{\infty} \frac{(1)_{n,\lambda}}{(n-1)!n^k} x^n, (\lambda \in \mathbb{R}), \quad (1.4)$$

where $(x)_{0,\lambda} = 1$, $(x)_{n,\lambda} = x(x-\lambda) \cdots (x-(n-1)\lambda)$, $(n \geq 1)$. From (1.4), we note that

$$Ei_{1,\lambda}(x) = \sum_{n=1}^{\infty} \frac{(1)_{n,\lambda}}{n!} x^n = e_{\lambda}(x) - 1. \quad (1.5)$$

The degenerate exponential functions are given by $e_{\lambda}^x(t) = (1 + \lambda t)^{\frac{x}{\lambda}}$, $e_{\lambda}(t) = e_{\lambda}^1(t)$, and the degenerate logarithm functions are defined by $\log_{\lambda}(t) = \frac{1}{\lambda}(t^{\lambda} - 1)$, which is the compositional inverse of $e_{\lambda}(t)$.

The degenerate poly-Genocchi polynomials are defined by

$$\frac{2Ei_{k,\lambda}(\log_{\lambda}(1+t))}{e_{\lambda}(t) + 1} e_{\lambda}^x(t) = \sum_{n=0}^{\infty} g_{n,\lambda}^{(k)}(x) \frac{t^n}{n!}, (k \in \mathbb{Z}). \quad (1.6)$$

For $x = 0$, $g_{n,\lambda}^{(k)} = g_{n,\lambda}^{(k)}(0)$ are called the degenerate poly-Genocchi numbers.

Here $g_{n,\lambda}^{(1)}(x) = G_{n,\lambda}(x)$ are the degenerate Genocchi polynomials given by

$$\frac{2t}{e_\lambda(t) + 1} e_\lambda^x(t) = \sum_{n=0}^{\infty} G_{n,\lambda}(x) \frac{t^n}{n!}, \quad (\text{see [12, 13]}). \quad (1.7)$$

More generally, for $r \in \mathbb{N}$, the Genocchi polynomials of order r are defined by

$$\left(\frac{2t}{e_\lambda(t) + 1} \right)^r e_\lambda^x(t) = \sum_{n=0}^{\infty} G_{n,\lambda}^{(r)}(x) \frac{t^n}{n!}, \quad (\text{see [13]}). \quad (1.8)$$

From (1.6), we note that

$$g_{n,\lambda}^{(k)}(x) = \sum_{l=0}^n \binom{n}{l} g_{l,\lambda}^{(k)}(x) {}_{n-l,\lambda}, \quad (n \geq 0). \quad (1.9)$$

We will need the Carlitz's degenerate Euler polynomials $\mathcal{E}_{l,\lambda}^{(r)}(x)$ of order r given by

$$\left(\frac{2}{e_\lambda(t) + 1} \right)^r e_\lambda^x(t) = \sum_{n=0}^{\infty} \mathcal{E}_{n,\lambda}^{(r)}(x) \frac{t^n}{n!}. \quad (1.10)$$

For $k_1, k_2, \dots, k_r \in \mathbb{Z}$, we define the degenerate multiple polyexponential function as

$$Ei_{k_1, k_2, \dots, k_r, \lambda}(x) = \sum_{0 < n_1 < n_2 < \dots < n_r} \frac{(1)_{n_1, \lambda} \cdots (1)_{n_r, \lambda} x^{n_r}}{(n_1 - 1)! (n_2 - 1)! \cdots (n_r - 1)! n_1^{k_1} \cdots n_r^{k_r}}, \quad (1.11)$$

where the sum is over all integers n_1, n_2, \dots, n_r , satisfying $0 < n_1 < n_2 < \dots < n_r$.

2. Degenerate multi-poly-Genocchi polynomials

For $\lambda \in \mathbb{R}$, the degenerate Stirling numbers of the first kind are defined by

$$(x)_n = \sum_{l=0}^n S_{1,\lambda}(n, l)(x)_l, \quad (n \geq 0), \quad (\text{see [1, 2, 5 - 18]}), \quad (2.1)$$

where $(x)_0 = 1$, $(x)_n = x(x - 1) \cdots (x - n + 1)$, $(n \geq 1)$.

From (2.1), we note that

$$\frac{1}{k!} (\log_\lambda(1+t))^k = \sum_{n=k}^{\infty} S_{1,\lambda}(n, k) \frac{x^n}{n!}, \quad (k \geq 0), \quad (2.2)$$

and that $\lim_{\lambda \rightarrow 0} S_{1,\lambda} = S_1(n, l)$, where $S_1(n, l)$ is the Stirling number of the first kind.

For $k_1, k_2, \dots, k_r \in \mathbb{Z}$, we consider the degenerate mulit–poly–Genocchi polynomials which are given by

$$\frac{2^r Ei_{k_1, k_2, \dots, k_r, \lambda}(\log_\lambda(1+t))}{(e_\lambda(t) + 1)^r} e_\lambda^x(t) = \sum_{n=0}^{\infty} g_{n, \lambda}^{(k_1, k_2, \dots, k_r)}(x) \frac{t^n}{n!}. \quad (2.3)$$

For $x = 0$, $g_{n, \lambda}^{(k_1, k_2, \dots, k_r)} = g_{n, \lambda}^{(k_1, k_2, \dots, k_r)}(0)$ are called the degenerate multi–poly–Genocchi numbers.

From (2.3), we note that

$$g_{n, \lambda}^{(k_1, \dots, k_r)}(x) = \sum_{l=0}^n \binom{n}{l} g_{l, \lambda}^{(k_1, k_2, \dots, k_r)}(x)_{n-l, \lambda}, \quad (n \geq 0). \quad (2.4)$$

From (2.3), we have

$$\begin{aligned} & \sum_{n=0}^{\infty} g_{n, \lambda}^{(k_1, k_2, \dots, k_r)}(x) \frac{t^n}{n!} \\ &= \left(\frac{2}{e_\lambda(t) + 1} \right)^r e_\lambda^x(t) \sum_{0 < n_1 < n_2 < \dots < n_r} \frac{(1)_{n_1, \lambda} \dots (1)_{n_r, \lambda} (\log_\lambda(1+t))^{n_r}}{(n_1 - 1)! \dots (n_r - 1)! n_1^{k_1} \dots n_r^{k_r}} \\ &= \sum_{l=0}^{\infty} \mathcal{E}_{l, \lambda}^{(r)}(x) \frac{t^l}{l!} \\ &\quad \times \sum_{0 < n_1 < n_2 < \dots < n_r} \frac{(1)_{n_1, \lambda} \dots (1)_{n_r, \lambda}}{(n_1 - 1)! \dots (n_{r-1} - 1)! n_1^{k_1} \dots n_{r-1}^{k_{r-1}} n_r^{k_r-1}} \frac{(\log_\lambda(1+t))^{n_r}}{n_r!} \\ &= \sum_{l=0}^{\infty} \mathcal{E}_{l, \lambda}^{(r)}(x) \frac{t^l}{l!} \\ &\quad \times \sum_{0 < n_1 < n_2 < \dots < n_r \leq m} \frac{(1)_{n_1, \lambda} \dots (1)_{n_r, \lambda} S_{1, \lambda}(m, n_r)}{(n_1 - 1)! \dots (n_{r-1} - 1)! n_1^{k_1} \dots n_{r-1}^{k_{r-1}} n_r^{k_r-1}} \frac{t^m}{m!} \\ &= \sum_{n=r}^{\infty} \left(\sum_{l=0}^{n-r} \binom{n}{l} \mathcal{E}_{l, \lambda}^{(r)}(x) \sum_{0 < n_1 < n_2 < \dots < n_r \leq n-l} \frac{(1)_{n_1, \lambda} \dots (1)_{n_r, \lambda} S_{1, \lambda}(n-l, n_r)}{(n_1 - 1)! \dots (n_{r-1} - 1)! n_1^{k_1} \dots n_{r-1}^{k_{r-1}} n_r^{k_r-1}} \right) \\ &\quad \times \frac{t^n}{n!}. \end{aligned} \quad (2.5)$$

Comparing the coefficients on both sides of (2.5), we have the following theorem.

Theorem 2.1. For $k_1, k_2, \dots, k_r \in \mathbb{Z}$, and $n, r \in \mathbb{N}$ with $n \geq r$, we have

$$\begin{aligned} & g_{n,\lambda}^{(k_1, k_2, \dots, k_r)}(x) \\ &= \sum_{l=0}^{n-r} \binom{n}{l} \mathcal{E}_{l,\lambda}^{(r)}(x) \sum_{0 < n_1 < n_2 < \dots < n_r \leq n-l} \frac{(1)_{n_1,\lambda} \cdots (1)_{n_r,\lambda} S_{1,\lambda}(n-l, n_r)}{(n_1-1)! \cdots (n_{r-1}-1)! n_1^{k_1} \cdots n_{r-1}^{k_{r-1}} n_r^{k_r-1}}, \end{aligned}$$

Further, we have $g_{n,\lambda}^{(k_1, k_2, \dots, k_r)}(x) = 0$, for $0 \leq n \leq r-1$.

It is immediate to show that

$$\begin{aligned} \left(\frac{2}{e_\lambda(t)+1} \right)^r e_\lambda^x(t) &= \frac{1}{t^r} \left(\frac{2t}{e_\lambda(t)+1} \right)^r e_\lambda^x(t) \\ &= \frac{1}{t^r} \sum_{n=r}^{\infty} G_{n,\lambda}^{(r)}(x) \frac{t^n}{n!} = \sum_{n=0}^{\infty} \frac{G_{n+r,\lambda}^{(r)}(x)}{r!(\frac{n+r}{n})} \frac{t^n}{n!}. \end{aligned} \quad (2.6)$$

Thus, by (2.6), we get

$$\mathcal{E}_{n,\lambda}^{(r)}(x) = \frac{1}{r!(\frac{n+r}{n})} G_{n+r,\lambda}^{(r)}(x), \quad (n, r \geq 0). \quad (2.7)$$

Therefore, by (2.7), we obtain the following corollary.

Corollary 2.2. For $k_1, k_2, \dots, k_r \in \mathbb{Z}$, and $n, r \in \mathbb{N}$ with $n \geq r$, we have

$$\begin{aligned} g_{n,\lambda}^{(k_1, \dots, k_r)}(x) &= \sum_{l=0}^{n-r} \frac{\binom{n}{l}}{r!(\frac{l+r}{l})} G_{l+r,\lambda}^{(r)}(x) \\ &\times \sum_{0 < n_1 < n_2 < \dots < n_r \leq n-l} \frac{(1)_{n_1,\lambda} \cdots (1)_{n_r,\lambda} S_{1,\lambda}(n-l, n_r)}{(n_1-1)! \cdots (n_{r-1}-1)! n_1^{k_1} \cdots n_{r-1}^{k_{r-1}} n_r^{k_r-1}}. \end{aligned}$$

From (2.3), we note that

$$\begin{aligned} \sum_{n=0}^{\infty} g_{n,\lambda}^{(k_1, \dots, k_r)}(r) \frac{t^n}{n!} &= 2^r \left(1 - \frac{1}{2} \frac{2}{e_\lambda(t)+1} \right)^r Ei_{k_1, \dots, k_r, \lambda}(\log_\lambda(1+t)) \\ &= \sum_{m=0}^{\infty} \left(\sum_{l=0}^r \binom{r}{l} (-1)^l 2^{r-l} \mathcal{E}_{m,\lambda}^{(l)} \right) \frac{t^m}{m!} \\ &\times \sum_{0 < n_1 < n_2 < \dots < n_r} \frac{(1)_{n_1,\lambda} \cdots (1)_{n_r,\lambda} (\log_\lambda(1+t))^{n_r}}{(n_1-1)! \cdots (n_{r-1}-1)! n_1^{k_1} \cdots n_{r-1}^{k_{r-1}} n_r^{k_r-1} n_r!} \end{aligned} \quad (2.8)$$

$$\begin{aligned}
&= \sum_{m=0}^{\infty} \left(\sum_{l=0}^r \binom{r}{l} (-1)^l 2^{r-l} \mathcal{E}_{m,\lambda}^{(l)} \right) \frac{t^m}{m!} \\
&\quad \times \sum_{0 < n_1 < n_2 < \dots < n_r \leq j} \frac{(1)_{n_1,\lambda} \cdots (1)_{n_r,\lambda} S_{1,\lambda}(j, n_r)}{(n_1 - 1)! \cdots (n_{r-1} - 1)! n_1^{k_1} \cdots n_{r-1}^{k_{r-1}} n_r^{k_r-1}} \frac{t^j}{j!} \\
&= \sum_{n=r}^{\infty} \left(\sum_{m=0}^{n-r} \sum_{l=0}^r \sum_{0 < n_1 < n_2 < \dots < n_r \leq n-m} \right. \\
&\quad \left. \frac{(1)_{n_1,\lambda} \cdots (1)_{n_r,\lambda} \binom{r}{l} \binom{n}{m} (-1)^l 2^{r-l} \mathcal{E}_{m,\lambda}^{(l)} S_{1,\lambda}(n-m, n_r)}{(n_1 - 1)! \cdots (n_{r-1} - 1)! n_1^{k_1} \cdots n_{r-1}^{k_{r-1}} n_r^{k_r-1}} \right) \frac{t^n}{n!}.
\end{aligned}$$

Therefore, by (2.8), we obtain the following theorem.

Theorem 2.3. *For $k_1, k_2, \dots, k_r \in \mathbb{Z}$, and $n, r \in \mathbb{N}$ with $n \geq r$, we have*

$$g_{n,\lambda}^{(k_1, \dots, k_r)}(r) = \sum_{m=0}^{n-r} \sum_{l=0}^r \sum_{0 < n_1 < n_2 < \dots < n_r \leq n-m} \frac{(1)_{n_1,\lambda} \cdots (1)_{n_r,\lambda} \binom{r}{l} \binom{n}{m} (-1)^l 2^{r-l} \mathcal{E}_{m,\lambda}^{(l)} S_{1,\lambda}(n-m, n_r)}{(n_1 - 1)! \cdots (n_{r-1} - 1)! n_1^{k_1} \cdots n_{r-1}^{k_{r-1}} n_r^{k_r-1}}.$$

By (2.3), we get

$$\begin{aligned}
\sum_{n=0}^{\infty} g_{n,\lambda}^{(k_1, \dots, k_r)}(x+y) \frac{t^n}{n!} &= \frac{2^r Ei_{k_1, \dots, k_r, \lambda}(\log_{\lambda}(1+t))}{(e_{\lambda}(t) + 1)^r} e_{\lambda}^x(t) e_{\lambda}^y(t) \\
&= \sum_{l=0}^{\infty} g_{l,\lambda}^{(k_1, \dots, k_r)}(x) \frac{t^l}{l!} \sum_{m=0}^{\infty} (y)_{m,\lambda} \frac{t^m}{m!} \\
&= \sum_{n=0}^{\infty} \left(\sum_{l=0}^n \binom{n}{l} g_{l,\lambda}^{(k_1, \dots, k_r)}(x) (y)_{n-l,\lambda} \right) \frac{t^n}{n!}.
\end{aligned} \tag{2.9}$$

Thus, by comparing the coefficients on both sides of (2.9), we get the following proposition.

Proposition 2.4. *For $k_1, k_2, \dots, k_r \in \mathbb{Z}$, and any nonnegative integer n , we have*

$$g_{n,\lambda}^{(k_1, \dots, k_r)}(x+y) = \sum_{l=0}^n \binom{n}{l} g_{l,\lambda}^{(k_1, \dots, k_r)}(x) (y)_{n-l,\lambda}.$$

3. Conclusion

As we mentioned in the Introduction, studying various versions of some special polynomials and numbers has drawn attention of some mathematicians, and many interesting results about those polynomials and numbers have been obtained. To state a few, these include the degenerate Stirling numbers of the first

and second kinds, degenerate central factorial numbers of the second kind, degenerate Bernoulli numbers of the second kind, degenerate Bernstein polynomials, degenerate Bell numbers and polynomials, degenerate central Bell numbers and polynomials, degenerate complete Bell polynomials and numbers, degenerate Cauchy numbers, and so on (see [11,12,14] and the references therein). We note here that the study has been carried out by using several different tools, such as generating functions, combinatorial methods, p -adic analysis, umbral calculus, differential equations, probability theory and so on.

In this paper, we introduced the degenerate multiple polyexponential functions which are multiple versions of the degenerate modified polyexponential functions. Then we defined the degenerate multi-poly-Genocchi polynomials by means of those functions. In addition, we derived some explicit expressions for the degenerate multi-poly-Genocchi polynomials and certain properties related to those polynomials.

It is one of our future projects to continue this line of research, namely to study degenerate versions of certain special polynomials and numbers and to find their applications in physics, science and engineering as well as in mathematics.

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